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CURVES AND SURFACES*

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1. Opening remarks. It is difficult to imagine a title more deceptive than the one chosen for this discussion, and yet the choice is necessary since no other phrase describes precisely the content of this paper.

A primary source of confusion is the fact that the words curve and surface have been used in the literature in many different ways. For example, one meaning which has been assigned to these terms has been considered in an elegant expository paper by Whyburn [1]† entitled "What is a curve?" But the curves of Whyburn's paper are sets of points, and whatever the curves of this discussion are, they are not point sets. This last phrase reads like a printer's error, but it is made deliberately. What then is the word curve or surface to mean if not some sort of a point set? And this brings us to a second difficulty.

There is exhibited in the literature a singular reluctance towards defining these terms explicitly. One can search in vain for an explicit definition of curve and surface in the standard texts. The definition to be presented here is not the creation of a single mind; it has evolved through the years.

2. The minimum objective. These opening remarks serve to indicate the minimum objective of this paper. It is to present as clear and precise a definition of curve and surface as possible. One has a right to expect no less, but in this case I do not mean by a minimum objective a minor one, for we shall see that having obtained a precise definition of these concepts we hold a salient far into the theory of curves and surfaces.

The paper is planned therefore to have three parts. The first (§1–§8) will be devoted to defining the fundamental terms while the other two portions (§9–§18 and §19–§23) will deal with the analytic and topological applications. I hope it can be said that the first part is readily accessible even to undergraduates, for in this part I make no attempt to appeal to the experts. To make up for this the last two parts contain problems which might be of interest to all. They certainly contain problems which have yet to be solved.

3. The map. But more of this later. For the present, let A denote a metric space and suppose $f(a)=b$ is a continuous transformation from A into another metric space. This generality is not introduced to confuse and if it seems preferable it is perfectly proper to think of A as the unit interval and the transformation as a triple of continuous functions

$$x = x(u), \quad y = y(u), \quad z = z(u) \qquad 0 \leqq u \leqq 1. \tag{1}$$

* An address delivered at the annual meeting of the Mathematical Association of America in New Brunswick, New Jersey, on September 11, 1943.

† Numbers in brackets refer to the bibliography at the end of the paper.

Or one may think of A as the unit square and the transformation as a triple of continuous functions:

$$(2) \qquad \begin{aligned} x &= x(u, v), \\ y &= y(u, v), \\ z &= z(u, v), \end{aligned} \qquad \begin{aligned} 0 \leqq u \leqq 1, \\ 0 \leqq v \leqq 1, \end{aligned}$$

But however one thinks of A, if B is the graph of the transformation—that is, the image of A under f—it is convenient to write $f(A)=B$ and say that f is a map (or mapping) from A onto B.

4. Equivalence. An indication can now be given as to the definition of a curve or surface. A curve or surface is a *class* of such maps, in fact, an *equivalence class* of such maps. Here again, as happens so often, the concept of an equivalence plays a fundamental role. It cannot be dismissed with a cursory mention.

Given a collection $\mathcal{J}$ of elements f and a relation $\sim$ over $\mathcal{J}$, the relation is an equivalence if it is reflexive ($f\sim f$ for any element f), symmetric ($f_1\sim f_2$ implies $f_2\sim f_1$) and transitive ($f_1\sim f_2$ and $f_2\sim f_3$ imply $f_1\sim f_3$).

One might think it inadvisable to spend any time on so elementary a concept and yet I do so without apology since it is precisely the concept of an equivalence which clinches the definition of curve and surface, and surprisingly enough, has only recently been mentioned in the literature.

Let us therefore pause for an illustration. Suppose that the collection $\mathcal{J}$ is the collection of members of the Association. Further, suppose that member f_1 is called equivalent to member f_2 if the surname of f_1 has the same number of letters as the surname of f_2. It is an easy matter to see that this is an equivalence relation over the Association.

But it is really what an equivalence relation does to a collection that is important, for it partitions the collection into mutually exclusive classes. Each class is generated as follows. The element f_1 generates a class $[f_1]$ which consists of all the elements f which are equivalent to f_1. It follows that $f_1\in[f_1]$‡ and if $f_1\sim f_2$ then $[f_1]=[f_2]$.

Perhaps this "curdling" into mutually exclusive classes can be observed best by referring again to our example. Each member of the Association generates an equivalence class; namely, the class of all those members whose surnames have the same number of letters as his own. Each member is in the class he generates and two equivalent members generate the same class. For convenience we can name these equivalence classes the five-letter class, the six-letter class, in general the n-letter class.

In precisely the same fashion the collection $\mathcal{J}$ of maps f will be divided into equivalence classes. Some of these equivalence classes will be called curves, others surfaces, just as an n-letter class of the preceding paragraph might be called *short* if $n\leqq 5$ and *long* if $n\geqq 10$.

‡ The notation is to be read f_1 is a member of the class $[f_1]$.

5. The equivalence relation. It remains but to choose the equivalence relation one wishes to employ. It is clear that the words curve and surface will have meaning relative to some equivalence relation. For example it can be argued that the relation used by Lebesgue—though these ideas are only implied by his work—is the following.

$f_1(A_1) \sim f_2(A_2)$ if and only if there is a topological§ transformation $T(A_1) = A_2$ such that $f_1(a_1) = f_2(T(a_1))$ for every a_1 in A_1.

This definition has the advantage of extreme simplicity. Perhaps the most pleasing feature is the fact that, given one map, it is easy to get all the others which are equivalent to it. For if $f_1(A_1)$ is a given map, the map $f_2(A_2) = f_1(T(A_2))$ is equivalent to it for any topological transformation $T(A_2) = A_1$, and further, any map equivalent to $f_1(A_1)$ can be so obtained. In spite of this tremendous advantage this notion of equivalence is no longer used, principally because it is difficult to get a satisfactory concept of *distance* between equivalence classes.

Admirably suited to this last purpose is a notion of equivalence due to Fréchet.||

$f_1(A_1) \sim f_2(A_2)$ if and only if there is, for every $\epsilon > 0$ a topological transformation $T_\epsilon(A_1) = A_2$ such that $\rho\{f_1(a_1), f_2(T_\epsilon(a_1))\} < \epsilon$ for every a_1 in A_1.¶

For the present let us not be worried as to what this descriptive definition means from a constructive point of view. No one knows the complete answer to this, but roughly speaking, the relation demands that the maps can be matched as closely as we please through the medium of a topological transformation.

Now consider the collection $\mathcal{J}$ of maps f. Any map f_1 generates a Fréchet equivalence class $[f_1]$ consisting of all maps Fréchet equivalent to $[f_1]$. Any map is a member of the Fréchet equivalence class it generates, and two Fréchet equivalent maps generate, of course, the same Fréchet equivalence class. Any map in a class will be said to *represent* the class or be a *representation* of it.

6. The definitions. The crucial point is that if two maps $f_1(A_1)$ and $f_2(A_2)$ are in the same Fréchet equivalence class $[f]$ then the definition requires that the base spaces A_1 and A_2 be the same from the point of view of topology. Thus we may classify the equivalence classes in terms of the topological character of the base space. *It is now possible to define curve and surface.*

§ A transformation $T(A_1) = A_2$ is topological if it is biunique and bi-continuous. That is, if in addition to the fact that with each a_1 of A_1 there is associated a single a_2 of A_2 it is true that with each a_2 of A_2 there is exactly one a_1 of A_1 such that $T(a_1) = a_2$. Thus there must exist an *inverse* transformation which is denoted by $T^{-1}(A_2) = A_1$. The bi-continuity condition demands that $T^{-1}(A_2) = A_1$ as well as $T(A_1) = A_2$ be continuous.

It is necessary to prove that the relation of Lebesgue is indeed an equivalence. *Reflexivity* follows from the fact that the transformation $T(a_1) = a_1$ is a topological map of A_1 onto itself: *symmetry* from the fact that the inverse of a topological map is topological: *transitivity* from the fact that a topological map followed by a topological map defines a transformation which is topological.

|| An important equivalence, due to McShane [1], requires, in addition, that the degree of the mapping T_ϵ be $+1$. This has meaning, of course only for special types of spaces. See also Radó [1].

¶ See footnote §. The notation $\rho\{a, b\}$ means the distance from a to b.

Consider a class $[f]$ with a base space A. Then $[f]$ will be called a *curve* or *surface* depending on the topological character of A. If A is, topologically speaking, a *closed segment*, then $[f]$ is called a *curve*, more specifically a *curve of the type of a closed segment*. If A is the circumference of a circle, then $[f]$ is called a *curve of the type of a circle*. On the other hand, if A is, topologically speaking, a closed square, then $[f]$ is called a *surface*, more specifically, a *surface of the type of a closed square*. If A is a sphere, then $[f]$ is called a *surface of the type of a sphere*.

7. Specialization. In this discussion we shall deal exclusively with curves of the type of a closed segment and surfaces of the type of a closed square and so refrain from pursuing a digression which might be both pleasant and profitable. In fact, we shall go further in this anti-generalization movement and for the sake of expository clarity consider only mappings into Euclidean 3-space and these only of the special forms (1) and (2) already tabulated.

That is, if we now speak of a map $f(A)=B$ it is automatically understood to be of form (1) or (2). If of form (1) the Fréchet equivalence class it generates is called a curve, if of form (2) the Fréchet equivalence class it generates is called a surface.

8. Historical comment. It is well to notice a sort of historical inevitability in the development of the definition. Consider for example the fact that no doubt a surface was first conceived of as a point set; a plane, a sphere or a cylinder. Topologists branched off at this point with two dimensional manifolds, but analysis began to think of a surface in terms of its description, it was thought of as a triple of continuous functions of the second type. On the other hand, certain pairs of triples were said to describe, or define, or represent the same surface. We have finally come to the place where the loose concept "describing the same surface" has been made rigorous through the notion of Fréchet equivalence and the equivalence classes so obtained are called surfaces.

9. The problems. Having spent this time in clarification of concepts it is gratifying to observe that the momentum acquired carries us naturally towards two objectives. Since we speak of curves and surfaces, can we in some manner speak of length and area? On the other hand, since the subject matter of this paper has been revealed as the study of Fréchet equivalence classes of maps, what can be said of the structure of these classes? The first will be called the *analytic*, the second the *topological* problem.

The analytic problem will be considered first, and a good many of the statements will be made for surfaces only. Many of the details will be omitted since it is the purpose of the last two sections of the paper to indicate merely the spirit of the theory.

10. Polyhedra. A mapping $p(A)=B$ is called quasi-linear (the terminology is due to McShane [1]) if A can be decomposed into a finite number of triangles $\Delta_1, \cdots, \Delta_n$ on each of which the coordinate functions x, y, and z are linear in u and v.

Now $p(\Delta_k)$, the image of Δ_k is a triangle in 3-space which in special cases, it is true, may reduce to a segment or a single point. Without exception, however, it is possible to speak of the area of $p(\Delta_k)$, *an* area which may be zero, and to designate it by $E(p(\Delta_k))$. Let $E(p) = \sum_{k=1}^{n} E(p(\Delta_k))$ be called the *elementary area* of the map p.

In the event a surface $[f]$ has a quasi-linear representation p it will be called a *polyhedron* and designated by $[p]$. The collection of polyhedra is fundamental in what follows.

11. Elementary area. It is perhaps worth noting that a polyhedron will certainly have representations which are not quasi-linear. In fact, a polyhedron will have more than one quasi-linear representation. But, if p_1 and p_2 are two quasi-linear representations of the same polyhedron $[p]$ then it can be shown that $E(p_1) = E(p_2)$. This justifies speaking of this common value as the *elementary area* of the polyhedron and designating it by $E([p])$. Notice that we have gone from a function of a map to a function of an equivalence class of maps. We have, in fact, defined an area for certain surfaces. To extend the definition to all surfaces we need the concept of *distance*.

12. Distance. If $f_1(A)$ and $f_2(A)$ are two maps then there is, for any fixed value a in A, the ordinary Euclidean distance from $f_1(a)$ to $f_2(a)$. Denote it by $|f_1(a) - f_2(a)|$. In a sense, this number measures the deviation of f_1 from f_2 at a. It is natural to say that the deviation of f_1 from f_2 over A is $\max |f_1(a) - f_2(a)| = d(f_1, f_2)$, where the maximum is taken for all $a \in A$. In fact, it is but the matter of a moment to see that this deviation is a bona fide metric over the collection of maps $f(A)$. In other words, this deviation is a non-negative, symmetric function of two maps which is zero if and only if the maps are identical, and the triangle inequality is satisfied. It remains only to assign meaning to the phrase, *distance from the class* $[f_1]$ *to the class* $[f_2]$. Each member of $[f_1]$ has a deviation or distance from each member of $[f_2]$. Nothing could be more natural than to say that the distance from $[f_1]$ to $[f_2]$ will mean the infimum of such deviations. Explicitly $\rho\{[f_1], [f_2]\} = \inf d(f_1^*, f_2^*)$ where the infimum is taken over all $f_1^* \in [f_1]$, and all $f_2^* \in [f_2]$.

This is a metric over the set of equivalence classes $[f]$.

13. Density. Let us make an immediate application of this concept. Suppose $[f]$ is any surface and $f(A) = B$ is one of the representations. Subdivide the square A into n^2 congruent squares and then in each of these draw the diagonal with slope 1. The square A is now decomposed into $2n^2$ congruent triangles. Let Δ be the typical triangle with vertices a_1, a_2, a_3. Consider the points $f(a_1)$, $f(a_2)$, $f(a_3)$. These determine a possibly degenerate triangle Δ^* in 3-space. Let $p(\Delta)$ be the unique linear map from Δ to Δ^*. The map $p_n(A) = B$ which agrees with $p(\Delta)$ on Δ for each Δ can be seen to be quasi-linear. Hence p is a representation of a polyhedron $[p]$. From the uniform continuity of f it follows that p_n converges uniformly to f over A, hence the distance from $[p_n]$ to $[f]$ converges

to zero. We say that $[p_n]$ converges to $[f]$, and use the notation $[p_n] \to [f]$.

What has been shown is that, given any surface $[f]$, there is a sequence of polyhedra $[p_n]$ such that $[p_n] \to [f]$. Each of the polyhedra we have defined is said to be *inscribed* in the surface. It should be mentioned that there are sequences of polyhedra converging to the surface such that none of the polyhedra is inscribed—but more of this later.

14. Area. In view of the convergence of the polyhedra it is a natural inclination to feel that $\lim E([p_n])$ exists. If it does, then the limit should certainly be called the area of $[f]$. Some sixty years ago, in a letter of Genocchi, Schwartz showed that this was not the case even for the simplest possible surfaces. On the other hand $\underline{\lim}\, E([p_n])$ does exist though it depends, of course, on the particular sequence of polyhedra chosen. (The limit may be infinite.) To hurdle the obstacle presented by the example of Schwartz, Lebesgue [1], some twenty years later, considered $\inf \underline{\lim}\, E([p_n])$ where the infimum is taken for all sequences of polyhedra (whether inscribed or not) which converge to $[f]$. This number, possibly infinite, certainly depends only on the equivalence class $[f]$. This number is the celebrated Lebesgue area of the surface $[f]$.

An immediate question arises. The Lebesgue area of the surface $[f]$, denoted by $L([f])$, is defined for any surface $[f]$. If the surface is a polyhedron, how does the Lebesgue area compare with the elementary area? One of the important theorems of the theory, and of course the theorem which makes the whole theory useful, is that the two are equal. This fact enables us to speak of the Lebesgue area as an *extension* of the elementary area.

15. Problem of Geöcze. In the definition of Lebesgue the approximating polyhedra are not necessarily inscribed. If we insist that each polyhedron of the sequence be inscribed in the surface the collection of available sequences is decreased and so the new infimum would certainly be no smaller than the old one. If we denote the new infimum by $\bar{L}([f])$ then it is a conjecture of the Hungarian mathematician de Geöcze that $\bar{L}([f]) = L([f])$. For recent developments along this line one should read Radó [3] and Huskey [1].

16. Length. To return to the definition of *area*, it is clear that in precisely the same way it is possible to define the Lebesgue *length of a curve* $[f]$. We use the same notation $L([f])$. No confusion can arise since in each case we state the character of the equivalence class $[f]$. For curves the de Geöcze conjecture is true.

17. Formulas. Let us halt the development of the theory at this stage to review some of the facts presented in most undergraduate courses on this topic.

You will recall that, in the elementary texts, if a map is given by a triple of type (1) then the map or its graph, there is usually some doubt, is called a curve and its length is given by an integral formula $I(f) = \int_0^1 [x'^2 + y'^2 + z'^2]^{1/2} du$. On the other hand, if a map is given by a triple of type (2) then the map or its graph, again there is some doubt, is called a surface, and its area is given by the

integral formula $I(f)=\int_0^1\int_0^1[X^2+Y^2+Z^2]^{1/2}dudv$, where X, Y and Z are the usual Jacobians.

It is, I believe, a fair test of an advanced theory to ask how well it deals with the naive, the freshman problems. At any rate, matching the Lebesgue theory to these integral formulas constitutes one of the most fascinating problems in the theory—and one which is by no means completely solved.

It is clear, of course, that the first mapping is, from our point of view merely a *particular representation* of a curve; the second mapping merely a *particular representation* of a surface. It is equally clear that the formulas may be meaningless because the derivatives may not exist. It is therefore extremely unlikely that the integral formulas, when applied to a particular member of an equivalence class, should give the Lebesgue length or area of that equivalence class. In fact, it is more than unlikely, it is definitely false. On the other hand some startling things are true.

Suppose we have a triple f representing a curve $[f]$. A classical theorem [Saks 1] states that:

(1) $L([f])<\infty$ if and only if f is of bounded variation; that is, if and only if each coordinate function is of bounded variation. Hence it is possible to determine conclusively if a curve is of finite length or not simply by examining any one of its representations. Nothing like this is known for surfaces.

Further, if $L([f])<\infty$, then

(2) The integral formula has meaning for every representation and $I(f) \leqq L([f])$.**

(3) The sign of equality holds if and only if f is absolutely continuous; that is, if and only if each of the coordinate functions is absolutely continuous.

(4) There is always a particular representation f^* such that $I(f^*)=L([f])$.

In comparison with these facts for curves the latest results for surfaces are those of Radó [1]:

(1) If a surface $[f]$ has a representation f^* for which $I(f^*)$ is finite then $I(f^*)\leqq L([f])$.

(2) The sign of equality holds if and only if there is a sequence of polyhedra $[p_n]\rightarrow[f]$ such that $E([p_n])\rightarrow I(f^*)$.

18. Curves vs. surfaces. Those who are unfamiliar with the field will perhaps be astonished at the fact that for two types of equivalence classes of maps so similar as to definition the results should be so different. It appears that the theory for curves is relatively simple principally because there is a systematic process whereby one can obtain a polygon close to a given curve but not longer than the curve. Nothing similar has been done for surfaces.

As a matter of fact, if $[f]$ is a *curve* and $[p]$ an inscribed polygon, then $L([p])\leqq L([f])$. The corresponding inequality for surfaces does not hold. This is, indeed, the source of the previously mentioned example of Schwartz. The result should not have been as astonishing to mathematicians as it appears to have

** The integrals are to be taken now in the sense of Lebesgue.

been, though it is all too easy to make such an observation after the event. There is a fundamental difference between inscribed polygons and inscribed polyhedra, neither of which, one will notice, has been yet defined explicitly.

In view of the historical importance of this difference a slight digression may now be permitted.

A polygon $[p]$ is said to be *inscribed in a curve* $[f]$ if there is a representation f of $[f]$ and a quasi-linear representation p of $[p]$ such that on each of the intervals $\Delta_1, \cdots, \Delta_n$ of the decomposition of A required in the definition of p, it is true that $f=p$ on the frontier of Δ_k (notation $F(\Delta_k)$) for each k.

Suppose we make the corresponding statement for surfaces. A polyhedron $[p]$ is said to be *inscribed in a surface* $[f]$ if there is a representation f of $[f]$ and a quasi-linear representation p of $[p]$ such that on each of the triangles $\Delta_1, \cdots, \Delta_n$ of the decomposition of A required in the definition of p, it is true that $f=p$ on the frontier of Δ_k for each k.

This statement is of course the sheerest nonsense since it requires two conditions to be satisfied which are, in general, mutually incompatible. On Δ_k the map p is supposed to be linear which implies that the image of $F(\Delta_k)$ is the boundary of a triangle (possibly degenerate); at the same time p is supposed to agree with the map of f on $F(\Delta_k)$, and the map of f on this set may be incredibly complicated.

Actually an inscribed polyhedron is defined by making a tremendous modification in the last condition. It is required merely that $f=p$ at *three* points; namely, the *vertices* of $\Delta_k (k=1, \cdots, n)$.

It should not be astonishing then that for a surface $[f]$ and an inscribed polyhedron $[p]$ instead of $L([p]) \leqq L([f])$ we usually have $L([p]) > L([f])$.

For curves the ideal character of inscribed polygons can be seen almost at once. Let A be the unit interval and consider a decomposition $D: \Delta_1, \cdots, \Delta_n$. Suppose $\mu(\tilde{D})$ is any mapping of $\tilde{D}=\sum_1^n F(\Delta_k)$ into 3-space and $f(A)$ is a map such that $f(a)=\mu(a)$, for $a \in \tilde{D}$. In this case the curve $[f]$ is said to be *supported by the network* $\mu(\tilde{D})$.

Consider the diameter of $\mu(F(\Delta_k))$, denoting it by $\delta(\mu, \Delta_k)$. The number $\sum_1^n \delta(\mu, \Delta_k)$ is of fundamental importance, for no matter what curve is supported by $\mu(\tilde{D})$, and no matter how "length" is defined, it is clear that the "length" of the curve should be no less than $\sum_1^n \delta(\mu, \Delta_k)$. This leads at once to the fact that an inscribed polygon is no longer (in the sense of Lebesgue) than the curve in which it is inscribed, since the curve is supported by a network of vertices of the polygon.

How is the case with surfaces? The statements of the penultimate paragraph can be reproduced word for word.

So far so good, but what notion is now to replace the diameter of a set so that analogous statements can be made throughout? There is an obvious danger in over simplification, but from one point of view it can be said that completing this analogy was the end towards which de Geöcze directed his efforts. Tremendous strides in this direction have been taken by Radó [4] and [5] and Reichelder-

fer [2], culminating in the elegant definition of an "essential area" by Reichelderfer [1]. The Lebesgue approach certainly does not supply an analogy.

19. The need for topology. In the course of comments in §17 it will have been noticed that some of the statements depend upon a particular representation of the equivalence class while others are independent of the representation. This makes it all the more important to investigate the structure of these classes, a study which has interested Kerékjártó [1 and 2], Morrey [1 and 2], Morse [1], and Tompkins [1].

20. Topological problems. This portion of the paper will be considered in two parts each of which suggests itself on a brief examination of the facts. Whenever one has a class of objects it is interesting to know something about properties which are enjoyed by every member of the class. Such properties are called *invariant*. On the other hand, there are always properties which are not invariant, and this makes it possible to say that some members of the class are better, with respect to a certain property, than others, for some members will possess a property not common to all. In the search for these good members it would naturally be useful to have a *constructive process* whereby one could obtain other members of a class of which one representation is known.

The *problem of invariants* is considered first and then the *problem of representations*.

21. Invariants. One invariant has already been mentioned since it has been observed, for curves, that if a single member of the equivalence class is of bounded variation, then every member has this property.

A simple and interesting invariant is the graph. A map $f(A)=B$ has a graph B which is also the graph of all the maps which are Fréchet equivalent to f. Hence B is invariant and may be called the graph of the equivalence class $[f]$.

But the graph of an equivalence class gives absolutely no indication as to the character of the class; for we have the somewhat astonishing fact that *every graph of a curve is also the graph of some surface, and conversely*. This simply serves to highlight an early statement that a curve or surface is not a point set.

As a matter of fact, any Peano space in 3-space is the graph of some curve and so also the graph of some surface. This leads to such peculiarities as a curve or surface whose graph fills out a cube.

Very well, one will say, the graph of an equivalence class gives no clue as to whether the equivalence class is a curve or a surface, but surely the graph is important in matters of length and area. The facts do not support this reasonable conjecture, for it is true, on the contrary, that *any graph is the graph of some surface of zero area*.

A final invariant can be described only after a somewhat technical discussion. The reader is here invited to consult Whyburn [2]. The map $f(A)=B$ induces a decomposition of the set A. For each point $b\in B$ there is an inverse set $f^{-1}(b)$ in A consisting of the totality of points which are mapped onto b. The set $f^{-1}(b)$ will not be connected in general. Let α be the notation for a component of

$f^{-1}(b)$, and use $\sum$ to denote the totality of α's. Each $a \in A$ is in exactly one α, so we may define a single valued association $m(a) = \alpha$ to mean $a \in \alpha$. The important thing is that the set $\sum$ can be topologized in such a way that $m(a) = \alpha$ is a continuous map. Moreover it is clearly monotone; that is, each inverse set is connected.

The abstract space $\sum$ can be thought of as the graph of the monotone part of $f(A)$. *This graph is invariant.* The statement should be interpreted to mean that if $f_1(A)$ is Fréchet equivalent to $f_2(A)$ and $\sum_1$ and $\sum_2$ are the corresponding abstract spaces, then there is a topological transformation $T(\sum_1) = \sum_2$.

22. Representations. As previously noted, the fact that not all properties are invariant brings us to the question of representations. This is the problem of constructing other members of an equivalence class of which one representation is at hand.

Suppose that $f(A) = B$ is a known representation of a class $[f]$. Then $g(A) = f(T(A))$ is a member of the class $[f]$ for any topological transformation $T(A) = A$. This is obvious on looking at the definition, for the topological transformation required for Fréchet equivalence is precisely the above T regardless of the size of ϵ.

A somewhat more sophisticated construction is the following. A map $g(A) = f(m(A))$ is a member of the class $[f]$ for any monotone map $m(A) = A$. This construction does not, however, yield all the members of the class f though its implications are not without interest.

Let me add that the interest in monotoneity does not end here. It has been shown that with every map $f(A) = B$ there is associated a monotone map $m(A) = \sum$. In fact, the space $\sum$ is associated with the equivalence class $[f]$, and whereas in this never-never land the graph of $[f]$ has very little to do with the length or area of the equivalence class $[f]$, the space $\sum$ which is the graph of m has a considerable say in the matter.

If $[f]$ is a curve, its length is zero if and only if $\sum$ is a single point. This theorem is obvious, but for surfaces there is the truly magnificent theorem of Morrey [2] and Radó [2]. The area of the surface $[f]$ is zero if and only if $\sum$ is a tree. (A tree, or dendrite, is a special type of Peano space which has a "slender" structure).

23. Deformations. A discussion of the problem of representations would not be complete without mentioning the work of Morse [1] and Tompkins [1] on deformations.

Suppose that for each t, $0 \leqq t \leqq 1$, there is a map $f(a, t)$ of the unit segment $0 \leqq a \leqq 1$. Suppose, further, that $f(a, t)$ is continuous in both a and t. In other words, if $a_n \to a$ and $t_n \to t$, then $f(a_n, t_n) \to f(a, t)$. In this case $f(a, t)$ is called a *homotopy* and the map $f(a, 0)$ is said to be *homotopic* to the map $f(a, 1)$. For each t, $0 \leqq t \leqq 1$, the map $f(a, t)$ is a representation of some curve $[g_t]$. The continuity condition guarantees that if $\tau \to t$, then $[g_\tau] \to [g_t]$. The converse question is this: Suppose we have a class of curves $[g_t]$ for each t, $0 \leqq t \leqq 1$; and suppose

that if $\tau \to t$, then $[g_\tau] \to [g_t]$. Is it possible to choose a representation $f(a, t)$, $0 \leqq a \leqq 1$, of $[g_t]$ for each t such that $f(a, t)$ is continuous in both variables?

The answer to this question is in the affirmative as has been shown by Morse [1]. Tompkins [1] extends this result to curves of the type of a circle.

In the course of his discussion, Morse shows that, for any curve $[f]$ there is a master representation $\phi(A) = B$, having the property that for any representation $g(A) = B$ of $[f]$ there is a monotone map $m(A) = A$ such that $g(a) = \phi(m(a))$ for each a in A.

Steenrod has asked about similar theorems for surfaces. Not much is known here, though it will be proved elsewhere that there is no such thing as a master representation for surfaces. Since this is of essential use in the argument employed in the homotopy result one is tempted to conjecture that the theorems do not carry over. At any rate we have here an unsolved problem to add to the spectrum of questions still open for investigation.

BIBLIOGRAPHY

Fréchet, M. [1] Sur la distance de deux surfaces. Annales de la Société Polonaise de Mathématique, v. 3, 1924, pp. 4–19.

Huskey, Harry D. [1] Contributions to the problem of Geöcze. Duke Mathematical Journal, v. 10, 1943, pp. 249–257.

Kerékjártó, B. [1] Involutions et surfaces continues I, Acta Szeged, v. 3, 1927, pp. 49–67.

[2] Involutions et surfaces continues II, Acta Szeged, v. 3, 1927, pp. 242–249.

Lebesgue, H. [1] Intégrale, Longueur, Aire. Annali di Matematica Pura ed Applicata, v. 7, 1902, pp. 231–359.

McShane, E. J. [1] On the semi-continuity of double integrals in the calculus of variations. Annals of Mathematics, v. 33, 1932, pp. 460–484.

Morrey, C. B. [1] The topology of (path) surfaces. American Journal of Mathematics, v. 57, 1935, pp. 17–50.

[2] An analytic characterization of surfaces of finite Lebesgue area, II. American Journal of Mathematics, v. 58, 1936, pp. 313–322.

Morse, Marston. [1] A special parametrization of curves. Bulletin of the American Mathematical Society, v. 42, 1936, pp. 915-922.

Radó, Tibor. [1] On the semi-continuity of double integrals in parametric form. Transactions of the American Mathematical Society, v. 51, 1942, pp. 336–361.

[2] On continuous path-surfaces of zero area. Annals of Mathematics, v. 44, 1943, pp. 173–191.

[3] On a problem of Geöcze. American Journal of Mathematics, v. 65, 1943, pp. 361–381.

[4] On absolutely continuous transformations in the plane. Duke Mathematical Journal, v. 4, 1938, pp. 189–221.

[5] With Reichelderfer, P. A theory of absolutely continuous transformations in the plane. Transactions of the American Mathematical Society, v. 49, 1941, pp. 258–307.

Reichelderfer, P. [1] On bounded variation and absolute continuity for representations of continuous surfaces. Transactions of the American Mathematical Society, v. 53, 1943, pp. 251–291.

[2] With Radó, Tibor. See Radó [5].

Saks, S. [1] Theory of the Integral, Warsaw, 1937.

Tompkins, C. [1] Fréchet deformations and homotopy. Annals of Mathematics, v. 40, 1939, pp. 392–399.

Whyburn, G. T. [1] What is a curve? This MONTHLY, v. 49, 1942, pp. 493–497.

[2] Analytic Topology, American Mathematical Society, Colloquium Publication, v. 28, 1942.

MATHEMATICS TEXTBOOKS FOR THE INTRODUCTORY COLLEGE COURSES*

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1. Introduction. The accelerated production of elementary books on college mathematics during the past few years raises a number of questions concerning the mathematical curriculum and bears indirectly on larger questions concerning the aims of mathematical education. Perhaps the textbook plays too great a role in our elementary courses, but it does seem to be true that the teaching of mathematics is influenced to a considerable extent by the character of the available textbooks. By way of examples, the content of the course in the theory of equations as given in the majority of our colleges has been determined to a large extent by Dickson's First Course, and the spirit in which the calculus has been taught in this country for many years has been influenced greatly by Granville's Calculus and several similar books. One wonders how many of the more recent books are destined to play similarly significant roles in mathematical education.

Certainly the bewildering array of survey texts, basic texts, unified texts, brief texts, comprehensive texts, *etc.*, leads to an embarrassment of riches for the instructor who is responsible for selecting a book for a particular course or purpose. The situation becomes even worse when one is confronted with the tidal wave of war course books relating to navigation, map making, artillery fire, training for pilots, *etc.* These books no doubt have their proper place, but it is unlikely that courses based on them will have a permanent place in the undergraduate curriculum in mathematics. In this connection it is of interest that both the Navy V-12 and the AST programs in mathematics stress, with slight variations, topics which have been considered standard in elementary collegiate courses for many years.

If we assume that, for the most part, the content of the standard courses up to and including the calculus is essentially fixed, much latitude remains with regard to the interweaving of the various topics, the degree of rigor, and the style of presentation. The textbook writers have been thoroughly cognizant of all of these possibilities, and the resulting multiplicity of treatments, to say nothing of the various permutations and combinations of topics, is becoming somewhat appalling. It is proposed here to discuss briefly some of these trends in their relation to the curriculum and to the larger question of the purpose of mathematical education.

2. The general course. There is much difference of opinion as to what constitutes the most desirable course for the general student who, presumably, will elect but one year of mathematics. The question as to the most desirable type of course for such students is a moot one and the textbooks reflect a diversity of opinions. If quantity of production is the criterion the conclusion is

* Presented at the meeting of the Mathematical Association of America at Chicago, Illinois, November 28, 1943.

inescapable that the traditional separate courses in college alegbra, trigonometry, and analytic geometry are to be preferred. A number of writers, however, contend that there is much to be gained by a unification and interweaving of these topics into a single course. Still others believe that the elements of calculus should be introduced, but there seems to be a considerable difference of opinion as to the amount of calculus which should be included in such a course. Then there is the survey-type text which scans the gamut of graduate and undergraduate work in mathematics, emphasizing fundamental concepts, cultural aspects, and historical background, but excluding much of the traditional drill work. Although some of these books are admirable, it is believed that their greatest value is to supplement rather than to replace books of a more conventional type.

Some of the survey-type books invite the neophyte to retread the more or less familiar ground of his high school algebra but to use a postulational approach in order that he may see the reasons underlying the rules of the operations. Then additional applications may be made, as for example, to the algebra of sets or to various kinds of geometries. Historical incidents and applications of various kinds are discussed in order that the student may better appreciate the significance of mathematics in a broader sense. In such a course few direct demands are made on the students' previous training in mathematics and but little time is devoted to the development of the techniques which are necessary for further work in mathematics. This last consideration is a very important one in view of the fact that many entering freshmen expect to take but one year of mathematics and later, through choice or necessity, take additional courses in the subject.

It should not be necessary for a college course in mathematics to offer an apology for requiring as prerequisite the elements of plane geometry and a knowledge of algebra through the simpler types of quadratic equations, although it is recognized that many colleges must of necessity admit students who do not have this minimum preparation. The beginning course in mathematics for such students should be primarily a remedial course and if a postulational approach to elementary algebra provides the most efficient means for enabling the student to overcome the handicap of a poor preparation then this method should certainly not be neglected. The survey-type books are not designed for the brief remedial course in elementary algebra which should be the most vital concern of poorly prepared students. The twin objectives of preparing a student for further work in mathematics and of giving him some appreciation of the values of the subject in a year course are believed not to be inconsistent. For the poorly prepared student a considerable amount of drill work in connection with a review of elementary algebra appears unavoidable if a permanent handicap in mathematics is to be precluded, but material which is new and significant to the student should be introduced as early as is feasible. Certainly the well prepared student should be introduced to new and undiluted material almost from the start if his interest in mathematics is to continue to develop.

At risk of being trite we reaffirm the tenet that the first collegiate course in mathematics should aim to acquaint all students of the subject with the nature of precise mathematical definitions and proofs of theorems and, also, to help them to acquire those basic skills without which further work in either mathematics or its applications would be unthinkable. The purpose of the course should not be to entertain the student, or to impress him with the diverse applications of mathematics, or to attempt to increase the popularity of the subject by watering it down so as to make it easy for the indolent to grasp. It seems entirely possible that many of the brilliant men and women who have avoided mathematics after their first year in college have done so because of a lack of interest in a diluted first course which consisted largely of an embellishment of their high school work in the subject. Diverse applications, historical background (if not confined to mere footnotes), *etc.*, undoubtedly add interest to the course, but they are of secondary importance, even as motivating influences, as compared with precision of definitions and clearness and elegance of proofs. And yet a cursory examination of a number of recent books of various types revealed that some authors were more concerned with rearranging subject matter, adding historical data, increasing the number of applications, *etc.*, than in avoiding the perpetration of unsatisfactory definitions, incomplete statements of theorems, and clumsy proofs. Fortunately there are notable exceptions, and within the last few years excellent books in almost all of the afore-mentioned categories have appeared.

In the light of the foregoing discussion, and for other reasons, it seems erroneous to assume that different groups of students with the same mathematical preparation should have different types of beginning courses in college mathematics. Naturally this conclusion is not applicable to terminal courses which are required for some specific professional preparation, as, for example, the mathematical theory of investment.

3. Courses in college algebra and trigonometry. College algebra, or a combination of college algebra and trigonometry, is perhaps the most commonly offered introductory course in college mathematics. In the case of college algebra the pattern of the course reveals a rapid succession of heterogeneous topics such as quadratic functions and equations, progressions, the binomial theorem, inequalities, complex numbers, logarithms, theory of equations, ratio and variation, permutations and combinations, determinants, *etc.* It has been pointed out by several writers that college algebra, as commonly presented, is the most heterogeneous of all the subjects constituting elementary college mathematics. Although a remedy for this state of affairs was proposed by G. A. Bliss* as early as 1911, and an excellent book by Wilczynski and Slaught† appeared in 1916, the majority of books on college algebra continued to treat the subject as a collection of miscellaneous topics, and in many cases with little considera-

* Monographs on Topics of Modern Mathematics, edited by J. W. A. Young, page 264.
† College Algebra, with Applications, Wilczynski and Slaught.

tion given to the reasoning upon which the algebraic processes depend. The proposal by Bliss and the book referred to both place the function concept in a central position and classify the various topics in relation to the types of functions involved. While this program does not seem to have been fully carried out in books on algebra, there has been a notable increase in emphasis on functional relationships, and it is no longer uncommon to find linear and quadratic equations playing a subsidiary role to the corresponding functions. This in turn has led to a greater use of graphical representations and geometric interpretations of algebraic operations, all of which appears highly desirable in a first course in college mathematics.

The trend of recent books on college algebra, as well as the more comprehensive unified books, toward a greater emphasis on the functional viewpoint is not entirely free from criticism. Purely algebraic concepts, for example, are usually relegated to a secondary place in books of this type, despite the increasing importance of these concepts in modern algebra and in theoretical physics. The crux of the matter seems to be that there are too many important topics in algebra to be included in a beginning course of one semester's length. For definiteness let us consider a group of students who are fairly well prepared in intermediate algebra through the elements of quadratic equations, so that it will not be necessary to consider the question of remedial work in algebra. To the writer it seems that the first course for these students should (in the case of a four or five semester hour course) be a course combining algebra and trigonometry and should emphasize the functional point of view. The algebra part of the course should be quite restricted in scope and would take up linear and quadratic functions, ratio and variation, progressions, and the binomial theorem. This would leave ample time in a four or five semester hour course for a thorough treatment of the essentials of trigonometry, including that portion of the theory of logarithms needed for the computational work in trigonometry. If the course under consideration is a five semester hour course, there would be time, also, for the elements of spherical trigonometry.

During the trigonometry part of the proposed course in algebra and trigonometry polar coordinates might well be introduced in connection with the derivation of the addition formulas, following either the method proposed by E. J. McShane* or the methods used in several recent textbooks on trigonometry.† Due to the importance of these formulas another proof of them might be given at the time when oblique triangles are studied. References may be made to a recent suggestion by A. A. Albert‡ and to the section on trigonometry by H. S. White in the Encyclopaedia Britannica. The use of mathematical induction to generalize the proofs obtained by oblique triangle methods would correlate well with its previous use in the course in proving the binomial theorem.

* The addition formulas for the sine and cosine, this MONTHLY, vol. 48, 1941, page 688.

† Essentials of Trigonometry, Curtiss & Moulton. Plane and Spherical Trigonometry, Rickey & Cole.

‡ A suggestion for a simplified trigonometry, this MONTHLY, vol. 50, 1943, p. 251.

These examples are cited merely to indicate the spirit in which the writer believes the course should be taught. The instructor who feels that the topics in this proposed first course in algebra and trigonometry are of insufficient interest to the general student may find in the various survey-type books a rich mine for collateral reading assignments to whet the students' interest without detracting from the main purpose of the course.

With the exception of determinants the course in algebra and trigonometry just considered provides the necessary background for fairly complete courses in analytic geometry and elementary calculus.

Students who take a course such as the one described, and who plan to specialize in mathematics or physical science, should have a more sophisticated course in college algebra added to their curriculum. Furthermore, the traditional course in the theory of equations does not satisfy this requirement. The proposed course in college algebra could be taken in the sophomore year concurrently with the calculus, or in the junior year. It would start with the number system and develop the principles of college algebra through complex numbers from a postulational point of view and might include topics in the elementary theory of numbers such as Euclid's algorithm, the theory of partial fractions, linear dependence, and the principles of matrices necessary for the complete discussion of a system of linear equations, as well as much of the classical theory of equations preceded by a careful discussion of the division transformation and the properties of polynomials. Topics which recently have suffered notable neglect, such as the multinomial theorem, finite differences, and summation of series might well find a place in such a scheme. Several of the older books on college algebra and a few recent books are available as reference books to be used in connection with such a course.

4. The course in analytic geometry. Analytic geometry, whether given as a separate subject or scattered through a unified course, should receive the most attention in the first year of college mathematics. This is natural in view of the inherent unity of its content, the opportunity it affords for the development of both intuition and rigor, and the excellent means it provides for exploiting the students' previously gained knowledge of elementary algebra and trigonometry. Unfortunately these desiderata have been achieved only in part by some of the standard and most popular textbooks on analytic geometry. This is evidenced by failures to emphasize the meaning of the converse of a theorem, by restrictions of proofs to special cases, and by tendencies to avoid parts of the students' previous mathematical experience as, for example, the addition theorems of trigonometry. However, some of the more recent books show marked improvements over many of their predecessors in all of these directions. In particular, the tendency to make greater use of the elementary properties of determinants in both plane and solid analytic geometry is to be commended. In most textbooks on analytic geometry a knowledge of determinants gained from the college algebra course is assumed, but in some respects a better plan is to introduce the elementary properties of determinants in the course on analytic geometry

as this makes the work with determinants more meaningful from the start. This has been recognized by the authors of several recent textbooks, and it is hoped that the plan will be adopted more generally.

For the student who needs the first semester to prepare himself in algebra and trigonometry, the introductory course in analytic geometry is limited in most cases to a four or five semester hour course if he is to begin the study of calculus in the first semester of his sophomore year. For reasons previously indicated, the course in analytic geometry, if properly taught, has great educational value, and no apology is needed for giving it this amount of emphasis, notwithstanding the conglomerate variety of topics which of late have been finding their way into first year courses at its expense. Moreover, a few of the new books on analytic geometry have added to the supply of interesting problems of the type which arouse the interest of those students who are intellectually susceptible, whether their main interest is in mathematics, economics, or medicine.

A number of available books on analytic geometry are readily adaptable to a two semester course and such a program is an excellent one for the student with a good high school preparation in algebra and trigonometry. The inclusion of work in the method of least squares and curve fitting in some texts adds to interest in the course for science students without detracting from the main objectives. In fact, the method of least squares provides a natural application of previous results on quadratic functions, while the graphs of the power function and exponential function using logarithmic and semi-logarithmic scales, respectively, follow naturally from the students' previous study of the graphs of these functions using uniform scales. Another trend in a two semester course is to introduce the elements of calculus in connection with the course in analytic geometry. This varies from the practice of introducing the derivative exclusively for the purpose of studying tangents and normals to conics, to the other extreme where the differentiation of algebraic functions is presented in considerable detail and applications are made to maxima and minima problems, to time rates of change, to curve tracing, *etc.* Some writers, especially of the so-called unified texts, seem to have emphasized these applications at the expense of the underlying theory of limits, a procedure which seems deplorable even though there are those who maintain that students should be shielded from precise definitions and proofs of theorems involving limits and continuity until they have completed a year's work in the calculus.

5. The course in calculus. The more or less popular belief that the first course in the calculus should deal almost entirely with formal manipulations and problem solving has resulted in the production of an array of textbooks which meet these demands. As a result of this attitude the serious student has his introduction to analysis deferred until he studies advanced calculus, and in some instances until the course in function theory is reached. On the other hand the student who takes but one year of calculus all too often remembers the subject only as a collection of rules and formulas and has few if any impressions concerning the nature of the reasoning employed and the elegance of the meth-

ods. It is encouraging that some of the better books published very recently indicate a definite trend away from this defeatist attitude.

The insertion of an isolated chapter on integration early in the calculus text is a practice which has lately become vogue. The chief advantages claimed for such a procedure seem to be that interesting applications may be made earlier in the course and that concurrent courses in physics or engineering may be taken with greater profit. An obvious disadvantage is that time must be taken for a review of this early chapter on integration if the subsequent chapters on integration are to be preceded by chapters which deal only with differential calculus.

The tendency to emphasize numerous applications early in the course in calculus has led some writers to take up the calculation of the area under a curve prior to the definition of the definite integral, whereas, if hazy notions are to be avoided the definite integral should be used to define the area under a curve. Justification by logic rather than by magic of the method of integration by substitution, the correct proof of the formula for the derivative of a function of a function, and the distinction between multiple and iterated integrals are other cases in point where proper care is not always exercised. Relatively few additions are being made to the hierarchy of books on introductory calculus which pay careful attention to questions of this kind. A detailed discussion of these and other matters will not be attempted here, but it is the writer's belief that the introductory and advanced courses in the calculus should be presented in the same spirit and that the two courses should form a unified two years' course in the subject.

6. Conclusion. The ever increasing multiplicity of books and treatments relating to introductory college mathematics has an important relation to methods of mathematical instruction and to the shaping of the curriculum, and it is believed that more definite aims and norms should be established. It is suggested that precision of definitions, simplicity and elegance of proofs, and significance of the work for the further study of mathematics be made to sound the dominant notes. It is believed that emphasis on such matters should not be inconsistent with the purposes of a general course in mathematics, especially if the best books of the survey type are used for collateral reading assignments.

It is undeniably true that students have difficulty with courses of the type which have been advocated in this discussion. Textbooks which develop the concepts and theorems carefully and completely are by their very nature difficult for students to read and make great demands on their powers of concentration. This, however, is as it should be, if the thesis is valid that the main function of the instructor is to elaborate on the subject and to clarify the points of difficulty encountered by the student in his reading. The roles of entertainer, taskmaster, and drillmaster should be subsidiary to that of interpreter if the true aims of mathematical education are to be realized.

DIFFERENTIATION OF FOURIER SERIES AND INTEGRALS

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1. Introduction. The object of this paper is to state precise necessary and sufficient conditions under which the Fourier series of a function f may be formally differentiated, term by term, to obtain the Fourier series of the derivative f'. There is an analogous consideration of Fourier integrals. The theorems and proofs are all elementary in character. They are of such a nature as to be readily adapted to the needs of a course at the advanced calculus level, or slightly beyond, where students meet the problem of Fourier representation as it arises in connection with the solution of physical problems.

It should be pointed out at once that the questions at issue here are easily disposed of with the aid of the Lebesgue integral. One may readily prove that if f and f' belong to the class $L(-\pi, \pi)$ the Fourier series of f' results from formally differentiating the Fourier series of f if and only if $f(x)$ is almost everywhere equal to a suitably chosen indefinite integral of f', and if in addition,

$$\int_{-\pi}^{\pi} f'(x)dx = 0.$$

The necessity of these conditions hinges upon the theorem that two functions are equivalent if they have the same Fourier coefficients.

2. Fourier series and the class S. We shall denote by S the class of real functions f defined for all values of the real variable x, and subject to the following limitations:

(a) $f(x+2\pi)=f(x)$;

(b) *to each f in S there corresponds a finite number of points $x_0, x_1, \cdots, x_m$ $(-\pi=x_0<x_1<\cdots<x_m=\pi)$, such that the derivative f' exists throughout each open interval (x_{k-1}, x_k) $(k=1, 2, \cdots, m)$. Furthermore, it is required that f' be bounded and Riemann-integrable over each of these subintervals.*

It follows from (b) that f is continuous in the open interval (x_{k-1}, x_k). Furthermore, by the law of the mean, if M is the least upper bound of $|f'(x)|$ on (x_{k-1}, x_k), and x, x' are any two points within that interval, $|f(x)-f(x')| \leq M|x-x'|$. Therefore, by Cauchy's criterion for the existence of a limit, the function approaches limits $f(x_{k-1}+0)$ and $f(x_k-0)$ at the left and right ends, respectively, of the interval. Then, by a standard argument,*

$$\int_{x_{k-1}}^{x_k} f'(x)dx = f(x_k - 0) - f(x_{k-1} + 0). \tag{1}$$

It is clear that if f is in S and g is a continuously differentiable function with period 2π, the product function fg is in the class S.

* See Philip Franklin, A Treatise on Advanced Calculus, New York, 1940, p. 202. Observe that it is not necessary to know anything about the differentiability of $f(x)$ at the ends of the interval (x_{k-1}, x_k).

THEOREM 1. *For a function f of class S, the Fourier series of f' results by formal differentiation of the Fourier series of f if and only if $f(x_k-0)=f(x_k+0)$ when $k=1, 2, \cdots, m$.*

Proof. If a_n, b_n are the Fourier coefficients of f, and a_n', b_n', are those of f', the statement, that the Fourier series of f' is obtainable by formally differentiating the Fourier series of f, is expressed by the equations

$$a_n' = nb_n, \qquad b_n' = -na_n, \qquad n = 0, 1, 2, \cdots.$$

These may be written conveniently in the form

$$a_n' + ib_n' = -in(a_n + ib_n),$$

or, taking cognizance of the integral formulas for the coefficients,

$$\int_{-\pi}^{\pi} f'(x)e^{inx}dx = -in\int_{-\pi}^{\pi} f(x)e^{inx}dx, \qquad n = 0, 1, 2, \cdots. \tag{2}$$

We transform these conditions as follows:

$$\int_{-\pi}^{\pi} f'(x)e^{inx}dx = \sum_{k=1}^{m}\int_{x_{k-1}}^{x_k} f'(x)e^{inx}dx;$$

$$\int_{x_{k-1}}^{x_k} f'(x)e^{inx}dx = f(x_k-0)e^{inx_k} - f(x_{k-1}+0)e^{inx_{k-1}} - in\int_{x_{k-1}}^{x_k} f(x)e^{inx}dx.$$

Here $x_0, x_1, \cdots, x_m$ have the meaning indicated in condition (b) of the definition of class S; the formula of integration by parts takes the above form in view of formula (1).

Let us now write

$$c_k = f(x_k-0) - f(x_k+0), \qquad k = 1, 2, \cdots, m.$$

Then we see that, since $f(x_0+0)=f(x_m-0)$,

$$\begin{aligned}\int_{-\pi}^{\pi} f'(x)e^{inx}dx &= \sum_{k=1}^{m}\{f(x_k-0)e^{inx_k} - f(x_{k-1}+0)e^{inx_{k-1}}\} - in\int_{-\pi}^{\pi} f(x)e^{inx}dx \\ &= \sum_{k=1}^{m} c_k e^{inx_k} - in\int_{-\pi}^{\pi} f(x)e^{inx}dx;\end{aligned}$$

the sum has been rearranged, and an appeal made to periodicity. Equations (2) now take the form

$$\sum_{k=1}^{m} c_k e^{inx_k} = 0, \qquad n = 0, 1, 2, \cdots. \tag{3}$$

The determinant of this linear homogeneous system in the c's, when n is allowed the range $0, 1, \cdots, m-1$, is well known. Its value is

$$\Pi(e^{ix_k} - e^{ix_j}),$$

the product being taken for j and k from 1 to m, with $j<k$. The determinant is thus obviously different from zero. It follows that the system of conditions (3) holds if and only if all the c's are zero. This completes the proof of the theorem.

3. Fourier integrals and the class T. Let T denote the class of real functions f defined for all values of the real variable x, and subject to the limitations:

(a) *To each* f *in* T *there corresponds an enumerable set of points* $\{x_n\}$, $n=0, \pm 1, \pm 2, \cdots$, *so arranged that* $x_n<x_{n+1}$, *and that* $x_n \to +\infty$ *as* $n \to +\infty$, *while* $x_n \to -\infty$ *as* $n \to -\infty$. *The derivative* f' *exists except perhaps at the points* x_n, *and is bounded and Riemann-integrable over each interval* (x_n, x_{n+1});

(b) *the following integrals are convergent:*

$$\int_{-\infty}^{\infty} | f(x) | \, dx, \qquad \int_{-\infty}^{\infty} | f'(x) | \, dx.$$

To obtain, for Fourier integrals, an analogue of Theorem 1, we consider, instead of Fourier coefficients, Fourier transforms. The analogue of conditions (2) is then the equation

$$\int_{-\infty}^{\infty} f'(x)e^{itx}dx = - it \int_{-\infty}^{\infty} f(x)e^{itx}dx, \tag{4}$$

where t is a real parameter. If (4) is true for all values of t then formally, at least, the Fourier integral representation of f' results by differentiating under the integral in the Fourier integral representation of f.

THEOREM 2. *If* f *is in class* T *and if*

(a) $f(x_n-0) \to 0$ *as* $n \to +\infty$,

(b) $f(x_n+0) \to 0$ *as* $n \to -\infty$,

(c) *the series* $\sum_{-\infty}^{\infty} |f(x_n-0) - f(x_n+0)|$ *is convergent*,

then (4) *is an identity in* t *if and only if* $f(x_n-0) = f(x_n+0)$ *for all integral values of* n.

Proof. We transform equation (4) in a manner similar to the transformation of (2), as carried out in the proof of Theorem 1. From the formula

$$\begin{aligned}\int_{x_\mu}^{x_\nu} f'(x)e^{itx}dx = - it \int_{x_\mu}^{x_\nu} f(x)e^{itx}dx + f(x_\nu - 0)e^{itx_\nu} \\ + \sum_{n=\mu+1}^{\nu-1} [f(x_n - 0) - f(x_n + 0)]e^{itx_n} - f(x_\mu + 0)e^{itx_\mu},\end{aligned}$$

together with the use of conditions (a), (b), (c) of the theorem, it is clear that (4) is equivalent to the equation

$$\sum_{-\infty}^{\infty} \{f(x_n - 0) - f(x_n + 0)\} e^{itx_n} = 0, \tag{5}$$

where the series is absolutely convergent, and hence uniformly convergent with respect to t.

To complete the proof, let us introduce the operation

$$M[\phi] = \lim_{A\to+\infty} \frac{1}{2A}\int_{-A}^{A} \phi(t)dt.$$

This operation is certainly meaningful if ϕ is a bounded function which is integrable over every finite interval. Moreover, if $\phi_n(t)\to\phi(t)$ uniformly in t, then $M[\phi_n]\to M[\phi]$, for

$$| M[\phi_n] - M[\phi] | = | M[\phi_n - \phi] | \leqq \text{l.u.b.} \ | \phi_n(t) - \phi(t) | .$$

The operation M has the further property that

$$M[e^{it\lambda}] = \begin{cases} 0 & \text{if} \quad \lambda \neq 0, \\ 1 & \text{if} \quad \lambda = 0. \end{cases}$$

Now suppose that

$$\phi(t) = \sum_{-\infty}^{\infty} c_n e^{itx_n},$$

where $c_n = f(x_n - 0) - f(x_n + 0)$. It follows from the foregoing considerations that, for any fixed m,

$$M[\phi(t)e^{-itx_m}] = \sum_{-\infty}^{\infty} c_n M[e^{it(x_n - x_m)}] = c_m.$$

In this way we see that (5) is an identity in t if and only if all the c's are zero. This completes the proof of Theorem 2.

The meaning of conditions (a)–(c) of Theorem 2 can best be made clear with the aid of the concept of the total variation of a function. Let us denote by f^* the function defined as follows:

$$f^*(x) = f(x) \text{ except at the points } x_n,$$
$$f^*(x_n) = \tfrac{1}{2}[f(x_n + 0) + f(x_n - 0)].$$

Then, for any f in S, conditions (a)–(c) hold if f^* is of bounded variation in the range $(-\infty, \infty)$. This is not difficult to show, and we omit the proof. On the other hand, if f is in S, and (c) holds, then f^* is of bounded variation, and therefore (a) and (b) hold as well. To see this, let

$$F(x) = \int_{-\infty}^{x} f'(t)dt, \tag{6}$$

and define $g(x) = f^*(x) - F(x)$. The total variation of F is

$$\int_{-\infty}^{\infty} | f'(t) | \, dt,$$

and g is a step-function whose total variation is given by the series in condition (c). Thus, if (c) holds, g, and hence f^*, is of bounded variation.

It is of interest to remark that there exist functions in the class T such that the series in condition (c) is divergent. For example, let

$$x_0 = 0, \quad x_1 = 1, \quad x_n = x_{n-1} + \frac{1}{n}, \qquad n = 1, 2, \cdots.$$

If n is of the form 2^k, $k=1, 2, 3, \cdots$, define $f(x)=1$ in the interval $x_{n-1}<x<x_n$. For all other values of n define

$$f(x_{n-1}+0) = \frac{(-1)^{n+1}}{n}, \qquad f(x_n - 0) = \frac{(-1)^{n+1}}{n+1},$$

and define f as a linear function between x_{n-1} and x_n. Finally, let f be an even function. It is then verifiable that f is in T, and that the series in (c) diverges more rapidly than the series whose general term is $2/(n+1)$.

In the above example the lower limit of $x_{n+1}-x_n$ is zero. The following lemma shows that if the lower limit of $x_{n+1}-x_n$ is positive, then conditions (a)–(c) are fulfilled, and a great deal more.

LEMMA. *If f is in T, and if m_n is the greatest lower bound of $|f(x)|$ in the open interval (x_n, x_{n+1}), then the series*

$$\sum_{-\infty}^{\infty} m_n, \quad \sum_{-\infty}^{\infty} |f(x_n+0)|, \quad \sum_{-\infty}^{\infty} |f(x_n-0)|$$

are either all convergent or all divergent. If there exists a positive constant h such that $h \leqq x_{n+1}-x_n$, $n=0, \pm 1, \pm 2, \cdots$, they are all convergent.

Let us temporarily, for convenience of writing, assume that $f(x)$ is defined at the ends of the interval (x_n, x_{n+1}) by its limiting values from within that interval. Then there must be some ξ_n for which $x_n \leqq \xi_n \leqq x_{n+1}$ and $|f(\xi_n)| = m_n$. We have

$$|f(\xi_n) - f(x_n+0)| = \left| \int_{x_n}^{\xi_n} f'(x)dx \right|,$$

$$|m_n - |f(x_n+0)|| \leqq \int_{x_n}^{\xi_n} |f'(x)|\, dx \leqq \int_{x_n}^{x_{n+1}} |f'(x)|\, dx,$$

whence, since the series

$$\sum_{-\infty}^{\infty} \int_{x_n}^{x_{n+1}} |f'(x)|\, dx$$

is convergent (by the definition of class T), we conclude, by the comparison test, that the first two series in the lemma converge or diverge together. The proof as regards the first and third series is entirely similar. Finally, if $0<h \leqq x_{n+1}-x_n$,

$$\int_{x_n}^{x_{n+1}} |f(x)|\, dx \geqq (x_{n+1} - x_n)m_n \geqq hm_n.$$

Since $\int_{-\infty}^{\infty}|f(x)|dx$ is convergent, it follows that the series $\sum_{-\infty}^{\infty} m_n$ converges.

The conclusion of Theorem 2 may be drawn from less restrictive hypotheses.

THEOREM 3. *If f is in the class T, and if the improper integral*

$$\int_{-\infty}^{\infty} |F(x)|\, dx \tag{7}$$

is convergent, where $F(x)$ is defined by (6), *then* (4) *is an identity in t if and only if $f(x_n-0)=f(x_n+0)$ for all integral values of n.*

To see that the hypothesis here is weaker than that of Theorem 2, it suffices to consider the following function. Let

$$x_0 = 0, \quad x_1 = 1, \quad x_n = x_{n-1} + \frac{1}{n}, \qquad n = 1, 2, \cdots.$$

If n is of the form 2^k, $k=1, 2, 3, \cdots$, define $f(x)=1$ in the interval $x_{n-1}<x<x_n$. Otherwise let $f(x)$ equal zero. Then the integral (7) is zero, but the series in (c) of Theorem 2 is divergent.

Proof of Theorem 3. The function $F(x)$ is continuous, and $F(x)\to 0$ as $x\to -\infty$. $F(x)$ approaches a limit when $x\to+\infty$, because the integral of f' from $-\infty$ to $+\infty$ is convergent. This limit must be zero, since otherwise the integral (7) would diverge. Now

$$\int_{x_{k-1}}^{x_n} f'(x)e^{itx}dx = F(x_n)e^{itx_n} - F(x_{n-1})e^{itx_{n-1}} - it\int_{x_{n-1}}^{x_n} F(x)e^{itx}dx.$$

Hence, summing on n from $\mu+1$ to ν,

$$\int_{x_\mu}^{x_\nu} f'(x)e^{itx}dx = F(x_\nu)e^{itx_\nu} - F(x_\mu)e^{itx_\mu} - it\int_{x_\mu}^{x_\nu} F(x)e^{itx}dx.$$

If now $\nu\to+\infty$ and $\mu\to-\infty$, we see that equation (4) takes the form

$$it\int_{-\infty}^{\infty} g(x)e^{itx}dx = 0, \tag{8}$$

where $g(x)=f(x)-F(x)$. The integral in (8) is absolutely, and hence uniformly, convergent with respect to t, because of our hypotheses.

Let us now suppose that (8) holds for all values of t. Then, if $t\neq 0$,

$$\int_{-\infty}^{\infty} g(x)e^{itx}dx = 0, \tag{9}$$

and (9) is also true if $t=0$, because the function defined by the integral is continuous. Now let us multiply (9) by e^{-ivt}, and integrate from $-p$ to p, where $p>0$. Then

$$0 = \int_{-p}^{p} e^{-iyt}dt \int_{-\infty}^{\infty} g(x)e^{itx}dx = \int_{-\infty}^{\infty} g(x)dx \int_{-p}^{p} e^{it(x-y)}dt,$$

$$0 = 2\int_{-\infty}^{\infty} g(x)\,\frac{\sin p(x-y)}{x-y}\,dx.$$

But it is well known* that the limit of this last integral, as $p\rightarrow\infty$, is $\pi g(y)$, provided that g has a derivative at y. In particular, then, $g(y)=0$ if $x_{n-1}<y<x_n$. This implies that $f(x_n+0)=f(x_n-0)$, since F is continuous.

On the other hand, suppose that $f(x_n-0)=f(x_n+0)$, $n=0, \pm 1, \pm 2, \cdots$. This implies that $g(x_n-0)=g(x_n+0)$. But g is constant in each interval $x_{n-1}<x<x_r$, since $g'(x)=0$ there. Hence we infer that g has the same constant value, say c, within each interval. The only possibility is that $c=0$, since the integral

$$\int_{-\infty}^{\infty} |\, g(x)\,|\, dx$$

is convergent. Thus (9) is an identity in t. But then so is (4), which is equivalent to (8). This completes the proof of Theorem 3.

MATHEMATICS, 300 B.C.–200 B.C.

MAX DEHN, St. John's College

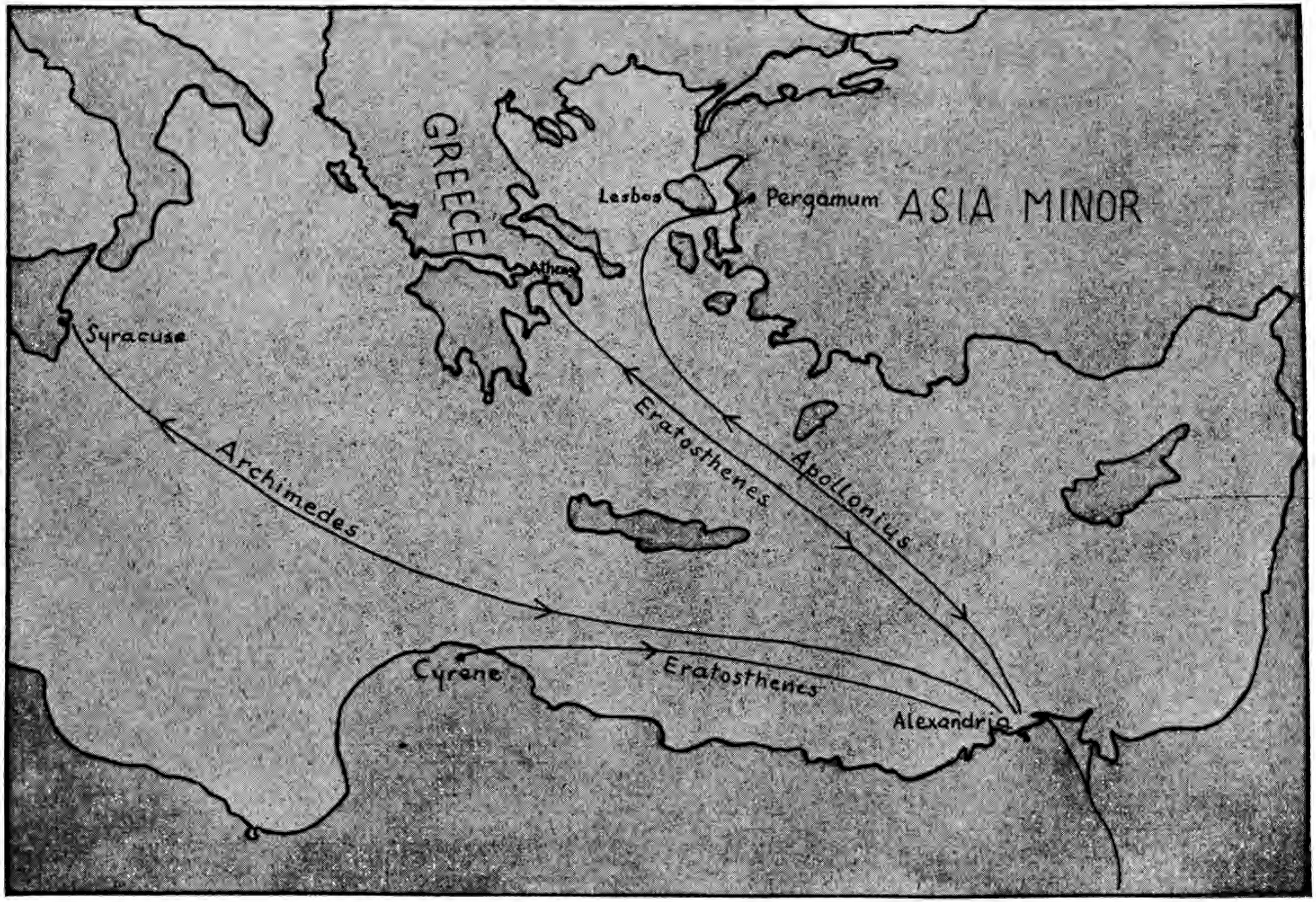
1. The life of Archimedes. The third century before Christ was dominated by the achievements of two men, Archimedes and Apollonius.

Archimedes has been considered, during his life and since his death up to this very day, the best known mathematician. He died in the year 212 B.C. when the Roman army conquered Syracuse. About fifty years later, the Greek historian Polybius, living in Rome and writing on Roman history, related how Archimedes, by his mechanical inventions, became a formidable adversary to the Roman general who besieged Archimedes' native city. "The soul of one man," writes Polybius, "created almost insurmountable obstacles for the Roman army."

Archimedes saw the mathematical structure of static phenomena. This insight enabled him to do important engineering work in war and peace, and to bring to light the basic principles of the statics of rigid bodies and ideal fluids. He does not seem, however, to have attached too high a value to these achievements, since he wanted on his own tombstone a figure symbolic of his measurement of the sphere: a sphere with the circumscribed cylinder.

He was of noble birth, a kinsman and friend of Hiero, king of Syracuse. Syracuse had been for many centuries a center of commerce and culture, a point

* See, for example, R. V. Churchill, Fourier Series and Boundary Value Problems, New York, 1941, pp. 89–90.


GREECE
Lesbos
Pergamum
ASIA MINOR
Athens
Syracuse
Eratosthenes
Apollonius
Archimedes
Cyrene
Eratosthenes
Alexandria

where Greeks and Phoenicians met. It is very probable that Archimedes went to Alexandria and studied there with the successors of Euclid.

A great part of his works is extant. There are no textbooks among them; many of them are concerned with theories or single problems. He was well aware of the value of discoveries and claimed the priority for his own. To show his superiority, he proposed to his Alexandrian competitors certain problems to be solved and certain theorems to be proved, intentionally including among the theorems some wrong ones. To take pride in one's own discoveries is perhaps not the sign of a philosophical mind, but it is certainly characteristic of times of great scientific progress. Archimedes was primarily a research man. He founded no school and had, as far as we know, no personal followers.

Archimedes died in the year 212 B.C. at the hand of a common Roman soldier. His tomb was found and restored by Cicero one hundred and fifty years later, when the latter was governor of Sicily. The Roman statesman did not share our reverent feeling for the unique genius of Archimedes, but spoke with condescending pity of the "modest man operating with sand and writing stylus" (paper and pen).

2. The achievements of Archimedes. (a) *Foundations of mathematics.* He was probably the first to emphasize the axiomatic foundation of continuity, stating the following postulate of basic importance for all non-algebraic operations: The difference of two unequal quantities of the same kind, when added to itself a sufficient number of times, will exceed any other quantity of the same kind. This postulate was called, in the nineteenth century, the postulate of Archimedes. But Archimedes himself did not call it an axiom. He says only that he had to assume it for his deductions and that the mathematicians before him, by tacitly using it, had achieved results universally acknowledged as right. This attitude again shows Archimedes' non-philosophical trend of mind. He did not concern himself with eternal truths and ideas; he preferred to reach his aim by assuming the obvious as true.

Of similar character was the other assumption enabling him to assign to a plane (convex) curve a length, and to a curved (convex) surface an area. We may formulate the assumption this way: if one closed convex surface (curve) is completely inside another closed convex surface (curve), then the former has a smaller area (length) than the latter. It would be interesting to know how he would have assigned a length to a non-plane curve or an area to a surface of negative Gaussian curvature; for example, to the hyperboloid of one sheet.

An application of the "Archimedian" axiom is the computation of the number of grains of sand sufficient to fill the "universe" in the sense of the astronomers of his time. He generated this number by means of exponential operations. This semi-popular work, addressed to Hiero's son Gelo, shows the surprising power of mathematical symbols.

An application of the assumption concerning the length of a plane curve was the approximation of π, the ratio of the circumference of a circle to the diameter. We have seen that, in Euclid, there are no approximations of irrational numbers.

Archimedes computed the length of the perimeter of the regular polygons of 6×2^n sides, and in this manner obtained (in modern notation)

$$2^n \frac{a_n}{b_n} > \frac{\pi}{6} > 2^{n-1}a_n, \qquad (n = 1, 2, \cdots),$$

where

$$b_n = \sqrt{2 + b_{n-1}},$$
$$a_n = \sqrt{2 - b_{n-1}},$$

with

$$b_0 = \sqrt{3}.$$

Archimedes gives approximate values for the resulting algebraic numbers for $n=1, 2, 3, 4$ and obtains

$$\frac{22}{7} > \pi > \frac{223}{71}\cdot$$

The history of the measurement of the circle goes back to very ancient times. The number 3 as an approximate value of π is used by people of low scientific standing. The comparison between the simple experiment yielding this approximation and Archimedes' procedure enabling us to determine π to any degree of accuracy demonstrates the wide distance between two levels of human thinking.

(b) *Problems of tangents and areas.* The major part of the mathematical work of Archimedes is related to what we now call Calculus. For example, the determination of the *tangents* to the "Archimedian" spirals is related to the differential calculus. In the work preserved by Eutokius (500 A.D.) we find the solution of the problem of determining the maximum of the function $x(s-x)^2$. This is done by determining a hyperbola of the type $xy=c$ which touches the parabola $y=(s-x)^2$. We see that Archimedes solved the problem of finding the maximum through the use of tangents as we do it now.

Much more important are the problems connected with the process of *integration.* We have already mentioned the determination of the area of the surface of a sphere with given radius, and the determination of its volume. Archimedes started with the computation of the area of the surface and, for this purpose, had to compute the integral of sin x. He achieved this integration by establishing the identity

$$\sin\frac{\pi}{2n} + \sin\frac{2\pi}{2n} + \sin\frac{3\pi}{2n} + \cdots + \sin\frac{(2n-1)\pi}{2n} = \cot\frac{\pi}{4n}\cdot$$

Characteristic for the trend of Greek mathematics is the formulation of this problem: It is required to construct a plane figure having the same area as the surface of a sphere with given radius. The answer is: The plane figure is a circle with radius twice that of the given sphere. Having found the area of the surface, Archimedes determined the volume.

Further we find Archimedes dealing with the construction of a square with

area equal to that of a plane figure bounded by arcs of a parabola and by straight lines. For the squaring of the parabola, he had to determine the asymptotic value of the sum

$$(1^2 + 2^2 + \cdots + n^2)/n^3.$$

In another solution of the problem, Archimedes makes use of mechanical notions, primarily of the center of gravity and its obvious properties. (Incidentally, it is not easy to take the existence of the center of gravity of an arbitrary body for granted.) He uses these properties to evaluate, as we shall say, an integral.

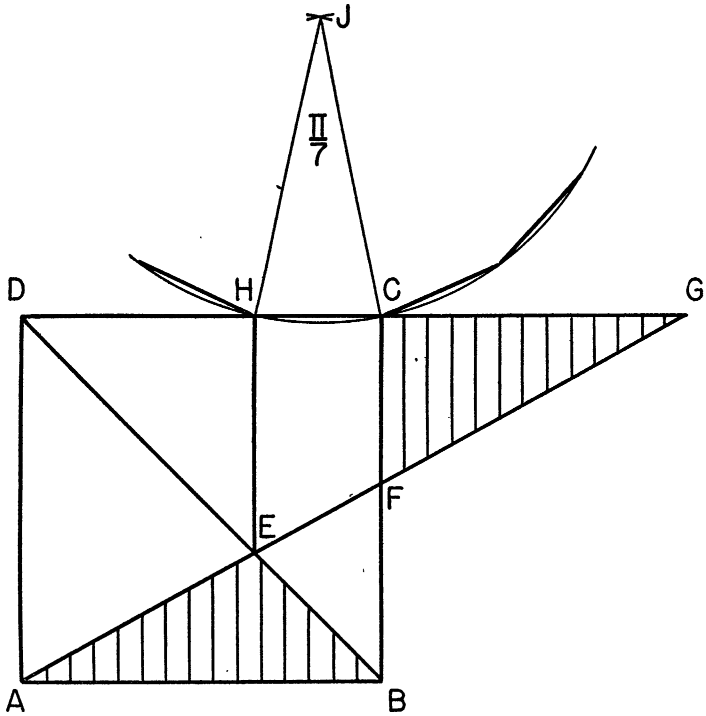


FIG. 1

This is also the method he uses in a work written for Eratosthenes, an outstanding astronomer and mathematician of his time. This work was found only recently in Istanbul, written on parchment that was later used for a liturgical text. Here Archimedes combines the mechanical method with the device of taking areas or volumes as if these were aggregates of lines or surfaces. Everywhere else in his works he uses the exact method of exhaustion. In this latter one, dedicated to a fellow master of mathematics, he does not hesitate to bring into play the relations between properties of areas and properties of a sum of lines. In this work he finds the *volume* of a sphere directly (thereby avoiding the integration of the sine function) reducing the problem to that of the mensuration of the cone.

There are probably extant in Arabic translation some other works of Archimedes. Not long ago there was discovered in the work of an Arabic author a remarkable construction of the regular heptagon by Archimedes. It is as follows: $ABCD$ is a square. A line $AEFG$ is so constructed that it meets the diagonal DB in E, the side BC in F, the side DC in G, so that the triangles AEB and FCG have the same area. H is a point on DC such that EH is parallel to CB. Then HC is the side of the regular polygon with 14 sides in the circle with the radius CG.

Such constructions with a moving ruler meeting two given lines in a prescribed way were often used in Greek mathematics.

3. The achievements of Apollonius. Apollonius, some forty years younger than Archimedes, was born in Pergamum, about 250 B.C. Pergamum was, at this time, beginning to be a center of culture. There are still extant great works of art produced there at the time of Apollonius. It is probable that he lived a part of his life in Alexandria, but he was closely connected with at least one of the rulers of Pergamum, Attalus, to whom he dedicated one book of his great *treatise on conics.*

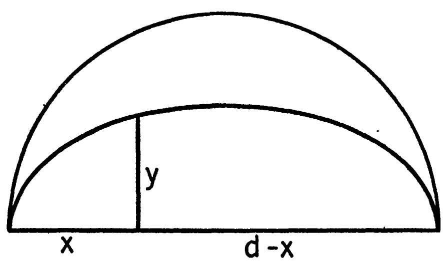


FIG. 2

Whereas Archimedes made his investigations in regions hitherto untrodden and used new methods, Apollonius uncovered in his principal work, the treatise on conics, a field of geometry already partially known, as a grand structure of admirable wealth and beauty. Through the theory of conics the astonishing fecundity of mathematical thought became apparent, perhaps for the first time. One simple notion, in this particular case, that of the plane sections of a cone, produces a wealth of problems and theorems.

One aim of Apollonius was to cover in a systematic treatment all that was already known about conics. Originally, the conics were considered as spatial phenoma. But we find Archimedes already using a definition of the conics as plane loci. The points of these loci are defined by a pair of lines which are different from our common rectangular coordinates only in so far as they are not a pair of proportions (or numbers) obtained through the introduction of unit lines. In Archimedes the "equation" of the conic has the form

$$y^2 = kx(d - x)$$

where k is a proportion (or a number). For $k>0$ and $k\neq 1$ we get the ellipse immediately as the projection of the circle for which $k=1$. In Apollonius, the equation for the conics, the basis for all his investigations, is in oblique coordinates

$$y^2 = px + \frac{px^2}{a}$$

where the + sign gives the hyperbola (y^2 is "surpassing" px), the − sign characterizes the ellipse (y^2 is "deficient" with regard to px); for infinitely great a we get the parabola (y^2 "equals" px). These names for the different conics are probably Apollonius' own.

Apollonius had to solve many problems to achieve a systematic theory of the conics. In the original equation for the conic, there are three parameters: p, a, and the angle between the x line and the y line. From these data one had to find the position and the length of the axes. Further, to have theorems valid for both ellipse and hyperbola, it was necessary to take the two branches of the hyperbola together as one curve, a difficult abstraction since it seems to contradict the appearances. This abstraction became easy only after the introduction of infinitely distant points in the seventeenth century.

In the work of Apollonius there are found the elementary construction of tangents to a given conic through a given point or in a given direction, the discussion of the problems of normals through a given point, and geometric constructions carried out with the help of conics. These problems are related to problems of maxima and minima and to the discussion of the intersection of conics. This latter problem is equivalent to the determination of the number of real roots for an equation of the fourth degree by means of certain inequalities between the coefficients.

The problem of determining a conic by five of its points is not considered, in spite of the solution of equivalent problems. Probably the obstacle was that the problem could not be easily enunciated, because it was necessary to determine at the same time whether one could construct an ellipse, hyperbola, or parabola going through the five given points (the assumption being that no three of them are collinear).

It seems quite impossible to find out who it was that determined the foci. One wonders in what way the old mathematicians came to discover these characteristic points. Probably Euclid already knew that the conics were loci of points for which the distances from a fixed point and from a fixed line have a fixed ratio. Apollonius knew the most important properties of the foci of the ellipse and hyperbola, namely, that they are the centers of the orthogonal involutions, that the tangents make equal angles with the lines through the foci, and finally the theorem concerning the sum or difference of the focal distances.

We conclude our report on Archimedes and Apollonius with a remark by Leibnitz: He who understands Archimedes and Apollonius will admire less the achievements of the foremost men of later times.

DISCUSSIONS AND NOTES

EDITED BY MARIE J. WEISS, Sophie Newcomb College, New Orleans, La.

The department of Discussions and Notes is open to all forms of activity in collegiate mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

A REMARK ON INTEGRATION BY PARTS

J. L. BORMAN, Purdue University

In employing the formula $\int u dv = uv - \int v du$, one makes a skillful choice for u and dv so as to have a product vdu which is readily integrable. Once the choice for dv is made, the function v is obtained by integration. The constant of integration is invariably chosen as zero, since it is felt that this is the simplest choice to make. The point of this note is that the popular choice is, in some cases, the worst rather than the best.

For the sake of completeness let us examine the theory. Suppose it is required of us to integrate

$$\int f(x)g(x)dx$$

and we let

$$u = f(x), \qquad dv = g(x)dx.$$

If $G(x)$ is a function such that $G'(x) = g(x)$, then

$$\int f(x)g(x)dx = f(x)G(x) - \int f'(x)G(x)dx.$$

But it is equally true that

$$\int f(x)g(x)dx = f(x)[G(x) + c] - \int f'(x)[G(x) + c]dx$$

for any constant c.

The object is to choose c so that the last integration is as simple as possible. In most cases the choice $c=0$ is best; indeed, the choice is made automatically, but this is not a rule without exceptions as the following example shows.

In the integral

$$\int x \arctan x \, dx$$

let $u = \arctan x$ and $dv = xdx$, then

$$du = dx/(1 + x^2) \quad \text{and} \quad v = x^2/2 + c.$$

Now if we let $c=0$,

$$\int x \text{ arc tan } x\, dx = \frac{x^2}{2} \text{ arc tan } x - \int \frac{x^2 dx}{2(1+x^2)} \cdot$$

While the last integral is not difficult to evaluate we submit that a tremendous simplification can be produced by choosing $c = 1/2$. In this case

$$\int x \text{ arc tan } x\, dx = \frac{(x^2+1)}{2} \text{ arc tan } x - \frac{1}{2}\int dx.$$

And now the last integration is obvious.

The reader might be interested in constructing other examples of his own. The method seems specially suitable in cases where one is asked to integrate such functions as arc tan $\sqrt{x+k}$ and log $(x+k)$.

CONTINUOUS METHODS FOR AMORTIZING LIFE INSURANCES

C. W. Vickery, Washington, D. C.

The use of credit, with installment payments for repayment of the debt, for the purchase of real estate and durable consumers' goods, has become widespread during recent years. In case of death of the wage earner there may be a need for life insurance to pay off the outstanding debt and protect the estate in possession of the property. While term insurance has been used for this purpose, the necessary protection could be provided more economically through the use of a policy with a decreasing amount designed to be precisely, or approximately, equal to the outstanding balance of the debt at the time of death of the insured. This paper provides a solution of the problem which is exact in the case of a continuously amortizing debt. For the case of annual, monthly, or other periodical payments, usually encountered in practice, a correction is provided which makes the formulas approximately correct. By the device of introducing a new rate of interest, the calculations are reduced to the use of conventional tables except in cases where these may not be available for the required rate. The extensive use of these methods would be facilitated by the calculation of tables for the most commonly used rates of interest.

First we consider the case of a continuously amortizing debt, of present value 1, to be completely amortized at the end of n years, starting at the present time. The balance of principal remaining unpaid at any time t between 0 and n will be

$$\frac{\bar{a}_{\overline{n-t|}\,(r)}}{\bar{a}_{\overline{n|}\,(r)}},$$

where r denotes the annual effective rate of interest. The symbol $(\overline{A}\overline{A})^{1}_{x\overline{n|}}$ is used to denote the single premium of an insurance on a life aged x which will pay at the moment of death an amount precisely equal to the outstanding balance of the debt. The bar over the first A indicates that the amortization is continuous while the bar over the second A indicates that the amount of the insurance is payable at the moment of death of the insured. This notation can

be varied to suit other modes of amortization and payment as indicated below.

We have, then,

$$
\begin{aligned}
(\overline{AA})^{1}_{x\,\overline{n}|} &= \frac{1}{\bar{a}_{\overline{n}|(r)}}\int_0^n \bar{a}_{\overline{n-t}|(r)}(1+i)^{-t}\,{}_tp_x\mu_{x+t}dt \\
&= \frac{1}{\delta_r\bar{a}_{\overline{n}|(r)}}\int_0^n [1-(1+r)^{-(n-t)}](1+i)^{-t}\,{}_tp_x\mu_{x+t}dt \\
&= \frac{1}{\delta_r\bar{a}_{\overline{n}|(r)}}\left\{\int_0^n (1+i)^{-t}\,{}_tp_x\mu_{x+t}dt\right. \\
&\qquad \left. -\int_0^n (1+r)^{-(n-t)}(1+i)^{-t}\,{}_tp_x\mu_{x+t}dt\right\}.
\end{aligned} \tag{1}
$$

It follows that

$$
(\overline{AA})^{1}_{x\,\overline{n}|} = \frac{1}{\delta_r\bar{a}_{\overline{n}|(r)}}\left\{\bar{A}^{1}_{x\,\overline{n}|(i)} - (1+r)^{-n}\bar{A}^{1}_{x\,\overline{n}|(i')}\right\}, \tag{2}
$$

where

$$
i' = \frac{i-r}{1+r}. \tag{3}
$$

We have thus apparently reduced the problem to that of looking up in tables the single premiums of two term insurances with different rates of interest and performing some simple calculations. This is the case provided we have tables for the required rate of interest i'. It may happen in practice that r is greater than i and hence i' is negative. While this would prevent use of the conventional tables, it would cause no difficulty in the calculation of new tables. For practical use tables could be made up for a few rates i' depending on the rate i allowed by the insurance company, such as 3%, and two or three of the rates most commonly used in amortizing obligations, such as $4\frac{1}{2}$% and 5%.

Formula (2) is precise for the case of a continuously amortizing obligation. In the case of a debt being amortized by periodical payments, formula (2) is precise only in case death occurs immediately after a payment has been made. If death occurs immediately before a payment has been made, then the amount paid by the insurance is less than the total amount required by an amount equal to a single payment. We thus need to add to (2) the single premium of a term insurance with amount equal to a single payment. We thus have, to a close approximation, with payments made p times per year:

$$
(A\overline{A})^{1\,(p)}_{x\,\overline{n}|} = (\overline{AA})^{1}_{x\,\overline{n}|} + \frac{1}{pa^{(p)}_{\overline{n}|(r)}}\cdot\bar{A}^{1}_{x\,\overline{n}|}. \tag{4}
$$

For monthly, quarterly, or annual payments formula (4) is sufficiently close, for all moments of death, for practical purposes. When it is in error it is on the side of safety for the insurance company.

A DERIVATION OF CARDAN'S FORMULA

H. B. Curtis, Lake Forest College

Consider first the case of a double root of the cubic

$$x^3 + 3Hx + G = 0, \tag{1}$$

where for convenience G is taken positive. The double root found from the equation $3x^2+3H=0$ is $\sqrt{-H}$. Substituting this value for x in (1) we obtain

$$G^2 + 4H^3 = 0,$$

which is the condition for a double root. This gives $H = -\frac{1}{2}\sqrt[3]{2G^2}$, so that the double root in terms of G is $\frac{1}{2}\sqrt[3]{4G}$. Since the sum of the roots is zero the remaining root is $-\sqrt[3]{4G}$, which may be put in the form $2\sqrt[3]{-G/2}$. That is, the remaining root is

$$x = \sqrt[3]{-\frac{G}{2}} + \sqrt[3]{-\frac{G}{2}}\cdot$$

Now putting the function G^2+4H^3 equal to D, let us try to find what function of D exists so that

$$x = \sqrt[3]{-\frac{G}{2} + f(D)} + \sqrt[3]{-\frac{G}{2} - f(D)} \tag{2}$$

shall satisfy the general equation of the cubic (1). Substituting this value in (1) we obtain without much labor

$$\left(\sqrt[3]{-\frac{G}{2} + f(D)} + \sqrt{-\frac{G}{2} - f(D)}\right)\cdot\left(\sqrt{\frac{G^2}{4} - f^2(D)} + H\right) = 0.$$

Since in general the first factor is not zero we have

$$\sqrt{\frac{G^2}{4} - f^2(D)} + H = 0,$$

which on solving for $f(D)$ gives

$$f(D) = \pm\ \tfrac{1}{2}\sqrt{G^2 + 4H^3}.$$

Putting this value in (2) we have

$$x = \sqrt[3]{-\frac{G}{2} + \tfrac{1}{2}\sqrt{G^2 + 4H^3}} + \sqrt[3]{-\frac{G}{2} - \tfrac{1}{2}\sqrt{G^2 + 4H^3}}$$

which is Cardan's formula.

CLUBS AND ALLIED ACTIVITIES

EDITED BY J. S. FRAME

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to J. S. Frame, Michigan State College, East Lansing, Michigan.

A VARIATION OF NEWTON'S METHOD

J. S. FRAME, Michigan State College

To solve an equation $y=f(x)=0$ by Newton's method, we first find an approximation x_1 to the root r by graphical or other methods. If x_1 was chosen sufficiently close to r we then obtain a second and better approximation x_2, by finding the point $(x_2, 0)$ at which the tangent to the curve at the point (x_1, y_1) intersects the x-axis. The slope of this tangent is $y_1' = f'(x_1)$, so we have

$$\frac{0 - y_1}{x_2 - x_1} = y_1', \tag{1}$$

whence

$$\mathbf{x}_2 = \mathbf{x}_1 + \frac{-\mathbf{y}_1}{\mathbf{y}_1'} \quad \text{(Newton)}. \tag{2}$$

If $y_2=f(x_2)$ is not sufficiently small, we apply the method once more and compute a second correction $-y_2/y_2'$, and a third approximation $x_3=x_2-y_2/y_2'$. This second application of Newton's method usually gives sufficient accuracy in practical cases, but the approximation may be improved to any desired degree of accuracy by repeated applications of the method, provided that the values of $f(x)$ can be obtained to the required degree of accuracy. This is always possible, though perhaps laborious, if $f(x)$ is a polynomial. For many other functions $f(x)$, if the values are obtained from tables, the accuracy is limited to the number of places given in the tables.

A remarkable improvement in the accuracy of the second approximation x_2 can be obtained with but little extra computation by using the second derivative $y_1'' = f''(x_1)$. This will often insure sufficient accuracy to eliminate the necessity for a second application of Newton's method. We shall assume that y, y' and y'' are continuous functions of x, so that the given curve may be approximated by a parabolic arc near the root and we shall further assume that

$$|2yy''| < y'^2 \tag{3}$$

for all x between x_1 and some value $x=a$ beyond the root, so that the parabolic arc will cut the x-axis in real and distinct points. This excludes the consideration of multiple roots of $f(x)=0$.

A parabola with the same slope and curvature as the given curve at the point (x_1, y_1) has the equation

$$(4) \qquad y = y_1 + y_1'(x - x_1) + (y_1''/2)(x - x_1)^2.$$

To find where this parabola crosses the x-axis we set $y=0$, transpose the term y_1, factor out $x-x_1$, and obtain

$$(5) \qquad x - x_1 = \frac{-y_1}{y_1' + (y_1''/2)(x - x_1)}.$$

Assumption (3) insures that the second term in the denominator is small compared with y_1, so we may approximate the factor $x-x_1$ in this second term by Newton's value $-y_1/y_1'$. This then gives an approximation $x=X_2$ to the root r which is closer than the approximation x_2 of equation (2).

$$(6) \qquad \mathbf{X}_2 = \mathbf{x}_1 + \frac{-\mathbf{y}_1}{\mathbf{y}_1' + (\mathbf{y}_1''/2)(-\mathbf{y}_1/\mathbf{y}_1')}.$$

It may be seen that the denominator in (6) is equal to a slope about half way between the slope of the given curve at $x=x_1$ and the slope at $x=x_2$, and as such is a good approximation to the slope of the chord joining the points (x_1, y_1) and $(r, 0)$.

To illustrate the use of formula (6), we use it to find roots of three typical equations and compare results with the first and second stages of Newton's method.

EXAMPLE 1. To evaluate $\sqrt[3]{2}$.
Let $y=f(x)=x^3-2$, $y'=3x^2$, $y''/2=3x$.
Take $x_1=5/4$. Then $y_1=-3/64$, $y_1'=75/16$, $y_1''/2=15/4$.
A first application of Newton's formula (2) gives:

$$x_2 - 1.25 = \frac{3/64}{75/16 + 0} = .010,$$

$$(1.1) \qquad x_2 = 1.260. \qquad [\text{Error} = +\ .00007895.]$$

Formula (6) gives in one step the values

$$X_2 - 1.25 = \frac{3/64}{75/16 + (15/4)(.010)} = \frac{1}{100.8} = .00992063+.$$

$$(1.2) \qquad X_2 = 1.25992063+. \qquad [\text{Error} = -\ .00000042.]$$

A second application of Newton's formula involves cubing 1.26 to find y_2, but does give the more accurate value

$$x_3 - 1.26 = (-\ .000376)/(4.7628) = -\ .00007893.$$

$$(1.3) \qquad x_3 = 1.25992107. \qquad [\text{Error} = +\ .00000002.]$$

EXAMPLE 2. To evaluate sin 50° by solving the cubic equation $8x^3-6x+1=0$ of which it is a root.

Since $1/2<\sin 50^\circ<1$, let us take $x_1=3/4$. Then $y_1=-1/8$, $y_1'=15/2$, $y_1''/2=18$. A first application of Newton's formula (2) gives

$$x_2 - 3/4 = \frac{1/8}{15/2} = \frac{1}{60} = .016667.$$

$$x_2 = .766667. \tag{2.1}$$ [Error = + .000622.]

Formula (6) gives the values

$$X_2 - .75 = \frac{1/8}{15/2 + 18(1/60)} = \frac{1}{62.4} = .016026.$$

$$X_2 = .766026. \tag{2.2}$$ [Error = − .000019.]

A second application of Newton's formula involves cubing 23/30 to find y_2, and gives a result only slightly better than formula (6) gave in one step.

$$x_3 = .766050. \tag{2.3}$$ [Error = + .000005.]

EXAMPLE 3. To solve the equation $x-\cos x=0$.

To avoid the use of tables we may take a first value such as $x_1=\pi/4=0.785398$. Then $y=x-\cos x$, $y'=1+\sin x$, $y''/2=(\cos x)/2$

$$y_1 = \frac{\pi}{4} - \frac{\sqrt{2}}{2} = .078291, \quad y_1' = 1.70711, \quad y_1''/2 = .3536.$$

By Newton's formula (2) we obtain

$$x_2 - .78540 = -\frac{.07829}{1.7071} = -.04586.$$

$$x_2 = .73954. \tag{3.1}$$ [Error = + .000454.]

By formula (6) we obtain

$$X_2 - .785398 = -\frac{-.078291}{1.70711 + (.3536)(-.04586)}$$

$$= -\frac{.078291}{1.69091} = -.046301.$$

$$X_2 = .739097. \tag{3.2}$$ [Error = + .000011.]

It is worth noting how small the error is compared with the first correction. A second application of Newton's method would require the use of tables of cosines for angles in radian measure, whereas the only values needed above were π and $\sqrt{2}$.

CLUB REPORTS 1942–43

Mathematics Club, Hunter College

The Mathematics Club of Hunter College held biweekly meetings during the year 1942–43, at which papers were presented by students. Topics in the fall semester were

Affine transformations, by Ilse Novak

Factorization of determinants, by Barbara Samson

The summation of the P-series by means of the Fourier transformation, by Anna DiFiori

Probability, LaPlace, deMorgan and Venn, by B. O. Koopman of Columbia University, guest speaker.

A volley ball game between the Mathematics Club and Statistical Society was won by the latter.

At the beginning of the spring semester, Hunter's beautiful buildings and campus in the Bronx were taken over by the U. S. Naval Reserve for a training center for Waves and Spars. This resulted in combining the Bronx and Main Building units of the Math Club into one group. Due to crowded conditions and the difficulty of arranging meetings, the Statistical Society was also absorbed, and the three groups heard the following papers together:

Nets of triangles, by Melitta Loewy

Partial fractions, by Joyce Rubin

The path of a heavenly body in relation to another body, by Mary Dolciani.

The activities of the year were brought to a close by the annual Department Party, which this year was held to welcome the freshmen and acquaint them with the students in the Mathematics Department. Professor Louis Weisner directed the Club's activities. The officers for the fall semester were: President, Barbara Samson; Vice-President, Catherine Porcheddu; Secretary, Lucy LaSala; Treasurer, Anna DiFiori; Publicity Manager, Joyce Larick. The officers for the spring semester were: President, Gladys Henlein; Vice-President, Ilse Novak; Secretary, Mae Perlstein; Treasurer, Irene Silver; Publicity Manager, Augusta Schurrer.

Pi Mu Epsilon, University of Arkansas

In addition to the usual business meetings, Arkansas Alpha enjoyed an initiation banquet on November 23, at which nine new members took the *Pi Mu Epsilon* pledge and read essays on mathematics, and a talk was given entitled

Men of Mathematics, by Dean Hosford.

Nine new members were also initiated in the spring of 1943. Since the University is now operating on a full year basis, a picnic is scheduled in the summer term. Officers elected in May 1943 were: President, Jack Hine; Vice-President, Tom Wheat; Secretary, Gloria Heineman; Treasurer, Charles Oxford.

Kappa Mu Epsilon, Illinois State Normal University

At the first meeting in October, nine new members were initiated. Twenty-seven attended the annual Homecoming Breakfast on October 17, at which Mr. Henry Poppen spoke on his experiences at George Peabody College where he worked on his doctorate. In November a discussion of coding was given, entitled

There's secrecy in numbers, by Mildred Bauer and Mary Underwood.

A Christmas party was held in December at which mathematical games were played. The February meeting consisted of a program of talks and demonstrations on

Historical methods of computation, presented by members of the History of Mathematics class.

The topics at the program meetings in March and April were

Astronomy and Mathematics, with slides, by Dr. C. N. Mills

Mathematical recreations, discussed and illustrated by Eleanor Rae Lower, Edith Robinson, Marjorie Brakenhoff, and Rosemary Monnier.

The year's activities closed with the annual spring banquet in April and a picnic in May. At the banquet interesting talks were given by four graduates of 1942: Shirley Isaacson, Nancy Hightower, Geneva Meers, and Ensign Leo Montgomery. A mimeographed booklet, the *Newsletter*, recorded the activities of the year, and was dedicated to the thirty-nine alumni of *Kappa Mu Epsilon* in service. Officers were: Sponsor, Edith Atkin; President Gauss, Mary Underwood; Vice-President Pascal, Robert De Barr; Secretary Ahmes, Mildred Bauer; Treasurer Napier, Frances Marie Cyrier; Historian Cajori, Marjorie Brakenhoff; Social Director Lilavati, Amber Grauer; Corresponding Secretary Descartes, Dr. C. N. Mills.

Hall Mathematical Society, Lafayette College

The club held well-attended bimonthly meetings. Members of the club discussed the following topics:

The magnetic compass, by Dr. W. M. Smith, John Smith, Boyd Kenvin, and B. C. Youngman

Celestial navigation, by Professor John Cawley and John Attinello

Aerial navigation, by Professor W. G. McLean

The MacMillan system of navigation, by Hugh Mahaffy

The Lafayette College wind tunnel, by John Attinello.

At the November meeting the following movies were presented:

Origin of Mathematics

Rectilinear Coordinates

Scientific demonstrations in physics.

The officers for the year 1942 were: President, Calvin Eells; Vice-President, Boyd Kenvin; Secretary, Benjamin Youngman; Librarian, James Hindenach. The officers elected for the year 1943 were: President, James Hindenach; Secretary-Librarian, Frank Eisberg.

Pi Mu Epsilon, Hunter College

During the year 1942–43, *Pi Mu Epsilon* was first directed by Professor A. D. Bradley. Upon his entrance into the armed services, Professor L. S. Hill took over the directorship. Papers were presented by Gertrude Crosby, Marie Johnson, Fortunata Vitanza, Catherine Porcheddu, Mildred Ackerman, Mary Demitrack, Helen Trief, and Deborah Davidson, on various aspects of

Celestial Navigation, including *The Sumner line*, *Ageton's method of celestial navigation*, *Great circle sailing*, *Gnomonic and Mercator projections*.

The remainder of the year was devoted to the presentation of papers by Elaine Frankel, Harriet Eisenberg, Marie Johnson, and Barbara Samson on the subject of

Cryptography.

Many of the cryptographic systems were based on the work of Professor Hill. The fraternity had two social functions during the year, at which new members were initiated into the chapter and old members were welcomed back. The returning members gave reports of the interesting war positions which their mathematical training had helped them to obtain. The student officers were: Vice-Director, Harriet Eisenberg; Corresponding Secretary, Helen Trief; Recording Secretary, Lucy LaSala; Treasurer, Ilse Novak.

RECENT PUBLICATIONS

EDITED BY VIRGIL SNYDER, Cornell University

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York, N. Y. 27, and not to any of the other editors or officers of the Association.

College Algebra. By Frank Morgan. New York, American Book Company, 1943. 6+368 pages. $2.00.

The present book recognizes that the prospective student who reads it may have had one or two years of previous training in algebra. While preparing the reader with the smaller prerequisite, the book promises the representatives of the other category that they too may find the review worth while. The effort is made to present all new ideas in correct and concise language, this aim being retained throughout the book. A rapid review of elementary notions is followed by the new ideas of the course, which include systems of quadratic equations, ratio and proportion, logarithms, theory of investments, mathematical induction, permutations and combinations, probabilities, theory of equations, partial fractions, determinants, and scales of notation. A laudable feature is the frequent insertion of brief historical notes. Answers are not given. The type and press-work are strictly first-class.

VIRGIL SNYDER

Mathematical Recreations. By M. Kraitchik. New York, W. W. Norton and Company, Inc., 1942. 328 pages. $3.75.

The author states in his preface that this book may be regarded as a revised edition of his similar work, published in French, entitled, "La Mathematique des Jeux." How familiar the latter, published in Brussels, 1930, is to the American mathematical public, I am unable to say. My impression is that in general it has not been accessible to the average reader. Consequently a translation brings another book on a subject of wide appeal to both the layman and the mathematician. It may well take its place beside Ball's famous work. Like the latter it contains not only specific problems, to which answers are given and which are often placed in their historical setting, but also essays on subjects of mathematical interest.

The new edition is more than a translation, for the author has supplemented, pruned, rearranged, and rewritten the material. In the rewriting the author has been especially conscious of the fact that he is writing for the American public. As evidence of this change of point of view it is amusing to note that "un docteur de Sorbonne" has been replaced by "a president of the American Mathematical Society" and the metric system has been changed to the English system. Perhaps the inclusion of the game of craps and of questions relating to the game of bridge might also be mentioned in this connection.

The book opens with several amusing items exercising only the powers of reasoning. The author cleverly brings this chapter "Mathematics Without Numbers" from the second part of the old book to the beginning of the new. The second chapter entitled "Ancient and Curious Problems" continues with interesting problems of French, Hindu, and Arabian origin, and concludes with some amusing wrong arithmetical manipulations. The third chapter, "Numerical Pastimes," includes problems on Diophantine analysis, scales af notation, how to guess a selected number, cryptarithmetic, and a discussion of various topics in number theory such as figurate numbers, Fermat numbers, Mersenne numbers, *etc.* The chapter closes with arithmetical games such as Nim and descriptions of toys such as Chinese Rings.

There are two chapters on geometric questions, one devoted to the problem of finding geometric elements whose measures are integers, the other to geometric recreations. In the latter there is a short discussion of the famous topics of topology: the Königsberg bridges, the four-color problem, and the Möbius strip; a collection of problems on the dissection of plane figures; and an interesting description of mosaics. A new page of illustrations showing the use of modified geometric configurations in mosaics as well as a paragraph on a mosaic on a sphere is introduced.

Several chapters have been considerably shortened in the new book. These are the two chapters on chess devoted to the problem of the queens and the problem of the knight, and the chapter on magic squares. The latter is still quite extensive and in it some of the material has been systematized by the use of lattices, the word being used in its old mathematical sense. The author credits Mr. D. A. Flanders with this innovation.

The remaining chapters will be summarized briefly. The chapter on the calendar gives a short historical account of the development of the calendar and discusses and illustrates a perpetual calendar. The chapter on probability describes various games of chance together with problems arising therefrom. Under the chapter entitled "Permutational Problems" the author discusses problems of difficult crossings, shunting problems, and problems of distribution such as, for example, the arranging of schedules for sporting matches and tournaments. The book closes with a chapter on miscellaneous positional and permutational games.

MARIE J. WEISS

Prepare Yourself. Physics experiments with practical applications. By L. F. Tuleen, G. S. Porter, and A. Houston. Chicago, Scott, Foresman and Company, 1943. 6+298 pages. $0.96.

This book is another of the many simplified science texts which have appeared since the beginning of the war. It is intended as a laboratory manual for secondary school physics. Experiments have been chosen with a good deal of ingenuity to represent many basic principles of physics as they are applied to common household problems and appliances. A good variety of experiments is included covering most of the important branches of elementary physics. Some experiments are included which, because of the amount of theory covered, as, for instance, experiment 20 on simple machines, or because of their more complicated nature, as some of the experiments on radio, are well suited to those students who progress faster than most. With each experiment is a brief discussion of the theory to be studied together with references to a number of elementary texts.

Unfortunately the book contains many errors in theory and in laboratory practice. Thus the discussion on page 52 and in the first few experiments concerning significant figures and the accuracy of measured data is badly confused, and leaves the student with the mistaken idea that the accuracy of a number depends in some way on the location of the decimal point. This subject is so important to any laboratory method and is so easy to teach correctly that there is little excuse for misleading the student. Experiment 7, "How does water produce pressure?", would be better used to illustrate Archimedes' Principle than hydrostatic pressure. In an elementary manual it might be much better not to mention the Bernouilli Principle at all than to use it to explain the curving of a baseball—a phenomenon to which it does not apply. In experiment 32, page 155, a measurement of the velocity of sound which was made in air at room temperature is corrected to room temperature as if it had been made at 0° C.

It is because of errors like these and others not mentioned here that teachers of physics in colleges and universities have to spend such a large portion of their time teaching students to forget.

F. E. DART

Principios fundamentales de la division del trabajo. By José Barral Souto. (Cuadernas de trabajo No. 10.) Facultad de Ciencias Económicas. Universidad de Buenos Aires, 1941. 55 pp.

The present study by the South American Statistician and Economist Barral Souto has been published by the Instituto de Biometrica at the University of Buenos Aires. It is noteworthy that the Institute has made to the South American student available, under the directorship of Dr. José González Galé, in the same series also Charlier's Lectures and Darmois' Treatise on Mathematical Statistics, a fact that testifies to the keen awareness with which South America is following the development of technical literature abroad. (Neither of these two classics has as yet found its way into English translation.)

The topic of the pamphlet is a borderline case of theory of foreign trade and general economic theory: the problem of division of labor. As such it is a contribution to a field that, so far, has been neglected by mathematical economists. For more than half a century the only major investigation, treating the subject from a mathematical point of view, has been T. O. Yntema's Mathematical Reformulation of the General Theory of International Trade (1932), after the classical and still very readable analysis by Vilfredo Pareto: Teoria matematica dei cambi forestieri, had been published as early as 1884.

The strong influence of romance economic thought on contemporary South American literature manifests itself throughout the book. Starting from the problem as posed by David Ricardo the author continues in the best tradition of the so called School of Lausanne in applying the tools of equilibrium analysis to his object. This makes the mathematical approach tempting and useful.

The treatment itself is algebraic with a few geometrical illustrations that are original and differ from the well established pattern familiar to students of our more recent textbooks from the chapter on comparative costs (where the story is usually told in form of diagrams as they were originally developed in Enrico Barone's Principii di economia politica). Barral Souto's study does not aim, it should be understood, at foreign trade theory primarily but at the problem of division of labor in general. He starts, however, from the usual position as construed by the classical writers in the field of international trade, making the analysis interesting to students of general theory and of foreign trade.

The professional economist may object to the lack of realism in some of the author's symplifying assumptions that are made explicitly, and more frequently still implicitly; this goes especially for the author's neglect of the possibility of changes over time in the elasticities of the functions and of dynamic changes in the entire indifference structure.

To the mathematically interested reader, however, Barral Souto's contribution is a further indication of the considerable interest of our southern neighbors in the use of exact and quantitative methods in the social sciences and of the infiltration of mathematical patterns into fields that, as the present one, are of considerable practical interest to the economic politician as well as of academic interest to the school economist.

J. E. Morton

Commercial Algebra. By H. E. Stelson and H. P. Rogers. New York, The Macmillan Co., 1943. 11+283 pages. $2.50.

This book is concerned with intermediate algebra. It includes the topics ordinarily treated in this subject and also presents chapters of commercial arithmetic on buying and selling merchandise, simple interest, simple discount, compound interest, consumer credit and annuities, together some thirty pages. The chapter on logarithms also contains a discussion of the slide rule. A year of elementary algebra is presupposed. Following the text are a number of tables including logarithms of numbers, a mortality table, compound amount, present value amount of an annuity, amount of an annuity, and a table of powers, roots and reciprocals. Answers to the odd-numbered exercises are given. Others can be obtained on request.

The type and press-work are excellent, as are also the 57 figures in the text.

VIRGIL SNYDER

NEW BOOKS RECEIVED

Navigation. By L. M. Kells, W. F. Kern, and J. R. Bland. New York and London, McGraw-Hill Book Co., Inc., 20+479 pages. $3.75.

Vectors and Matrices. By C. C. MacDuffee. (Carus Mathematical Monographs, no. 7.) Ithaca, New York, Mathematical Association of America, 1943. 9+192 pages. $2.00.

Methoden der mathematischen Physik. By R. Courant and D. Hilbert. New York, Interscience Publishers, Inc. Reprint of Vol. 1, 1931; Vol. 2, 1937. 15+469+19+549 pages. Vol. 1, $8.00; Vol. 2, $8.00. Vols. 1 and 2, $14.00.

Business Mathematics for College Students. By G. S. Whiteskee. New York and London, McGraw-Hill Book Co., Inc., 1943. 12+184 pages. $1.50.

Navigational Trigonometry. By P. R. Rider and C. A. Hutchinson. New York, The Macmillan Company, 1943. 9+232 pages. $2.00.

Clear Thinking. By L. H. Schnell and M. Crawford. New York and London, Harper and Brothers, 1943. 14+358 pages. $1.96.

Solid Geometry and Spherical Trigonometry. By H. L. Leighton. New York, D. Van Nostrand Co., Inc., 1943. 19+210 pages. $2.20.

Plane Trigonometry. By A. W. Weeks and H. S. Funkhouser. New York, D. Van Nostrand Co., Inc., 1943. 8+193 pages. $1.75.

Calculus. By L. M. Kells. New York, Prentice-Hall, Inc., 1943. 8+509 pages. $3.75.

Random Sampling Distributions. By A. E. Treloar. Minneapolis, Burgess Publishing Co., 1942. 84 pages. $2.25.

Correlation Analysis. By A. E. Treloar. Minneapolis, Burgess Publishing Co., 1942. 96 pages. $1.50.

Analytic Geometry. By F. H. Steen and D. H. Ballou. Boston, Ginn and Co., 1943. 8+206+9 pages. $2.40.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND H. S. M. COXETER

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to H. S. M. Coxeter, 24 Strathearn Boulevard, Toronto 10, Canada.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 601. *Proposed by C. T. Tobin, St. Francis Xavier College, Antigonish, N. S.*

A baseball league is formed by seven teams belonging to towns located one mile apart around a circular railroad. In order that each team shall play each other, three games are held every day for a week. No team plays in any town more than once, nor plays more than one game a day. In each town, there is no more than one game a day, and there are never games on two consecutive days. There are no home games. At night each team travels to the scene of the next day's game, by the shortest route along the circular railroad. (Trains run in both directions.) A team that is going to be idle on any day returns (or remains) home the preceding night. When a team has completed its schedule, it returns home that night. The first games are played on Sunday; so the travelling starts on Saturday night and is completed on the following Saturday night. The towns are 1, 2, 3, 4, 5, 6, 7, though not in that order. Teams 3, 5, 6, never travel more than 2 miles any night. Teams 3, 4, 7 all travel the same total number of miles during the schedule. Teams 4 and 6 play at town 5 on Friday, and on that day there is also a game played at town 1.

From the above information, find the order of towns along the railroad, and draw up the schedule of games.

E 602. *Proposed by V. Thébault, San Sebastián, Spain*

With the ten different digits, taken once each, form two numbers which are respectively the square and the cube of two consecutive multiples of 3.

E 603. *Proposed by Free Jamison, U. S. Navy Air Navigation School*

Is there any numerical solution of E 552 [1943, 563] with $4<b<c<33800$?

E 604. *Proposed by H. W. Becker, Mare Island Training Division*

Give a combinatorial proof that

$$\sum_{r=0}^{\infty} \frac{1}{(r!)^2}\left(\frac{t}{2}\right)^{2r} \sim \frac{e^t}{\sqrt{2\pi t}} \quad \text{as} \quad t \to \infty.$$

E 605. *Proposed by L. M. Kelly, U. S. Coast Guard Academy*

Show that, if a set of four coplanar points has the property that the circumcircles of all subsets of three are equal, then the set is orthocentric. Establish the existence of an analogous set of five points in space, i.e., such that the circumspheres of all subsets of four are equal.

SOLUTIONS

The Incompatible Mechanics

E 565 [1943, 200]. *Proposed by H. W. Becker, Mare Island Training Division*

Show that the number of ways n men can be divided up in crews is N_n, in the notation of E 461 [1941, 701]. What is the number of ways if two particular men cannot stand the sight of each other, and must be kept in different crews? What is it if one must be segregated from each of m others?

Solution by D. H. Browne, Buffalo, N. Y. It is understood that, in the division of n men into r crews, some crews may contain more men than others. The number of such divisons is easily seen to be

$$\frac{\Delta^r 0^n}{r!}.$$

Summing for $r=1, 2, \cdots, n$, we obtain the desired number N_n.

Thus, for four men we have 1 crew ($ABCD$), 7 divisions into two crews (AB, CD; AC, BD; AD, BC; A, BCD; etc.), 6 divisions into three crews (AB, C, D; *etc.*) and 1 division into four crews: $N_4=15$ divisions altogether.

If one man is incompatible with one other, we have to subtract the N_{n-1} crew-divisions in which the two incompatibles occur together, leaving $N_n-N_{n-1}=\Delta N_{n-1}$, or, in Steffensen's notation,

$$\nabla N_n.$$

Similarly, if the incompatibility extends to m men in the group, the number of divisions is

$$\nabla^m N_n.$$

Also solved by the proposer.

The Regular Triacontagon

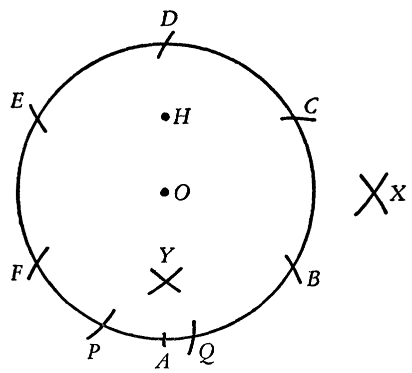
E 567 [1943, 260]. *Proposed by V. Thébault, San Sebastián, Spain*

Using compasses alone, construct a regular polygon of thirty sides.

Solution by E. P. Starke, Rutgers University. Draw a circle (O) with radius 1. Choose a point A on (O) and lay off on (O) the points B, C, D, E, F such that $AB=BC=CD=DE=EF=FA=1$. Locate an intersection X of the two circles having centers A and D and common radius equal to AC. Locate the intersection Y, within (O), of the two circles having centers C and E and common radius equal to OX. Lay off on (O) the points P, Q, such that $FP=PQ=OY$ and

P is between F and A. Then AQ is a side of the regular triacontagon.

To prove the construction, we have $AX=DX=AC=CE=\sqrt{3}$ and $CY=EY=OX=\sqrt{2}$. Let H be the midpoint of CE. Then $HC=\frac{1}{2}\sqrt{3}$ and $HO=\frac{1}{2}$. In the right triangle CHY, CY and HC are known, whence $HY=\frac{1}{2}\sqrt{5}$. Thus $OY=HY-HO=\frac{1}{2}(\sqrt{5}-1)=2\sin 18°$, which is the length of the side of an inscribed regular decagon. Finally, FQ subtends $\frac{1}{5}$ of the circumference while FA of course subtends $\frac{1}{6}$. Hence AQ subtends $\frac{1}{5}-\frac{1}{6}=1/30$, as required.



We note that the arc AQ has been constructed by the use of eleven circles having four different radii.

Also solved by Howard Eves, using twenty circles with nine different radii.

The Cardioid as an Envelope

E 569 [1943, 260]. *Proposed by David Matlack, Grinnell College*

Through a fixed point A on a circle (O), a line is drawn, parallel to a variable radius OP, meeting the circle again at Q. Find the envelope of the chord PQ.

Solution by E. P. Starke, Rutgers University. In the circle $x^2+y^2=1$, let the coordinates of A, P, Q be respectively

$$(-1,0), \quad (\cos t, \sin t), \quad (\cos 2t, \sin 2t).$$

The equation of the chord PQ is easily reduced to

$$x(\sin 2t - \sin t) - y(\cos 2t - \cos t) - \sin t = 0.$$

If this is differentiated partially with respect to t, we have

$$x(2\cos 2t - \cos t) + y(2\sin 2t - \sin t) - \cos t = 0.$$

The equation of the envelope of PQ could be obtained by eliminating t between these two equations; but it seems easier to solve for x and y, obtaining thus the parametric equations of the curve. We find, after various simplifications,

$$x = (\cos 2t + 2\cos t)/3, \qquad y = (\sin 2t + 2\sin t)/3.$$

This is a cardioid. If the origin is translated to $(-\frac{1}{3}, 0)$ the equation in polar coordinates is easily found to be

$$r = \tfrac{2}{3}(1 + \cos\theta).$$

Also solved by D. H. Browne, J. H. Butchart, and Howard Eves.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known text-books or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4104. *Proposed by E. T. Bell, California Institute of Technology*

Two symmetric functions, $M(x_1, \cdots, x_n)$, $S(x_1, \cdots, x_n)$, of n non-negative integers $x_1, \cdots, x_n$ are defined as follows

$$M(x_1, \cdots, x_n) \equiv M'(x_1) \cdots M'(x_n),$$

in which $M'(x)=1, -1, 0$, according as $x=0$, $x=1$, $x>1$; if $S_j(x_1, \cdots, x_n)$ is the jth elementary symmetric function of $x_1, \cdots, x_n$,

$$S(x_1, \cdots, x_n) \equiv 1 + \sum_{j=1}^{n} jS_j(x_1, \cdots, x_n).$$

Prove that

$$\sum M(x_1 - b_1, \cdots, x_n - b_n)S(b_1, \cdots, b_n) = n,$$

where the summation refers to all integers b_i such that

$$0 \leqq b_i \leqq x_i, \qquad i = 1, 2, \cdots, n.$$

4105. *Proposed by J. H. Butchart, Grinnell College*

Let ρ, ρ_s, R be respectively the radius of curvature of a curve, the radius of curvature of the locus of the center of the osculating sphere, and the radius of this sphere. Then the radius of geodesic curvature ρ_g of the locus of the center of curvature with regard to the polar developable is given by $\rho_g = R^3/(2R^2\text{-}\rho\rho_s)$.

4106. *Proposed by N. A. Court, University of Oklahoma*

If the Monge point of a tetrahedron lies on the circumsphere, show that (a) The line joining the circumcenter to the centroid of a face is equal to half the corresponding median of the tetrahedron; (b) Each median subtends a right angle at the Monge point. Conversely.

4107. *Proposed by V. Thébault, San Sebastián, Spain*

Let $N=123\cdots(n-3)(n-2)n$ be a number of $n-1$ digits in order of increasing magnitude in the system of base $n+1$. The product $P=N\cdot L$ is formed where $L=\alpha\beta$ consists of two digits whose sum $\alpha+\beta=\gamma$ is less than n and δ is the greatest common divisor of n and γ. Show that

$$P = N\cdot L = ab\cdots pqab\cdots pq\cdots ab\cdots pq,$$

where the δ periods $ab\cdots pq$ are formed by n/δ distinct digits.

Dedicated to E. P. Starke.

SOLUTIONS

Perfect Square Integers

4054 [1942, 549]. *Proposed by V. Thébault, San Sebastián, Spain*

Find the base less than 100 for which the number 2101 is a perfect square.

Solution by Irving Kaplansky, Cambridge, Mass. In the base r, the number is $2r^3+r^2+1=s(r+1)$, where $s=2r^2-r+1$. The H. C. F. of $r+1$ and s is a divisor of 4, so that they are either both squares or both doubles of squares. With $r=n^2-1$, $n=2$ or 3 are solutions and $n=4,\cdots,10$ can be rejected by direct test or as follows. Modulo 11, $n\equiv\pm 4$ or ± 5 makes $s\equiv 7$, a non-residue; modulo 17, $n\equiv\pm 7$ or ± 8 makes $s\equiv 5$, a non-residue. With $r=2m^2-1$, $2s$ must be a perfect square; but $m\equiv 0$ (3) yields $2s\equiv 2(3)$, a non-residue, while $m\equiv$ 2, 4, 5, 7 (17) yields $2s\equiv -3$, 10, 5, 10 (17), respectively, all non-residues. Hence we reject $m=2,\cdots,7$. Hence the only solutions with $r<100$ are $r=3$, 8. With the above congruences and a few others, it was easy to verify that there are no further solutions with $r<10{,}000$.

Solved also by the proposer giving the two answers without his method.

Editorial Note. It will be shown that, if $s=2r^2-r+1=2y^2$, the only solution is $r=1$ which is excluded. Solving for r we find that $4r=1+x$, where $16y^2-x^2=7$, or $4y+x=7$, $4y-x=1$, discarding negative values. This yields as the only positive integral solution $x=3$, $y=1$, $r=1$. We now turn to the only other case where $s=y^2$, and proceeding in the same manner we find that

$$x^2-8y^2=-7; \tag{1}$$

$$4r=1+x; \tag{2}$$

and we shall show how to find in succession all the positive integral solutions of (1). It is easily verified that, if x, y is an integral solution of (1), so are also x', y', and x'', y'', where

$$x'=3x+8y,\quad y'=x+3y, \tag{3}$$

$$x''=3x-8y,\quad y''=3y-x, \tag{4}$$

and (4) is the inverse of (3). If x, y are both positive then y'' is also positive; and, if also $x \geqq 11$, then x'' is positive, and we have a positive solution in x'', y'' with $x''<x$, $y''<y$. If on the other hand x, y is a positive solution with $x<11$ then x'' is negative and y'' is positive. With these facts it may be shown that the following process gives all of the positive solutions. From the solution 1, 1 of (1) we get from (4) the positive solution 5, 2, after change of sign of the x; and from (3) we get the solution 11, 4, We now use (3) and the last pair of solutions, to obtain a second pair 31, 11 and 65, 23; this second pair with (3) gives a third pair, and so on. We must reject 5, 2 and 65, 23 since $x \not\equiv -1 \bmod 4$, and in each pair one solution must be rejected for the same reason. We eliminate the rejected solutions by using the result of two applications of (3) which is

$$x''' = 17x + 48y, \qquad y''' = 6x + 17y,$$

applied to 11, 4 and 31, 11; and finally we put this transformation in the form

$$\begin{gathered} x_{i+4} = 34x_{i+2} - x_i, \qquad y_{i+4} = 34y_{i+2} - y_i, \qquad t_{i+4} = 34t_{i+2} - t_i - 40, \\ 4t_i = x_i + 5, \qquad r_i = t_i - 1, \end{gathered} \tag{5}$$

where x_1, y_1 is 11, 4 and the following solutions are arranged in the order

x:	11	31	379	1055 $\cdots$,
y:	4	11	134	373 $\cdots$,
t:	4	9	96	265 $\cdots$.

We have a solution of the problem if and only if t is a perfect square. For convenience we set $t_0=1$ and then the values of t_i may be computed in turn from t_0, t_1, t_2, t_3. By the use of quadratic residues, denoting a non-quadratic residue by N. R., it may be shown that there are only two solutions of the problem for r or t less than $4\cdot 10^{37}$. For (7), *i.e.*, mod 7, we have at once $t_{i+4} \equiv -t_{i+2} - t_i - 5$, $t_{i+6} \equiv t_i$; and then we find that the N. R. (7) are given by the t's with the subscripts $6k+3$, $6k+4$, $k=0, 1, 2, \cdots$. In a similar manner we have N. R. for

$$t_{12k+3,5,6,8,10}(11); \qquad t_{14k+3,4,9,10,11,12}(13);$$
$$t_{8k+3,4,5}(17); \qquad t_{20k+4,6,7,8,9,11,12,14,15,17}(19).$$

These five expressions give the subscripts from 3 to 49 inclusive but not 50.

Explicit formulas for t_i are as follows

$$t_{2j} = \left(\frac{2\sqrt{2}-1}{8}\right)(17+12\sqrt{2})^j - \left(\frac{2\sqrt{2}+1}{8}\right)(17-12\sqrt{2})^j + \frac{5}{4}, \tag{6}$$

and for t_{2j+1} replace the coefficients of the two jth powers by $(11+8\sqrt{2})/8$ and $-(8\sqrt{2}-11)/8$. Using only the first term of (6) we have approximately $t_{50}=4.3308\cdot 10^{37}$. If we consider also (23) it will be found that there are only two solutions for $r<6.65\cdot 10^{46}$. While these limits are quite large it would be much more interesting to have a process for determining whether or not there are more than two solutions.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to B. W. Jones, White Hall, Cornell University, Ithaca, New York.

Dr. G. E. Albert of Ohio State University has been promoted to an assistant professorship.

Dr. Warren Ambrose of Princeton University has been appointed to an assistant professorship at the University of Michigan.

Assistant Professor Max Astrachan of Antioch College has been promoted to an associate professorship and to the chairmanship of the department of mathematics.

Assistant Professor R. H. Bardell, chairman of the mathematics department of the University of Wisconsin at Milwaukee, is now acting director of that institution.

Assistant Professor A. F. Bernhart of Bucknell University has been appointed to an assistant professorship at the University of Oklahoma.

Drs. O. K. Bower, Leonard Bristow, M. M. Day and J. W. Peters of the University of Illinois have been promoted from the rank of instructor to that of associate.

Assistant Professors D. G. Bourgin, J. L. Doob, P. W. Ketchum and G. E. Moore of the University of Illinois have been promoted to associate professorships.

Assistant Professor M. G. Boyce of Western Reserve University has been promoted to an associate professorship.

Assistant Professors J. D. Burk and G. de B. Robinson of the University of Toronto have been promoted to associate professorships. The latter is on leave with the National Research Council, Ottawa.

Assistant Professor C. L. Buxton of Clarkson College of Technology has been promoted to the rank of associate professor of physics.

Professor Helen Calkins of Pennsylvania College for Women has been appointed lecturer at the University of Minnesota.

Assistant Professor R. H. Cameron of Massachusetts Institute of Technology has been promoted to an associate professorship.

Associate Professor E. J. Camp of Macalester College, St. Paul, has been promoted to a professorship.

Associates Josephine H. Chanler and H. J. Miles of the University of Illinois have been promoted to assistant professorships.

Drs. R. F. Clippinger and A. D. Hestenes of Carnegie Institute of Technology have been promoted to assistant professorships.

Assistant Professors N. B. Conkwright and L. A. Knowler of the University of Iowa have been promoted to associate professorships.

Professor J. A. Cooley of the University of Tennessee has been appointed head of the department.

Associate Professor T. F. Cope of Queens College has been appointed chairman of the department.

W. S. H. Crawford of the University of Minnesota has been appointed to an assistant professorship at Mt. Allison University, Sackville, New Brunswick.

J. C. Currie of Northeast Junior College of Louisiana State University has been promoted to an assistant professorship.

Associate Professor Wayne Dancer of the University of Toledo has been promoted to a professorship.

Associate Professor P. H. Daus of the University of California at Los Angeles has been promoted to a professorship.

Associate Professor J. E. Davis of the Central Y.M.C.A. College of Chicago has been appointed to the rank of associate in Pharmacy at the University of Illinois in Chicago.

U. P. Davis of the University of Florida has been promoted to an assistant professorship.

Associate Professor D. C. Dearborn at Catawba College, Salisbury, North Carolina, has been promoted to a professorship.

Professor R. C. Dragoo of Southern Institute of Technology (Oklahoma) has been appointed to an assistant professorship at the University of Oklahoma.

Assistant Professors L. T. Dunlap and H. L. Krall have returned to the Pennsylvania State College after service with the armed forces.

Associate Professor W. L. Duren, Jr. of Tulane University has been promoted to a professorship.

Associate Professors H. H. Germand and J. H. Kusner, Assistant Professor E. S. Quade, and Instructors R. D. Specht and T. S. George of the University of Florida are on leave in the armed services or in war work.

Assistant Professors W. O. Gordon, Evan Johnson, Jr., and W. O. Rogers of the Pennsylvania State College have been promoted to associate professorships.

Majors V. S. Lawrence and H. A. Robinson of the United States Military Academy have been promoted to the rank of lieutenant colonel.

Dr. A. T. Lonseth of the Iowa State College has been promoted to an assistant professorship.

Assistant Professor Neil Little of Purdue University now holds the same rank in the department of applied mechanics in that institution.

H. G. Means has been appointed to an associate professorship at Lehigh University.

Assistant Professor S. B. Myers of the University of Michigan has been promoted to an associate professorship.

Assistant Professor J. C. Oxtoby of Bryn Mawr College has been promoted to an associate professorship.

Dr. H. V. Price of the State University of Iowa has been promoted to an assistant professorship.

Associate Professor W. C. Randels of the University of Oklahoma has resigned to serve with the North America Aviation Company in Los Angeles.

Professor S. E. Rasor of Ohio State University has retired.

Professor H. W. Reddick of Cooper Union has been appointed adjunct professor of mathematics at New York University.

Assistant Professor George Sauté of Cleveland College has been appointed to an associate professorship at Rollins College.

Assistant Professor A. C. Schaeffer of Stanford University has been promoted to an associate professorship.

Dr. H. M. Schwartz of the University of Illinois has been appointed to an assistant professorship at the University of Idaho.

Associate Professor D. T. Sigley of Kansas State College is on leave of absence in war work at Silver Springs, Maryland.

Professor Emeritus Virgil Snyder of Cornell University is teaching at Rollins College.

Assistant Professor H. P. Thielman of the Iowa State College has been promoted to an associate professorship.

Professor C. B. Upton of Teachers College in Columbia University has been elected chairman of the board of the American Book Company.

Dr. P. R. Wallace of Massachusetts Institute of Technology is on leave of absence serving with the National Research Council of Canada.

Dr. George Whapples of Johns Hopkins University has been appointed to an assistant professorship at the University of Pennsylvania.

Associate Professor J. K. Whittemore of Yale University has retired.

The following appointments to instructorships are announced:
Barnard College: Louise Comer
Columbia University; mathematical assistant: Dr. George Piranian
Cornell University: Teaching assistants: Gunter Jahn, Irma R. Moses, A. E. Ventriglia
DePaul University (Chicago): Paul D'Arco
Greenbrier College: Margaret R. Davis
Kansas State College: W. V. Unruh, L. E. Milleson, D. K. Brooks
Lehigh University: R. C. King, S. W. Smith
Massachusetts Institute of Technology: W. S. Loud, Donald Thomson
Ohio State University: L. H. Miller
Pennsylvania State College: J. R. Kinney, Mrs. Nellie M. Krall
Purdue University: H. M. Anderson, F. C. Leone
Stanford University: Marjorie L. Hoffman, Mary V. Sunseri
University of Chicago: Dr. Anne L. Lewis, Dr. Janet MacDonald
University of Illinois: W. F. Atchison, Janie C. Lapsley
University of Kentucky: Mary H. Cooper, Mrs. J. C. Lamb, Mrs. C. G. Latimer
University of Michigan: Dr. B. J. Lockhart, Dr. C. A. Truesdell
University of Minnesota: H. D. Colson
University of Oklahoma: Phyllis Barclay, Mrs. Dorothea A. Sudduth
University of Pennsylvania: S. I. Askovitz, R. C. Cambell, Dr. H. D. Huskey C. W. Pflaum
Yale University: J. S. Blair, J. C. Caughlin, A. S. Day, R. E. Fullerton, Dr. W. C. G. Fraser, C. R. Kossack, J. R. Lee, Jr., J. S. Leech, J. G. Pocock, P. E. Poe, James Smillie, W. J. Strange

Professor Emeritus H. H. Dalaker of the Institute of Technology at the University of Minnesota died May 20, 1943. He was a charter member of the Mathematical Association.

Professor Leonard Richardson of the University of British Columbia died on October 23, 1943.

WAR INFORMATION

EDITED BY C. V. NEWSOM

Send news reports upon the utilization of mathematicians or mathematics in war activities to C. V. Newsom, University of New Mexico, Albuquerque, New Mexico.

NOTES FROM THE WAR DEPARTMENT

Pre-Induction Training. Civilian pre-induction training is to become part of the individual's military record when he is inducted into the Army, it has been announced by the War Department. This is to be accomplished by entering the selectee's pre-induction training experience upon his personal qualification card, which records his military qualifications and career, and follows him throughout his service.

Approximately 600,000 prospective inductees were enrolled last year in pre-induction training courses according to Merwin M. Peake, Director of the Civilian Pre-Induction Training Branch. It is to be emphasized that students who take such courses build a valuable basis for Army specialist training after their induction. "At the present time," Mr. Peake said, "900 out of every 1,000 men inducted must fill Army jobs requiring specialized training, as compared to but 630 men in every 1,000 only one year ago. In stressing vocational training, the War Department also emphasizes the great importance of basic education, especially in the fields of mathematics, English and social studies, as preparation for Army life. All men entering the Army should know why we are at war, be competent in basic mathematics and English, learn in advance what Army life is like, and be physically fit. Schools and other civilian training agencies can promote the nation's war effort by providing potential Army inductees with training in these fields."

A.S.T.P. Certificates. Certificates will be awarded to all soldiers who successfully complete their prescribed work in the Army Specialized Training Program at colleges and universities, the War Department announced in September. The certificate will list the curriculum number in which the soldier performed his work, the number of terms completed, and the date the training was completed; it will be signed by the appropriate authority at the college attended and by the Commandant of the Army Specialized Training Unit at that college. Certificates will be executed and forwarded to all enlisted men who have already successfully completed their work in the A.S.T.P.

The record of the soldier's performance will also be available at the university, on request by appropriate authority, for the purpose of determining his academic credit. In the opinion of the A.S.T.P. Advisory Committee, comprised of presidents of ten leading colleges and universities, all A.S.T.P. academic work is at the college level. It is anticipated, therefore, that appropriate college credits will be granted, enabling the soldier-trainee to complete his work for a

college degree if he returns to college as a civilian after the termination of his military service.

Deferment of A.S.T.P. Teachers. The following is a direct quotation from a notice sent to presidents of institutions participating in the Army Specialized Training Program.

"It is of the utmost importance that qualified and irreplaceable teachers who are at institutions under contract to the Army Specialized Training Program and who are providing instruction under the program be permitted to continue this work.

"The requirements of the Army Specialized Training Program indicate that very serious disruption is probable if the responsible officials of these institutions do not sufficiently emphasize to Selective Service Authorities the necessity for the occupational deferment of these teachers.

"As a first step, it is requested that all teachers in institutions under contract to the Army Specialized Training Program register their qualifications with the National Roster of Scientific and Specialized Personnel, Washington, D. C. The National Roster is concerned with the proper utilization of professional personnel and is an important source of information in this matter to the Selective Service System.

"The Selective Service Form 42-A, completely filled out, should be used for individual teachers for whom deferment is considered necessary, to furnish the appropriate local board with specific information reflecting the considered judgment of the institution with respect to qualifications, degree of training, and experience required to engage in the profession; and to inform the board either (1) that no replacement can be obtained and that replacement training time is too great to afford relief, or (2) that no replacement can be obtained immediately, in which case the time required for replacement should be indicated. Special attention should be devoted to furnishing complete information concerning the capacities and duties of the individual teacher. In cases where teachers can be replaced, it should be emphasized that the replacement should not break into the term during which the incumbent teacher is giving instruction.

"It is essential that all possible means of procuring replacement be employed, including the transfer of non-vulnerable teachers who are engaged in civilian training, the conversion of teachers from cognate fields or from administrative duties, the recruitment of teachers not now engaged in essential activities, and in general the maximum practical use of existing teaching personnel.

"Where deferment is not obtained at the local board level, appeal should be made within ten days to the area or district appeals board and if necessary the case should be called to the attention of the State Director with a request for a Presidential appeal.

"You are requested to adhere strictly to all Selective Service procedures regarding occupational deferment. You are also requested to advise the Military

Commandant of your institution immediately when any irreplaceable teacher is classified by Selective Service authorities as available for military service.

"In the event that the Military Commandant at the institution considers the deferment of the teacher to be essential to the success of the Army Specialized Training Program, representations to that effect may be made by him to the State Director of the Selective Service System. Such representations are not to be made, however, unless all appeal procedure provided under the Selective Service Act is being pursued and unless in fact the particular teacher is irreplaceable in the institution.

"In the event that the deferment is denied after all appeal steps through the state level have been complied with, the institution should immediately transmit the matter to the Labor Branch, Industrial Personnel Division, Headquarters, Army Service Forces, for representations at the Washington level.

"In the case of the irreplaceable teacher whose induction into the Army has taken place, causing serious jeopardy to the Army Specialized Training Program, institutions may submit a full statement of the facts to the Labor Branch, Industrial Personnel Division, Headquarters, Army Service Forces, Washington, D. C., for consideration of methods for remedial action."

THE COLLEGE PRE-FLIGHT PROGRAM

The College Training Program (Aircrew) of the Training Command of the Army Air Forces was established to give preliminary training to the lowest 80% of those prospective aviation cadets who have passed the screening test given by the Air Forces; upon graduation from the College Program, trainees are sent to an Army Pre-Flight School. Men in the upper 20% are sent directly to Army schools, and receive no specialized college training.

In spite of the screening test, a considerable number of men in the Program have little mathematical background, and sectioning of the students on the basis of ability is becoming common in many institutions. A similar situation is appearing in all the Army and Navy Educational Programs.

The curriculum recommended for the Army Aircrew Program includes a course in mathematics designed for eighty class hours, spread over a period of approximately sixteen weeks. The topical outline of the course in mathematics provided by the Training Command includes the following introduction.

"The topics listed in the . . . outline are mainly those needed directly by the student as a basis for subsequent training. There are some, however, which are given for background material. The presentation should be from the standpoint of preparation for aircrew training, with as much time as possible spent on supervised drill. Assignment of problems to be done outside of class is recommended. Examinations should be of the open-book type. The emphasis on theory should be limited to the minimum essential to enable the students to appreciate the content of the course. Applications of numerical operations should be emphasized whenever possible."

An abridgment of the recommended course outline appears below.

Unit 1. ARITHMETIC. Fourteen hours.

Addition, subtraction, multiplication and division of whole numbers; the use of hours, minutes, seconds; degrees, minutes, seconds, radians; circular and Air Corps mils; English measure; metric measure; the relations between different measures; common and decimal fractions; percentage; ratio and proportion; conversion tables. Emphasis should be placed on speed and accuracy in numerical computations. It is suggested that drill for speed be given, and tests designed accordingly.

Unit 2. ALGEBRA. Twenty hours.

The content should be selected from any reputable college text to emphasize the manipulative skills needed for numerical trigonometry, physics, and the elementary technical fields; graphical methods should be introduced. Practical problems in time, speed, distance, and fuel consumption should be especially emphasized. Statement problems should be given wherever practicable to exercise the student's ability to think.

Unit 3. GEOMETRY. Eighteen hours.

This course should be designed to create accurate conceptions of space on the part of the student, and to prepare him for trigonometry and certain phases of astronomy. The content should include a treatment of straight lines, planes, dihedral and trihedral angles, and the geometry of the sphere; other major parts of the usual geometry course may be omitted. Proofs should be held to a minimum; great emphasis should be placed on the drawing of figures and the making of simple tridimensional paper models. Content which will be used in spherical trigonometry should receive particular attention.

Unit 4. TRIGONOMETRY AND LOGARITHMS. Twenty-eight hours.

This is primarily a course in the numerical aspects of trigonometry with only that amount of analytical trigonometry which is essential for the solution of practical problems. Substantial emphasis should be given to slide-rule computation, stressing applications, especially those involving vector forces and velocities in problems peculiar to military service. Formulas for the solution of the right triangle and the general triangle on the celestial sphere should be treated. A major objective of the course is to give the student confidence in the later use of navigation tables, which frequently makes it unnecessary for the navigator to carry out the solution of spherical triangles.

THE PAPER SHORTAGE

Publishers in the United States are facing the apparent dilemma of an increasing shortage of paper and a rising demand for books. A very few publishing houses have already introduced rationing programs, but not of technical books of current importance. The policy generally is to produce more books out of less paper, and this will be increasingly evident in the books' appearance.

Through 1943, publishers were permitted to receive up to 90% of the paper which they used in 1942. Mr. H. M. Bitner, Director of the Printing and Publishing Division, War Production Board, has warned, however, that "a curtailment may be expected as of January 1, 1944, and that it will include the book industry's share of the general cut levied against the graphic arts." Mr. M. K. Dutton, also of the War Production Board, has asserted that another cut of possibly 10% should be anticipated, since inventories of wood-pulp are down as much as 33%.

Consequently, every skill at the command of the paper mills is being called upon to produce good printing papers at the lower weights. Also, publishers have developed model pages which permit more words on the page. Smaller print and smaller margins are becoming common, and there is considerable discussion in regard to the introduction of the two-column page into technical books. By the use of similar devices, the Federal Government has been able to curtail its use of paper by 25%.

CALENDAR OF FUTURE MEETINGS

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN, Pittsburgh, Pa., April, 1944
ILLINOIS
INDIANA
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI, Ruston, La.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK, New York, April 22, 1944
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA
NORTHERN CALIFORNIA, Berkeley, Jan. 29, 1944
OHIO, Columbus, April 6, 1944
OKLAHOMA
PHILADELPHIA, Philadelphia, November, 1944
ROCKY MOUNTAIN
SOUTHEASTERN
SOUTHERN CALIFORNIA, Los Angeles, March 11, 1944
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE
WISCONSIN, Milwaukee, May, 1944

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

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PART II

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JANUARY 1944

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The Chauvenet Prize

The Chauvenet Prize for a "noteworthy expository paper published in English during the five-year period by a member of the Association" was established in 1925 by a gift from Professor J. L. Coolidge, then president of the Association. Its frequency period was reduced to three years by an additional gift from Professor W. B. Ford during his presidency and by an anonymous gift in 1936.

As determined more recently by the Board of Governors, the prize is to be fifty dollars and is to be awarded for a noteworthy expository paper such as will come within the range of profitable reading of members of the Association. The purpose of the prize is to stimulate expository contributions in mathematical journals on the part of the younger American scholars. The next two awards will be made in December 1944 and December 1947, covering the periods 1941-43 and 1944-46 respectively.

Six awards have been made as follows:

1920-1924. G. A. Bliss, "Algebraic Functions and Their Divisors," *Annals of Mathematics.*

1925-1928. T. H. Hildebrandt, "The Borel Theorem and Its Generalizations," *Bulletin* of the American Mathematical Society.

1929-1931. G. H. Hardy, "An Introduction to the Theory of Numbers," *Bulletin* of the American Mathematical Society.

1932-1934. Dunham Jackson, "Series of Orthogonal Polynomials" and "Orthogonal Trigonometric Sums," *Annals of Mathematics;* "The Convergence of Fourier Series," *American Mathematical Monthly.*

1935-1937. G. T. Whyburn, "On the Structure of Continua," *Bulletin* of the American Mathematical Society.

1938-1940. Saunders Mac Lane, "Modular Fields," and "Some Recent Advances in Algebra," both in the *American Mathematical Monthly.*

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MARION. Porter.
MUNCIE. Edwards, Shively.
NOTRE DAME.
Univ. of Notre Dame. Caparó, Landin, Menger, Pepper, Utz.
REYNOLDS. Erwin.
TERRE HAUTE. Martin, Shriner, Sousley, Straw.
UPLAND. Draper.
WEST BADEN SPRINGS. Muehlmann.

IOWA

AMES. M. M. McKelvey.
Iowa State Coll. Anderson, Bancroft, Beach, Brandner, Daniells, Davis, Fleming, Gouwens, Herr, Lonseth, J. V. McKelvey, Robertso [illegible] Snedecor, Thielman.
CEDAR FALLS.
Iowa State Teachers Co [illegible] Kearney, Trimble, Van Engen
CEDAR RAPIDS. Coffin, Swanson.
CRESTON. Newlin.
DES MOINES. Kirchen, McClelland, Neff.
DUBUQUE. Burns.
FAYETTE. Deming.
GRINNELL.
Grinnell Coll. Butchart, McClenon, Rusk.
HOPKINTON. Earhart.
INDIANOLA. Emmons.
IOWA CITY. Price.
State Univ. of Iowa. Chittenden, Conkwright, Craig, Knowler, Oberg, Ward, Woods, Wylie.
IOWA FALLS. Kreider.
MOUNT VERNON.
Cornell Coll. Davis, Freese, McGaw, Moots.
SIOUX CITY. Rochford.
WAVERLY. Chellevold.

KANSAS

ATCHISON. Obrist, Pretz, Sullivan.
BALDWIN. Garrett.
EL DORADO. Wrestler.
EMPORIA. Peterson, Tucker.
HAYS. Colyer, Grabbe, McCormick.
HESSTON. Driver.
KANSAS CITY. Thornton.
LAWRENCE.
Univ. of Kansas. Babcock, Black, Dougherty, Jordan, Kalisch, Marm, Price, Smith, Stouffer, Ulmer, Wheeler.
MANHATTAN.
Kansas State Coll. Babcock, Daugherty, Hyde, Janes, Lewis, Mossman, Remick, Sigley, Stratton, White.
NORTH NEWTON. Richert.
OTTAWA. Bemmels.
PITTSBURG.
Kansas State Teachers Coll. Curfman, Shirk, Smith.
SALINA. Arnoldy.
STERLING. Bell.
TOPEKA. Eberhart.
WICHITA. Greer, Longenecker, Reagan.
Univ. of Wichita. Hanna, Hoare, Read, Wedel.
XAVIER. Ann Elizabeth.

KENTUCKY

BEREA.
Berea Coll. Hutcherson, Pugsley, Roberts.
BOWLING GREEN. Yarbrough.
COVINGTON. Thuener.
GEORGETOWN. Hatfield.
LEXINGTON. Wright.
Univ. of Kentucky. Boyd, Brown, Cohen,

Downing, John, Latimer, LeStourgeon, Pence, South.
LOUISVILLE. Bloom, Bullitt, Fields, Ford, Morrison, Schaeffer.
Univ. of Louisville. Moore, Simester, Stevenson.
MAPLE MOUNT. Sheeran.
MOREHEAD. Fair.
MURRAY. Carman.
RICHMOND. Jenkins, Park.

LOUISIANA

BATON ROUGE.
Louisiana State Univ. Dorroh, Freas, Karnes, Nichols, O'Quinn, Parker, Rickey, Rutt, Sanders, Smith, Thorne, White.
HAMMOND. Tucker.
LAFAYETTE.
Southwestern Louisiana Inst. Buchanan, Loflin, Rees.
LAKE CHARLES. Bradford.
MONROE. Currie.
NATCHITOCHES.
Louisiana State Normal Coll. Blair, Killen, Maddox.
NEW ORLEANS. Dickinson, Frankenbush, Monasterio, Stevens.
Tulane Univ. Beard, Buchanan, Cramer, Duren, Humphreys, Many, Riess, Spencer, Thomson, Weiss.
PINEVILLE. Temple.
RUSTON.
Louisiana Poly. Inst. Garrison, Kaltenborn, Schroeder, Smith.
SHREVEPORT. Banks, Cooper, Hardin.

MAINE

BRUNSWICK.
Bowdoin Coll. Christie, Hammond, Holmes, Korgan, Moody.
HOULTON. Morse.
LEWISTON. Ramsdell, Wilkins.
ORONO. Bryan, Kimball.
WATERVILLE. Ashcraft.

MARYLAND

ABERDEEN PROVING GROUND. Dederick, Green, Hart, Mills.
ANNAPOLIS. Bingley, Buchanan, Wilson.
U. S. Naval Acad. Ayres, Ball, Bleick, Bramble, Church, Clements, Currier, Dillingham, Gere, Kells, Lamb, Lyle, Moore, Phelps, Rawlins, Root, Scarborough, Sears.
BALTIMORE. Karl, Roman, Smith.
Goucher Coll. Bacon, Lewis, Torrey.
Johns Hopkins Univ. Bourne, Cohen, Morrill, Murnaghan, Reed, Zariski.
COLLEGE PARK.
Univ. of Maryland. Hall, Lancaster, Richeson.
EMMITSBURG. Burke.
FREDERICK. Brown.
FROSTBURG. Hallett.
TOWSON. Littauer.

MASSACHUSETTS

AMHERST. Esty, Kleene.
Massachusetts State Coll. Boutelle, Miller, Moore.
BELMONT. Jackson.
BOSTON. Brown, Bruce, Combellack, Gould, Hemenway, Hubbard, Laurentine, Miller, Mode, Spear, Weaver, Wilson.
BROOKLINE. McCarthy.
CAMBRIDGE.
Harvard Univ. Beatley, G. D. Birkhoff, G. Birkhoff, Boas, Buck, Coolidge, Emmons, Huntington, Kaplansky, Mac Lane, Rulon, Stone, Walsh, Widder.
Massachusetts Inst. of Tech. Cameron, Clifford, Douglass, Franklin, Harvey, Haskins, Moon, Salem, Woods, Zeldin.
CHESTNUT HILL. Marcou, O'Donnell.
CHICOPEE. Madden.
EAST BOSTON. Lennahan.
GROTON. Holt, Nash.
LYNN. Oergel, Taylor.
MILFORD. Dennison.
MOUNT HERMON. Lockwood.
NORTHAMPTON. Munroe.
Smith Coll. McCoy, Montgomery, Newton, Rambo.
NORTON. Garabedian, Watt.
PETERSHAM. Moriarty.
PITTSFIELD. Washburne.
SOUTHBOROUGH. Harrison.
SOUTHBRIDGE. Boeder.
SOUTH HADLEY. Litzinger.
SWAMPSCOTT. Evans.
TUFTS COLLEGE. Mergendahl, Ransom.
TYNGSBORO. Richmond.
WELLESLEY.
Wellesley Coll. Copeland, Merrill, Russell, Stark, Young.
WESTON. Burke.
WILLIAMSTOWN.
Williams Coll. Agard, Beer, Gillette, Wells.
WORCESTER. Melville, Wheeler.
Worcester Poly. Inst. Brown, Gay, Morley, Rice.

MICHIGAN

ALBION. Ingalls, Sleight.
ANN ARBOR.
Univ. of Michigan. Bradshaw, Cebula, Churchill, Coe, Copeland, Craig, Crispin, Dwyer, Faulkner, Field, Fischer, Goldstine, Hay, Hildebrandt, Hopkins, Johnson, Kaplan, Karpinski, Love, Nyswander, Rainich, Rainville, Rouse, Rufus, Running, Schorling, Thrall, Wilder.
DEARBORN. Jones.
DETROIT. Bagby, Denton, Goldman, Mary Paula, Sanders, Shires.
Univ. of Detroit. Hausmann, Johnston, McCarthy.
Wayne Univ. Baldwin, Borgman, Folley, Morrow, Nelson, Pixley.
EAST LANSING.
Michigan State Coll. Barbour, Baten, Bis-

singer, Frame, Grove, Hill, Nordhaus, Plant, Powell, Speeker, Toralballa, Van Schaack, Welmers.
FLINT. DeMoss, Swanson.
GRAND RAPIDS. Bellardo, Warren.
HART. Burdick.
HIGHLAND PARK. Peterson.
HILLSDALE. Beeler.
HOLLAND. Lampen.
IRONWOOD. Field.
KALAMAZOO. Walton.
Western Michigan Coll. Ackley, Bartoo, Blair, Butler, Cain, Everett.
KALKASKA. Dunlap.
MARQUETTE. Spooner.
MIDLAND. Spencer.
MILFORD. McNeal.
MOUNT PLEASANT. Foust, Richtmeyer.
MUSKEGON. Conger.
YPSILANTI. Erikson, Lindquist.

MINNESOTA

COLERAINE. Davis.
COLLEGEVILLE. Danzl, Winkelmann.
DULUTH. Cothran, Mercedes, Strane.
ELY. Cramer.
GILBERT. Schey.
HIBBING. Erickson.
MINNEAPOLIS. Shuman.
Univ. of Minnesota. Amundson, Brink, Brooke, Bussey, Campaigne, Carlson, Colson, Eggers, Fattu, Fischer, Gibbens, Hart, Hartig, Jackson, Jensen, Johnston, Kirchner, Koehler, McCutcheon, McEwen, Munro, Ness, Olmsted, Priester, Quaid, Saunders, Scammon, Shumway, Stigler, Thorp, Turner, Turrittin, Underhill.
MOORHEAD. Andersen, Mundhjeld.
NORTHFIELD. Carlson.
Carleton Coll. Gingrich, Milleson, Wegner.
ROCHESTER. Hickman.
ST. JOSEPH. Scoblic.
ST. PAUL. Camp, Thornton.
Coll. of St. Thomas. Bush, Spitzbart, Taylor.
ST. PETER. Nelson.
VIRGINIA. Hancock.
WINONA. De La Salle.

MISSISSIPPI

BILOXI. Gerst.
CLEVELAND. Ward.
CLINTON. Barnes.
JACKSON. Babbitt, McCoy, Mitchell, Van Hook.
STATE COLLEGE.
Mississippi State Coll. Coker, Murray, Ollivier, Pettis, Smith
UNIVERSITY.
Univ. of Mississippi. Bickerstaff, Hume, Quarles.

MISSOURI

CAPE GIRARDEAU. Michel.
CLAYTON. Lafferty.
COLUMBIA. Cosby, Callaway.
Univ. of Missouri. Blumenthal, Ewing, Ferguson, Haynes, Shanks, Wahlin, Wehausen, Westfall.
FAYETTE. Hadley.
FULTON. Sweazey.
HANNIBAL. Foreman.
JEFFERSON CITY. Jason, Talbot.
KANSAS CITY. Cutting, Doyle, Pierson.
KIRKSVILLE. Jamison.
MARYVILLE. Lane.
PARKVILLE. Crull, Messick.
ROLLA. Erkiletian.
ST. CHARLES. Karr.
ST. LOUIS. Bock, Callaghan, Case, Gatewood, Marth, Shreve, Siroky.
Washington Univ. Dunkel, Middlemiss, Rider, Roever.
SPRINGFIELD. Finkel, Graves, H'Doubler.
WARRENSBURG. Jacobson.
WEBSTER GROVES. Clarke.

MONTANA

BOZEMAN. Hurst.
BUTTE. Smith.
GARRISON. Canning.
HELENA. Topel.
MISSOULA. Carey, Merrill.

NEBRASKA

CHADRON. Berry.
GILEAD. Erwin.
HASTINGS. Hadlock, McDill.
LINCOLN. Arnold, Gass, Howie, Ogden.
Univ. of Nebraska. Basoco, Brenke, Camp, Candy, Congdon, Cox, Gaba, Harper, Huff, Hull, Linscheid, Runge.
OMAHA. Bettinger, Dwyer.
Univ. of Omaha. Coleman, Earl, Rice.
WAYNE. Boyce.
YORK. Feemster.

NEVADA

RENO. Beesley, Wood.

NEW HAMPSHIRE

CONCORD. Conwell.
DURHAM. Lewis, Slobin.
EXETER. Adkins, Funkhouser, Pennell.
HANOVER. Morgan.
Dartmouth Coll. Brown, Forsyth, Mathewson, Nordstrom, Perkins, Robinson, Silverman, Street, Wilder.
KEENE. Goodrich.
MANCHESTER. O'Leary.
PLYMOUTH. Smith.

NEW JERSEY

CLIFTON. Struyk.
CONVENT STATION. Kenna.
DOVER. Cohen.
EAST ORANGE. Nordgaard.
HIGHTSTOWN. Litterick.
HOBOKEN. Hazeltine, Murray, Schnefel.
JERSEY CITY. Kopp, McGrath.
LAKEWOOD. Wallick.

LAWRENCEVILLE.
Lawrenceville School. Durell, Kimball, Mikesh.
MADISON. Battin.
NEWARK. Karst, MacDonald, Mosesson, Strock.
NEW BRUNSWICK.
Rutgers Univ. Bunyan, Galbraith, Grant, Meder, Morris, Nelson, Starke, Walter.
NORTH PLAINFIELD. Rayher.
PATTERSON. McGlade.
PRINCETON. Flood.
Inst. for Advanced Study. Alexander, Morse, Veblen, von Neumann.
Princeton Univ. Adams, Bohnenblust, Brock, Eisenhart, Gillespie, Lefschetz, Mosteller, Rauch, Scheffé, Tompkins, Tucker, Tukey, Wedderburn, Wilks.
SOUTH ORANGE. Rauch, Stanwick.
TEANECK. Coleman.
TRENTON. Shuster.
UPPER MONTCLAIR.
New Jersey State Teachers Coll. Campbell, Clifford, Davis, Fehr, Mallory.
WEST ORANGE. Edison.

NEW MEXICO

ALBUQUERQUE. Bauer.
Univ. of New Mexico. Barker, Gentry, Larsen, Newsom, Rosenthal, Valuckas.
LAS VEGAS. Roberts, Rodgers.
PORTALES.
Eastern New Mexico Coll. Fleck, Rosen, Wallis.
ROSWELL. Harp.
SOCORRO. Reece.
STATE COLLEGE.
New Mexico Coll. of A. and M.A. Branson, Heinzman, Swingle, Wells.

NEW YORK

ALBANY.
New York State Coll. for Teachers. Beaver, Birchenough, DoBell, Lester, Snader, Stokes.
ALFRED.
Alfred Univ. Lowenstein, Nevins, Polan, Seidlin, Titsworth, Whitford.
AURORA.
Wells Coll. Clement, Hollcroft, Rusk.
BROOKLYN. Braverman, Charosh, Cowles, Eichert, First, Kalish, Koch, Kramer-Lassar, Lavoie, Levy, Lieber, Mandel, McCarthy, McMahon, Mehr, Miller, Palladino, Peters, Rush, Salkind, Thompson, Tolle, Treiber, Wallach.
Brooklyn Coll. Borofsky, Boyer, Douglas, Fleisher, Forman, Griffin, Harkin, Hurwitz, Johnson, Kennison, Landers, MacNeish, Maria, Moore, Prenowitz, Richardson, Singer, Smith, Wolfe, Woodbridge.
Poly. Inst. of Brooklyn. Berry, Foster, Reagan, Whitford.
BUFFALO. Bartram, Blanche, Browne, Harrington, Podmele.
Univ. of Buffalo. Brendel, Civin, Gehman, Montague, Ott, Pound.
CLINTON.
Hamilton Coll. Brown, Ferry, Johanson, McBrien, Patterson.
ELMIRA. Suffa.
FLUSHING.
Queens Coll. Archibald, Brown, Cairns, Cope, Feld, Raudenbush, Sard, Williamson.
GARDEN CITY. Bowden, Schultz.
GENEVA. Durfee, Hubbs.
HAMILTON.
Colgate Univ. Aude, Munshower, Wardwell.
HAYT CORNERS. Ford.
HEMPSTEAD. Ollmann, Stabler.
HOUGHTON. Davison.
ITHACA.
Cornell Univ. Agnew, Beinert, Carver, Curtiss, Firestone, Flexner, Gunderson, Hailperin, Hatfield, Hurwitz, Jones, Kac, Lee, Manning, Moses, Peiser, Rosser, Scott, Smith, Snyder, Walker.
KENMORE. Hodges.
LOUDONVILLE. Nickol.
NEW YORK. Berger, Bergstresser, Berkeley, Bernard, Boehm, Buell, Buikstra, Burgess, Crane, Darraugh, D'Atri, A. M. Ginsburg, J. Ginsburg, Gray, Hastings, Heath, Hlavaty, Jablonower, Joffe, Karapetoff, Katz, Keeler, Kirby, Kraitchik, Kubis, Lazar, Lehner, Longfellow, Martin, McKenna, Mirick, Molina, Moore, Nehrbas, Oehler, Phillips, Quilty, Roll, Ruderman, Schwartz, Sheridan, Skelding, Steinhaus, Stuckey, Swenson, Wayne, Weaver, Young.
Bell Telephone Labs. Jones, MacColl, Mead, Fry, Gray, Harvey, Riordan, Schelkunoff, Shewhart.
Coll. of the City of New York. Allen, Fagerstrom, Gill, Griffin, Grove, Hubert, Linehan, MacEwen, Milkman, Post, Robinson, Turner, Whitford, Wirth, Wright.
Columbia Univ. Comer, Fite, Gentzler, Hutchinson, Kasner, Ladue, Mullins, Piranian, Reeve, Ritt, Siceloff, Smith, Upton, Walker.
Cooper Union. Lehmann, Miller, Roth, Tanzola.
Hunter Coll. Anderson, Aroian, Bowden, Bradley, Mrs. J. H. Bushey, J. H. Bushey, Cooper, Darkow, Eisele, Hill, Kutman, Landers, Rees, Simons, Weisner, Whelan.
New York Univ. Adler, Bernstein, Cooley, Courant, Doermann, Flanders, Forray, Graham, John, Kline, Norman, Payne, Peters, Putnam, Reddick, Rehberg, Schlauch, Tilley, Wahlert, Yanosik.
NIAGARA UNIVERSITY. O'Connor.
ONEONTA. Callahan, Sanford.
PELHAM. Milos.
POTSDAM. Buxton, Waltz.

POUGHKEEPSIE.
Vassar Coll. Baker, Durand, Wells.
ROCHESTER. Chesna, Eastham, Fraser, Harding.
Univ. of Rochester. Atkins, Bernstein, Betz, Gale, Long, Seidel, Watkeys.
ST. BONAVENTURE. Scheier.
SCHENECTADY. Poritsky.
Union Coll. Burkett, Fox, Morse, Rhodes, Snyder.
STATEN ISLAND. Baez, Karnow.
SYRACUSE.
Syracuse Univ. Carroll, Celaúro, Decker, Eves, Halmos, Harwood, Martin, S. H. Taylor, W. E. Taylor.
TROY.
Rensselaer Poly. Inst. Allen, Biggerstaff, Brown, Campbell, Merrill, Nash, Stilwell.
WEST POINT.
U. S. Military Acad. Farnell, Hazlewood, Jones, Yates.
WYOMING. Hartnell.

NORTH CAROLINA

BUIE'S CREEK. Harris.
CHAPEL HILL. Carroll.
University of North Carolina. Browne, Cameron, Garner, Henderson, Hickerson, Hill, Lasley, Mackie, Reynolds.
CHARLOTTE. Douglass, Jones, Woodson.
DAVIDSON. McGavock, Mebane.
DURHAM.
Duke Univ. Dressel, Elliott, Gergen, Hickson, Patterson, Rankin, Roberts, Thomas.
GREENSBORO. Barton, Pegram, Strong.
GREENVILLE. Graham, ReBarker.
HIGH POINT. Adams.
MARS HILL. Howell, Robinson.
RALEIGH. Downing, Eason.
North Carolina State Coll. Bullock, Cell, Levine, Strobel.
RED SPRINGS. Prince.
SALISBURY. Dearborn.
WILMINGTON. Peebles.

NORTH DAKOTA

FARGO. Smith
GRAND FORKS. Mason, Staley.
JAMESTOWN. Jackson.

OHIO

ADA. Whitted.
AKRON. Bender, Selby.
ALLIANCE. Ruddick.
ATHENS.
Ohio Univ. Denbow, Marquis, Miller, Reed, Starcher.
BEREA.
Baldwin-Wallace Coll. Annear, Brown, Stright.
BOWLING GREEN. Mathias, Overman.
CHILLICOTHE. Mathias.
CINCINNATI. Reilly.
Univ. of Cincinnati. Barnett, Brand, Hancock, Justice, Lipsich, Lubin, Merriman, Moore, Smith, Szász, Taylor, Yowell.
CLEVELAND. Anderson, Bailey, Burwell, Garvin, Johnson.
Case School of Appl. Sci. Brown, Burington, Focke, Green, Justin, Morris, Nassau, Pierce, Rinehart, Thomas, Torrance.
Fenn Coll. Godfrey, Kelly, Patterson, Tolar, Van Voorhis.
John Carroll Univ. Dustheimer, Hunter, Joliat.
Western Reserve Univ. Boyce, Musselman, Simon.
CLEVELAND HEIGHTS. Irr.
COLUMBUS. Singer, Wildermuth.
Ohio State Univ. Albert, Bamforth, Bareis, Beatty, Blumberg, Caris, Jones, Kuhn, LaPaz, Manson, Maple, Mickle, Morris, Parrish, Radó, Rasor, Rice, Rickard, Synge, Toops, Woods, Wylie.
DAYTON. Eagle, Johnson, Schraut.
DEFIANCE. MacCullough.
DELAWARE. Crane, Rowland.
GAMBIER. Bumer, MacNeille.
GRANVILLE.
Denison Univ. Bright, Heyda, Ladner, Moeller, Stewart, Wiley.
HIRAM. Clarke.
KENT.
Kent State Univ. Brooks, Harshbarger, Manchester, Olson, Rogers, Stelson.
LAKESIDE. Wolfe.
LAURELVILLE. Reichelderfer.
MARIETTA. Bennett, Sandt.
MOUNT ST. JOSEPH. Corona.
NAPOLEON. Yeager.
NEW CONCORD. Knight.
NEW LEXINGTON. Hoops.
NORTH BALTIMORE. Blackall.
OBERLIN. Smyth.
Oberlin Coll. Cairns, Carr, Randolph, Sinclair, Vance, Wagner, C. H. Yeaton, M. M. Yeaton.
OXFORD. Goins, Tappan.
Miami Univ. Anderson, Pollard, Spenceley, Wolfe.
PAINESVILLE. Peters.
SPRINGFIELD. Krueger, Tripp.
TOLEDO. Koley, Mercedes.
Univ. of Toledo. Brandeberry, Dancer, Lemme, Welker.
WESTERVILLE. Glover.
WILBERFORCE. Toney.
WILMINGTON. Spinks.
WOOSTER.
Coll. of Wooster. Fobes, Hildner, Knight, Williamson, Yanney.
YELLOW SPRINGS. Astrachan, Helms.
YOUNGSTOWN. Foard, McKissock.

OKLAHOMA

ADA. Heimann.
ALVA. Hall.
NORMAN.
Univ. of Oklahoma. Brixey, Court, Duval, Hassler, LaFon, McFarland, Reaves, Springer.

SHAWNEE. Doerfler, Short.
STILLWATER.
Oklahoma A. and M. Coll. Allen, Barnett, Flanders, Smith, Zant.
TULSA. Ellis, Veatch.

OREGON

CORVALLIS.
Oregon State Coll. Beaty, Hammer, LaRoe, Milne, Sobczyk, Williams.
EUGENE.
Univ. of Oregon. DeCou, Maddaus, Moursund, Peterson, Poole, Scobert, Wood.
FOREST GROVE. Price.
MCMINNVILLE. Ramsey.
PORTLAND. Griffin, Hadley, Merriss.
SALEM. Luther.

PENNSYLVANIA

ALLENTOWN. Billig, Deck, Koehler, Kunkel.
ANNVILLE. Black.
BEAVER FALLS. Cleland.
BETHLEHEM. Ashbaugh, Rader.
Lehigh Univ. Cutler, Fort, Latshaw, Pitcher, Raynor, Reynolds, Shook, Smail, Smiley, Van Arnam.
BRISTOL. Downing.
BRYN MAWR. Allen, Atkinson, Williams.
Bryn Mawr Coll. Lehr, Oxtoby, Wheeler.
CAMBRIDGE SPRINGS. Hawthorne.
CARLISLE. Ayres, Stuart.
CHESTER. Campbell, Williams.
COLLEGEVILLE.
Ursinus Coll. Clawson, Dennis, Manning.
DREXEL HILL. Maddrill.
EASTON.
Lafayette Coll. Benner, Cawley, Hatch, C. V. L. Smith, W. M. Smith.
ELIZABETHTOWN. Heilman.
ERIE. Benedicta, Kraus, Sullivan.
GETTYSBURG. Clutz.
GREENSBURG. McNeil.
GROVE CITY. Carpenter, Renwick.
HARRISBURG. Whited.
HAVERFORD.
Haverford Coll. Allendoerfer, Oakley, Vedova, Wilson.
HAZELTON. Herpel.
HERSHEY. Haag.
HUNTINGDON. Stayer.
KUTZTOWN. Knedler.
LANCASTER.
Franklin and Marshall Coll. Ikenberry, Marburger, Murray.
LATROBE. Seubert.
LEWISBURG.
Bucknell Univ. Gold, MacCreadie, Miller, Richardson.
LOCK HAVEN. Smith.
MEADVILLE. Smith, Steen.
MERCERSBURG. Johnson.
MILLERSVILLE. Boyer.
NEW KENSINGTON. Sturm.
NEW WILMINGTON. Black.
OAKMONT. Petrie.
OXFORD. Haviland.
PHILADELPHIA. Brown, Cavalli, Cherry, Constable, Davis, Eggert, Epstein, Fudge, Latshaw, Leifer, McDonough, McNeary, Robertson, Russ, Slepin, Smith.
Univ. of Pennsylvania. Caris, Evans, Kline, Luginbuhl, Pflaum, Safford, Shohat, Wallace, Whitman, Wilson.
PICTURE ROCKS. Price.
PITTSBURGH. Arnold, Baird, Briant, Buker, Calkins, Donaldson, Frankel, Jamison, Mullan, Wells.
Carnegie Inst. of Tech. Dines, Hoover, Johnson, Levine, Moskovitz, Neelley, Olds, Rosenbach, Saibel, Smith, Starr, Whitman.
Univ. of Pittsburgh. Blumberg, Bryson, Foraker, Hovey, Wagner.
PLEASANTVILLE. Kerr.
POTTSVILLE. Knapper.
READING. Speicher.
SCRANTON. Bertrand, Mary Daniel.
SHENANDOAH. Bauser.
SHIPPENSBURG. Kieffer.
SLIPPERY ROCK. Lady.
STATE COLLEGE. H. B. Owens.
Pennsylvania State Coll. Cohen, Cunningham, Curry, Dunlap, Frink, Gordon, Gravatt, Graves, Hagen, Harrington, Johnson, F. W. Owens, Rogers, Rupp, Schwartz, Sheffer.
SWARTHMORE.
Swarthmore Coll. Brinkmann, Dresden, Marriott.
VILLANOVA. Crawford.
WASHINGTON.
Washington and Jefferson Coll. Bert, Dorwart, Shaub, Thomas.
WAYNESBURG. Moston.
YORK. Baker.

RHODE ISLAND

KINGSTON. Brown, Stauffer.
NEWPORT. Chase.
PROVIDENCE. McKenney.
Brown Univ. Adams, Archibald, Bennett, Carlen, Gaskell, Gilman, Manning, Richardson, Saltzer, Schmidt, Smiley, Tamarkin.

SOUTH CAROLINA

CHARLESTON.
The Citadel. Dye, Hair, Hutchison, Reves, Saunders.
CLEMSON. Stanley.
CLINTON. Spencer.
COLUMBIA.
Univ. of South Carolina. Dinkines, Jackson, Shuler, Williams.
GREENVILLE. Blackwell.
HARTSVILLE. Reaves.
NEWBERRY. Gaver.
ROCK HILL. Stokes.
SPARTANBURG. Pettis.

SOUTH DAKOTA

BROOKINGS.
South Dakota State Coll. MacDougal, Walder, Wente.

MITCHELL. Knox.
RAPID CITY. Swanson.
SIOUX FALLS. Winn.
SPEARFISH. Hesseltine.
SPRINGFIELD. Hoopes.
VERMILLION.
Univ. of South Dakota. Ekman, Howell, Sealander.

TENNESSEE

CHATTANOOGA. Hughes, Massey, Mays.
COOKEVILLE. Hutchinson, Moorman.
FOUNTAIN CITY. Keller.
HARROGATE. Bowling.
JEFFERSON CITY. Sloan.
JOHNSON CITY. Carson.
KNOXVILLE.
Univ. of Tennessee. Cooley, Eaves, Ficken, Gillis, Lee, Pepper, Pollard, Purviance.
MARYVILLE. Sisk.
MEMPHIS. Locke, Walbert.
NASHVILLE. Falvey, Jordan, N. P. Miser, Van Horn, Wren.
Vanderbilt Univ. Blair, Hyden, Lundberg, W. L. Miser, Morrel.

TEXAS

ABILENE. Burnam, Mullings, Tate.
ALPINE. Gilley.
AMARILLO. Layton, Whetstone.
ARLINGTON. Howard.
AUSTIN.
Univ. of Texas. Anderson, Batchelder, Beckenbach, Craig, Decherd, Ettlinger, Greenwood, Lubben, Moore, Nelson, Vandiver.
BONHAM. McLaughlin.
BROWNSVILLE. de la Garza.
CANYON. Murray.
COLLEGE STATION.
A. and M. Coll. of Texas. Basye, Blumberg, Daum, Edmonson, Klipple, Luther, McCulley, Moore, Pinkerton, Wapple.
DALLAS. Sorrells.
Southern Methodist Univ. Huff, Mouzon, Starr.
DENTON. White.
North Texas State Teachers Coll. Brown, Cooke, Hanson.
EL PASO. Schwid.
FORT WORTH. Ramsey, Sherer.
GEORGETOWN. Olson.
HOUSTON. Baker, Blau, Horn, Rees, Slotnick, Underwood.
Rice Inst. Bray, Brunk, Dean, Lovett, Mandelbrojt, Ulrich.
HUNTSVILLE. Querry.
KINGSVILLE. Kennedy.
LUBBOCK.
Texas Tech. Coll. Heineman, May, Michie, Parker, Rowland, Sparks, Thompson, Underwood, Webb.
MAYPEARL. Thomas.
PRAIRIE VIEW. Randall, Stephens.
SAN ANTONIO. Mary of Mercy, McNelly, Morgan, Newton, Schnepp.
STEPHENVILLE. McSweeny, Redden.
TEAGUE. Notley.
WICHITA FALLS. Adams.

UTAH

LOGAN. Bird, Calvert.
ST. GEORGE. Everett.
SALT LAKE CITY. Bridger.
Univ. of Utah. Biesele, Gibson, Hayes, Henriques, Horsfall, Pehrson.

VERMONT

BURLINGTON.
Univ. of Vermont. Bullard, Butterfield, Millington, Swift.
MIDDLEBURY.
Middlebury Coll. Ballou, Bowker, Hazeltine, Perkins.
NORTHFIELD. Dix.
RANDOLPH. Alliot.
SAXTONS RIVER. Belding.

VIRGINIA

ASHLAND. Blincoe, Simpson.
BLACKSBURG.
Virginia Poly. Inst. Hatcher, McFadden, O'Shaughnessy.
BUENA VISTA. Durham.
CHARLOTTESVILLE. Patten.
Univ. of Virginia. Aylor, Hedlund, Linfield, McShane, Oglesby, Whyburn.
EMORY. Miller.
ETTRICK. Hunter.
FARMVILLE. Taliaferro.
FREDERICKSBURG. Frick.
HAMPTON. Perkins.
HOLLINS. Allen.
LEXINGTON. Paxton, Smith.
Virginia Military Inst. Byrne, Knox, Purdie.
LYNCHBURG. Baker, Elder, Larew, Wiggin.
MIDDLEBURG. Keppler.
NEWPORT NEWS. Raine.
NORFOLK. Smith.
RICHMOND. Drew.
Univ. of Richmond. Coleman, Gaines, Harris, Wheeler.
SALEM. Carpenter.
STAUNTON. Taylor.
SWEET BRIAR.
Sweet Briar Coll. Bates, Cole, Morenus.
WILLIAMSBURG.
Coll. of William and Mary. Calkins, Gregory, Phalen, Stetson.

WASHINGTON

ABERDEEN. Porter.
CENTRALIA. Van Arkel.
CHENEY. Bell.
PULLMAN.
State Coll. of Washington. Butler, Hacker, Knebelman, Ogawa.
SEATTLE. Beegle, Muggli.
Univ. of Washington. Ballantine, Cramlet, Haller, Jerbert, McFarlan, Mullemeister, Winger, Zuckerman.
SPOKANE. Carlson.
TACOMA. Martin.

WALLA WALLA. Bratton, Stewart.
YAKIMA. Whitney.

WEST VIRGINIA

KEYSER. Krabill.
LEWISBURG. Davis.
MONTGOMERY. Reckzeh.
MORGANTOWN.
West Virginia Univ. Davis, Eiesland, Peters, Reynolds, Turner, Vehse, Vest.
WEST LIBERTY. Kiplinger.

WISCONSIN

APPLETON. Berry.
BELOIT. Conwell, Huffer.
CUDAHY. Sedlak.
EAU CLAIRE. Ullsvik.
LA CROSSE. Adkins.
MADISON. Felder.
Univ. of Wisconsin. Allen, Butter, Evans, Hart, Ingraham, Kenney, Langer, MacDuffee, March, Simpson, Sokolnikoff, Trump.
MILWAUKEE. Beckwith, Bigelow, Boehmer, London, Mary Felice, Norris.
Marquette Univ. Luteyn, Pettit, Wilczewski.
Univ. of Wisconsin Exten. Div. Bardell, Battig, Marden, Parkinson, Vass, Wolf.
OSHKOSH. Beenken, Price.
PLATTEVILLE. Harrell.
RIVER FALLS. Eide.
ST. FRANCIS. Fetterer, Jautz, Mary Gertrude.
SUPERIOR. Flogstad, Smith.
WAUKESHA. Dancey, Hopkins.
WEST ALLIS. Fitzpatrick.
WEST DEPERE. DeCleene.
WHITEFISH BAY. Anderson.

WYOMING

LARAMIE. Bellamy.
Univ. of Wyoming. Barr, Neubauer, Rechard, Varineau.

CANADA

EDMONTON.
Univ. of Alberta. Campbell, Cook, Sheldon.
FREDERICTON. Miller.
HAMILTON. Findlay.
KINGSTON.
Queen's Univ. Blyth, Jeffery, Miller.
LENNOXVILLE. Richardson.
LONDON. Cole, Kingston.
MONTREAL. Gauthier, Gough, Pelletier, Perry.
McGill Univ. Pall, Tate, Wood.
OTTAWA. Dubé.
QUEBEC. Pouliot, Tremblay.
REGINA. Oakland.
SACKVILLE. Crawford, McEwen.
SASKATOON. Pyke.
TORONTO. Dobson, Grant.
Univ. of Toronto. Beatty, Brauer, Burk, Coxeter, Pounder, Robinson, Rosebrugh.
VANCOUVER.
Univ. of British Columbia. Buchanan, Gage, James, Nowlan.
WELLS. Stewart.
WINNIPEG. Warren, Wilson.
WOLFVILLE. Macphail.

FOREIGN MEMBERS

ARGENTINA

BUENOS AIRES. Baidaff, Barral-Souto.

AUSTRALIA

ADELAIDE. Wilton.

BELGIUM

UCCLE. Errera.

BRITISH HONDURAS

BELIZE. Zimmerman.

CHILE

SANTIAGO. Moreno.

CHINA

CANTON. MacDonald

CUBA

HAVANA. Corral y Alemán, Gonzáles.

ECUADOR

QUITO. Thullen.

ENGLAND

CAMBRIDGE. Hardy.
CHIPPING NORTON. O'Hara.
LONDON. Dalal, Todd.

FINLAND

HELSINGFORS. Ahlfors.

HUNGARY

BUDAPEST. Arany.

INDIA

BANGALORE. Iyengar.
MADRAS. Durairajan.
POONA. Banerji.
SURAT. Shah.

IRELAND

BELFAST. McCrea.
DUBLIN. Rowe.

ITALY

BOLOGNA. Bortolotti.
NAPLES. Crudeli.
ROME. Enriques, Laboccetta.

MEXICO

MEXICO. Nápoles.

NEW ZEALAND

DUNEDIN. Martyn.

PANAMA

PANAMA CITY. Linares.

PERU

LIMA. de Losada y Puga.

PORTUGAL

LISBON. Caraça, Cunha.

ROUMANIA

TIMISOARA. Sergescu.

RUSSIA

MOSCOW. Kryloff.

SOUTH AFRICA

BLOEMFONTEIN. Arndt.

SPAIN

SAN SEBASTIAN. Thébault.

STRAITS SETTLEMENTS

SINGAPORE. Oppenheim.

SWITZERLAND

FRIBOURG. Bays.
NEUCHÂTEL. DuPasquier.

SYRIA

BEIRUT. Jurdak.

URUGUAY

MONTEVIDEO. Calcagno.

BY-LAWS OF THE MATHEMATICAL ASSOCIATION OF AMERICA (INC.)

(As amended to January 1, 1944)

ARTICLE I—NAME, PURPOSE AND CORPORATE SEAL

1. This organization shall be known as

THE MATHEMATICAL ASSOCIATION OF AMERICA (INCORPORATED)

2. Its object shall be to assist in promoting the interests of mathematics in America, especially in the collegiate field, by holding meetings in any part of the United States or Canada for the presentation and discussion of mathematical papers, by the publication of mathematical papers, journals, books, monographs and reports, by conducting investigations for the purpose of improving the teaching of mathematics, by accumulating a mathematical library and by coöperating with other organizations whenever this may be desirable for attaining these or similar objects.

3. The Corporate Seal of the Association shall have inscribed thereon the name of the Association and the words "Corporate Seal—Illinois."

ARTICLE II—MEMBERSHIP

1. Any person who is interested in the field of collegiate mathematics shall be eligible for election to membership in the Association.

2. Any institution in which the Calculus is regularly taught shall be eligible for election to institutional membership in the Association. Such an institution shall have the privilege of sending a voting delegate to the meetings of the Association.

3. Election to membership shall be by vote of the Board upon written application from the individual or institution seeking admission, endorsed in the case of individuals by two members of the Association.

4. Those who were admitted to membership in The Mathematical Association of America (unincorporated) prior to October 1, 1920, and were in good standing as such on that date, were thereby admitted to membership in this Association (Incorporated).

ARTICLE III—BOARD OF GOVERNORS AND OFFICERS

1. The Officers of the Association shall be a President, a First Vice-President, a Second Vice-President, an Editor-in-Chief of the Official Journal (hereinafter called the "Editor"), a Secretary-Treasurer, and an Associate Secretary.

2. There shall be a Board of Governors (hereinafter called the "Board"), to consist of the Officers, the Ex-Presidents for terms of six years after the expiration of their respective presidential terms, and of additional elected members (hereinafter called "Governors"). It shall be the function of the Board to supervise all scholarly and scientific activities of the Association, to administer and control these activities, and to authorize expenditures of funds of the Association, except that at the demand of ten or more members of the Board, or at the demand of forty or more members of the Association, any proposal to alter or initiate a matter of policy shall be referred to the general membership of the Association for its decision. All members of the Board shall hold over until their respective successors are selected or appointed and qualify.

3. There shall be an Executive Committee advisory to the Board, and consisting of the President, the two Vice-Presidents, the Editor and the Secretary-Treasurer. It shall be the function of this Committee to review continually the policies and activities of the Association, to plan and organize new activities, to formulate in broad outline the programs of meetings and of publications, and in general to consider all matters of importance or of interest to the Association. This Committee shall prepare the agenda for meetings of the Board, and shall analyze the implications and aspects of all matters which are to come before the Board for decision. It shall present to the Board the viewpoints suggested by such analyses, as well as all such facts as may seem pertinent, or as may in any way facilitate the Board's work.

4. A statement regarding any proposed action of the Board which makes or alters a question of policy shall be published in the official journal, or notice of such proposed action shall be mailed to each member, before final action has been taken, so that members of the Association may make known to the Board their individual views.

5. The Board shall have authority to fill vacancies *ad interim* in any office, including vacancies in the Board, and to make any other appointments necessary for the transaction of the business of the Association.

6. At all meetings of the Board of Governors a quorum shall consist of not less than five (5) members and no business may be validly transacted at a meeting at which less than a quorum is present; *provided* that any meeting of the Board, whether or not a quorum be pres-

ent, may be adjourned to a specified time and place by a majority of the members present without notice to the members at large other than announcement at such meeting. Informal action based on a mail ballot by the members of the Board, if ratified at a properly convened meeting of the Board, shall be as valid and effective as if originally authorized at such meeting.

7. There shall be a Finance Committee responsible to the Board; at the direction of the Board it shall receive and administer the funds of the Association, control its properties and investments, make its contracts, and exercise such powers as may be delegated to it by the Board. This committee shall consist of three members, of whom the Secretary-Treasurer shall be one.

8. (a) The Officers and Governors of the Association shall be elected in part by the Board, in part by the general membership, and in part by this membership in constituencies (hereinafter called "Regions") established by the Board.

(b) The membership at large shall elect in alternate years respectively a President and a First Vice-President, each for a term of two years, and shall elect each year two Governors, for terms of three years.

(c) The membership in each Region shall elect biennially a Governor for a term of two years. Nominations shall be made by the Section or Sections of the Association existing within the Region, or, in the absence of such Sections, by a committee appointed for that purpose by the Governor representing the Region.

(d) The Board shall elect at appropriate times by ballot and for the terms stated: a Second Vice-President for two years; an Editor, a Secretary-Treasurer, and an Associate Secretary, each for five years; and members of the Finance Committee (other than the Secretary-Treasurer) for four years.

(e) The President shall be ineligible for reëlection. The Vice-Presidents, the Editor, and the Governors shall be eligible for reëlection only after an interim equal to their respective terms of office.

(f) Elections by the Board shall be made from nominations by the Executive Committee. At least two nominations shall be made for each office to be filled in the case of the Second Vice-President and the members in the Finance Committee, and the Board may in any case reject all nominations made and call for a new list.

(g) The names of members to be printed upon the ballots, together with blank spaces in the case of elections by the general membership, shall be determined by a Nominating Committee to be appointed annually for that purpose by the Board. Approximately two months before the date of the annual meeting all members shall be given an opportunity to nominate by mail a candidate for each office to be filled by the members for the ensuing year. Approximately one month before the annual meeting the Board shall select a nominee for President out of the three persons who received the most votes for this office in the nominations; the Board shall furthermore select two candidates for each other office to be filled by the members, one being the person who received the highest vote in the nominations and the other being selected from among the several nominees next in order. The election shall be by mail or in person and shall close on the day of the annual meeting.

9. The President shall be the Executive Officer of the Association, shall preside at all meetings of the Board of Governors and at the annual meeting of the Association. He shall have the usual duties pertaining to his office and such other duties as may from time to time be assigned him by the Board of Governors.

10. The Vice-Presidents shall, in the absence of the President, have and exercise the powers of the President, their order being determined alphabetically. The Board of Governors may assign to the Vice-Presidents such duties as may from time to time be determined.

11. The Secretary-Treasurer shall have the usual duties pertaining to the office of Secretary and of Treasurer, including the custody of the records of the Association and of its Corporate Seal, the keeping of minutes of the meetings of the Board of Governors and of the annual meeting and special meetings of members, and giving of due notice of all regular and special meetings of the Association and of the Board of Governors, and the supervision and safekeeping of the funds of the Association. The Secretary-Treasurer shall also have the duty of seeing that whenever Governors are elected, including the election of Governors to fill vacancies, a Certificate, under the Seal of the Association, giving the names of those elected and the term of their office, shall be recorded in the Office of the Recorder of Deeds for Cook County, Illinois. Such Certificates shall be signed by the Secretary-Treasurer and verified by oath of the President.

Article IV—Meetings

1. A meeting of the Association shall be held annually, at such time and place as the Board may direct. Special meetings of the Association may be called from time to time by the Board, or while the Board is not in session by the President of the Association, to be held at such time and place as may appear from the call.

2. The outgoing Board shall hold a meeting immediately preceding the annual meeting of the Association next succeeding their election, and the members of the new Board shall

hold a meeting and organize, by completing the Board, immediately succeeding the annual meeting of the Association at which the new members thereof were elected. Further meetings of the Board may be held from time to time at the call of the President or of any three(3) members of the Board.

3. Notice of any meeting of members of the Association shall be given by the Secretary-Treasurer at least thirty (30) days prior to the date set for each meeting. Notice of all meetings of the Board other than the regular meetings provided in Section 2 shall be given to each member of the Board at least fifteen (15) days prior to the date set therefor.

4. Any member of the Association or of the Board may waive notice with the same effect as if due notice had been given him.

5. At all meetings of the Association a quorum shall consist of not less than twenty-five (25) members and no business may be validly transacted at a meeting at which less than a quorum is present; *provided* that any meeting of the Association, whether or not a quorum be present, may be adjourned to a specified time and place by a majority of the members present without notice to the members at large other than the announcement at such meeting.

6. Members may take part and vote in person or by proxy at all meetings of the Association.

Article V—Sections

1. Any group of not less than ten (10) members of this Association may petition the Board for authority to organize a Section of the Association for the purpose of holding local meetings. The Board shall have power to specify the conditions under which such authority shall be granted.

2. The Association shall not be obligated to pay from its treasury any of the expenses of such Sections.

Article VI—Official Publications

1. The Association shall publish an official journal, which shall be sent free to all members of the Association in accordance with Article VII.

2. The Board shall have full control of the publication and sale of the official journal and of all other official publications.

3. There shall be appointed by the Board a body of Associate Editors who shall give assistance in connection with the official journal.

4. The Board shall from time to time, as the need arises, make special provision for the management of any other official publications.

5. The Board shall fix the price of the official journal and of any other official publications of the Association, but in no case shall the journal be sold to non-members for less than the annual dues of individual members.

Article VII—Dues

1. Individual members of the Association shall pay an initiation fee of Two Dollars ($2) at the time of election.

2. The annual dues of each individual member shall be Four Dollars ($4), including a subscription to the official journal.

3. The annual dues of each institutional member shall be Seven Dollars ($7), including two (2) subscriptions to the official journal.

4. All dues shall be payable on the first of January of each year. Should the annual dues of any member remain unpaid beyond a reasonable time, his name shall be dropped from the list after due notice.

5. New members entering the Association after April 1 of any year shall have their dues pro-rated for the balance of the year, except when they desire to receive the full current volume of the official journal.

6. Any member who because of age is no longer in active service, who is in good standing at the time of his retirement and who has been a member of the Association for twenty years, may, upon notifying the Secretary of said retirement, be exempt from the payment of dues, with the privilege of obtaining the official journal at an annual cost of one dollar.

Article VIII—Amendments to the Articles of Association and By-Laws

1. Changes in the Articles of Association or amendments to the By-Laws may be made at any annual meeting of the Association, or at any adjourned session thereof, or at any special meeting of the Association called for such purpose, by a two-thirds ($\frac{2}{3}$) vote of those present and entitled to vote; *provided* that due notice concerning such amendment shall have been printed in the official journal, or mailed to each member, at least one (1) month before the date of such meeting.

2. No changes in the Articles of Association shall have legal effect until a certificate thereof, verified by oath of the President and under Seal of the Association, attested by the Secretary-Treasurer, shall be filed in the office of the Secretary of State of the State of Illinois and recorded in the office of the Recorder of Deeds for Cook County, Illinois.

PERIODS OF SERVICE OF FORMER OFFICERS OF THE ASSOCIATION

(The periods were for the calendar years except that after 1942 the terms of Regional Governors began and ended July 1.)

Honorary President for Life

H. E. Slaught, December 1933–May 1937

Presidents

E. R. Hedrick......1916
Florian Cajori......1917
E. V. Huntington......1918
H. E. Slaught......1919
D. E. Smith......1920
G. A. Miller......1921
R. C. Archibald......1922
R. D. Carmichael......1923
H. L. Rietz......1924
J. L. Coolidge......1925
Dunham Jackson......1926
W. B. Ford......1927–1928
J. W. Young......1929–1930
E. T. Bell......1931–1932
Arnold Dresden......1933–1934
D. R. Curtiss......1935–1936
A. J. Kempner......1937–1938
W. B. Carver......1939–1940
R. W. Brink......1941–1942

Vice-Presidents

E. V. Huntington......1916
G. A. Miller......1916
D. N. Lehmer......1917, 1918
Oswald Veblen......1917
J. W. Young......1918, 1926
R. G. D. Richardson......1919
H. L. Rietz......1919
Helen A. Merrill......1920
E. J. Wilczynski......1920
R. C. Archibald......1921
R. D. Carmichael......1921, 1922
B. F. Finkel......1922
A. B. Chace......1923
L. P. Eisenhart......1923
J. L. Coolidge......1924
Dunham Jackson......1924, 1925
A. A. Bennett......1925, 1933, 1934
W. B. Ford......1926
A. J. Kempner......1927, 1928, 1935
Clara E. Smith......1927
F. D. Murnaghan......1928, 1939
E. T. Bell......1929, 1930
W. C. Graustein......1929, 1930, 1940
Arnold Dresden......1931
C. N. Moore......1931
W. H. Bussey......1932
G. C. Evans......1932
E. B. Stouffer......1933
E. P. Lane......1934
L. L. Dines......1935
N. A. Court......1936
T. C. Fry......1936
T. H. Hildebrandt......1937
E. J. Moulton......1937, 1938
H. E. Buchanan......1938
W. L. Hart......1939
R. W. Brink......1940
B. H. Brown......1941–1942
R. E. Langer......1941
Tomlinson Fort......1942–1943

Secretary-Treasurer

(Appointed by the Board after 1918)

W. D. Cairns......1916–1942

Committee on Official Journal

(Appointed by the Board. Discontinued after 1939)

H. E. Slaught......1916–1937
R. D. Carmichael......1916–1918
W. H. Bussey......1916–1918, 1926–1931
R. C. Archibald......1919–1921
W. A. Hurwitz......1919–1921
A. A. Bennett......1922
H. P. Manning......1921–1922
W. B. Ford......1923–1925
J. L. Coolidge......1923
A. J. Kempner......1924–1939
W. B. Carver......1932–1936, 1937–1939
E. J. Moulton......1937–1939

Editors-in-Chief after 1939

E. J. Moulton......1940–1941

ADDITIONAL MEMBERS OF THE BOARD

D. N. LEHMER.......1916–1918, 1922–1924
1930–1932
R. E. MORITZ..................1916–1918
K. D. SWARTZEL...............1916
OSWALD VEBLEN..........1916, 1920–1922
1926–1931
R. C. ARCHIBALD.....1916–1917, 1923–1930
FLORIAN CAJORI..........1916, 1918–1923
1926–1930
M. B. PORTER.................1916–1917
J. W. YOUNG........1916–1917, 1920–1922
B. F. FINKEL........1916–1921, 1930–1935
E. H. MOORE........1916–1921, 1923–1928
ALEXANDER ZIWET..............1916–1918
E. R. HEDRICK.......1917–1922, 1924–1929
1932–1937
J. N. VAN DER VRIES...........1916–1918
HELEN A. MERRILL............1917–1919
D. E. SMITH.........1917–1919, 1921–1926
1937–1939
ELIZABETH B. COWLEY..........1918–1920
G. A. MILLER........1918–1920, 1922–1924
E. J. WILCZYNSKI....1918–1919, 1922–1926
L. P. EISENHART.....1919–1922, 1925–1930
E. V. HUNTINGTON........1917, 1919–1927
1933–1935
E. L. DODD....................1920
R. D. CARMICHAEL........1920, 1924–1929
1939–1941
A. A. BENNETT..1921, 1930–1932, 1939–1941
H. L. RIETZ.........1921–1923, 1925–1930
1934–1936
C. F. GUMMER.................1921–1925
DUNHAM JACKSON.............1923–1929
CLARA E. SMITH...............1923–1925
A. B. CHACE...................1924–1925
J. L. COOLIDGE................1926–1931
E. T. BELL....................1927–1928
E. P. LANE....................1928–1933
W. B. FORD...................1929–1934
E. R. SMITH...................1929
W. L. HART...................1930–1935
LAO G. SIMONS.................1930–1931
L. L. DINES..................1931–1933
T. C. FRY.....................1931–1933
J. W. GLOVER.................1931–1933
H. E. BUCHANAN...............1932–1937
W. R. LONGLEY......1932–1934, 1936–1938
E. J. MOULTON................1933–1936
R. W. BRINK..................1934–1939
D. R. CURTISS..1934, 1937–1939, 1940–1942
J. L. WALSH...................1934–1936
ARNOLD DRESDEN..............1935–1940
J. O. HASSLER.................1935–1936
F. D. MURNAGHAN.............1935–1937
G. C. EVANS...................1936–1941
MARY EMILY SINCLAIR..........1936–1938
J. M. THOMAS.......1937–1939, 1940–1941
MARIE J. WEISS...............1937–1938
WILLIAM BETZ.................1938–1940
A. B. COBLE...................1938–1940
J. H. WEAVER.................1938–1940
PHILIP FRANKLIN..............1940–1942
F. L. GRIFFIN.................1940–1942
MAYME I. LOGSDON............1940–1942
G. T. WHYBURN...............1940–1942
C. V. NEWSOM.................1940–1941
O. J. PETERSON................1940–1941
F. B. WILEY...................1940–1941
H. M. BACON..................1941–1943
H. J. ETTLINGER...............1941–1943
CORNELIUS GOUWENS...........1941–1943
W. C. KRATHWOHL.............1941–1943
E. J. MCSHANE................1941–1943
F. W. OWENS..................1941–1943
S. T. SANDERS.................1941–1943

The Carus Mathematical Monographs

The CARUS MATHEMATICAL MONOGRAPHS are an expression of the desire of Mrs. Mary Hegeler Carus, and of her son, Dr. Edward H. Carus, to contribute to the dissemination of mathematical knowledge by making accessible at nominal cost a series of expository presentations of the best thoughts and keenest researches in pure and applied mathematics.

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FEBRUARY 1944

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COLLEGE MATHEMATICS DURING RECONSTRUCTION*

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1. Introduction. If we pause for a little while amid the turmoil of the war-training courses in order to look forward to the period just after the war, it is not because we feel that the war is over nor even that its end is hidden from us by being just around the corner. The war-training programs will doubtless be with us for a long time to come. Doubtless they will be subject to more of the sudden and arbitrary changes that we have all found so baffling. Doubtless we shall continue to be faced with shortages of staff, irregular schedules, budgetary difficulties, and elusive academic standards, as in the recent past. Yet I feel that the general pattern of training in the colleges is fairly well established or at least that we are psychologically better prepared than we were twelve months ago to meet such wartime changes as may occur. On this plateau, then, rather than summit, that we have reached I should like to pause to take stock of our gains and losses, and to look forward toward the land ahead. Perhaps it is not too early to see something of this land of college mathematics after the war is won and of the problems we shall find there. Perhaps, by looking back at the way we have come, we can find help in solving those problems.

2. College enrollments after the war. In order to form some notion of the size of the task ahead of us, let us first consider the probable general university and collegiate enrollment, without special regard to its effect on mathematics. We shall almost certainly have a flood of former students returning to the colleges from their service with the armed forces, governmental agencies, and defense plants. To these will be added the recent high school graduates of perhaps several years, who will be entering college for the first time.

Another force, also tending in the direction of increased enrollments, and, to my mind, at least equally important for the long haul, is the probable broader base of education that will result from the war. American education, in constrast to that in most European countries, has long been characterized by the theory of the broad base. Not for us has been the system of higher education only for the intellectually or economically elite. Of course, this has resulted in the attempted education of many young persons never intended by their Maker to breathe the rare atmosphere of the intellectual life. Yet if, as educators would probably be the first to assert, education is a chief bulwark of democracy, it would seem to be a sound theory which both permits and encourages each citizen to carry his education as far as his native gifts permit, regardless of his economic status and arbitrary academic standards. In the experiences that our young men are undergoing in the training camps and on the battlefields of the world, I can see a great levelling force at work. It will level upward, I hope, as well as downward. After observing that some eighty per cent of their officers

* Retiring presidential address delivered to the Mathematical Association of America, Chicago, Ill., November 27, 1943.

have had some college background, after having taken part in Army Institute and other courses preparing them for their immediate jobs or more remotely for their places as citizens, after having worked and fought on even terms with comrades who expect to return to college, is it not likely that many thousands of our young soldiers will demand for themselves the privileges of further education?

When they knock at our college doors, I hope that we shall apply some intelligent guidance, not so much in the form of outright rejection as in advice as to the most suitable types of training. But somewhere in our educational system we shall accept them without too rigid an application of scholastic requirements. For one thing, during our recent experiences, we have learned not to be too choosey about our raw material. Many colleges that previously accepted only students in the upper twenty per cent of their high school classes have been delighted to take men selected for them on the doubtful basis of an Army General Classification Test. Good men are still good men, and stand out as such. But we have found that with the aid of proper refresher courses even men of inferior preparation can be given useful training at a level beyond what might at first be expected from their previous experience. I shall revert to this point later on.

Most of all we will give an ungrudging welcome to these young men and women because we think that it will be advantageous to the nation to do so. Much as we may fear future wars and foreign aggression, our greatest dangers after the war will come from within. They will come from the disunity of one special interest opposing others. To dispel such dissensions we must teach as large a part of your young population as possible to distinguish information from dishonest propaganda, to seek logical instead of emotional bases of action, to understand the essentially cooperative nature of society, to respect he other man's manner of thought, and to earn a livelihood in socially useful work.

Not all of the tendencies will be in the direction of increased enrollments. Many of the returning men will be impatient to secure permanent positions. Many will have new opportunities open to them as a result of their war training and experience. Others, who married on the strength of wartime salaries and find that they married on a shoestring, will be forced to seek work to support their families.

But, on the whole, the signs point to enrollments far exceeding anything that the colleges have known before. A committee at one of our large middle-western state universities* has estimated that during the first year after the war its enrollment will be from eleven per cent to seventeen per cent, depending on the war's duration, greater than it ever was before, and that during the second year after the war it will be from twenty-eight per cent to thirty-four per cent greater. By 1950–1951, the committee predicts an enrollment forty per cent greater than the maximum hitherto. These estimates do not take into ac-

* Report of the Senate Committee on Education, University of Minnesota, Senate Minutes, October 21, 1943.

count the stimulus of any possible plans for federal subsidization of students after the war. If these plans develop, it is likely that the forty per cent increase may occur within a year or two after the war's end.

With reference to the share that returning service men will occupy in our load, Colonel John N. Andrews of the Reemployment Division of Selective Service Headquarters states that the general indication now is that of the eleven to twelve million men who will have been in service by the time the war ends, at least one million will wish to return to vocational schools or colleges and universities to pursue further study. Many of these entered the service directly from high schools. Large numbers will have completed from one to three years of college work, and will wish to continue in their original courses or in other directions. And the indications are that those who desire to return to school will be given financial support to enable them to do so.

3. Plans for federal subsidization of education. Without yet knowing the exact form that a scheme of federal subsidies will assume, we can take it as probable that some scheme will develop. The most definite suggestions from any authoritative source are contained in the recommendations of the Conference on Postwar Readjustment of Civilian and Military Personnel under the auspices of the National Resources Planning Board.* The general principles of these recommendations were contained in the report of a special committee on education appointed by the President of the United States. On October 27, 1943, the President embodied these principles in a message† to Congress with a request for the necessary legislation. In brief outline, the recommendations for subsidy of education call for two plans. Under the first plan of general education free tuition and reasonable allowances for maintenance would be provided for a period not exceeding one year to any honorably discharged ex-service man who might apply for it. Both general and vocational or professional education and training would be provided, but the latter would not be provided in those fields and for those occupations in which the supply of trained personnel is already sufficient to meet anticipated requirements. Training and education would begin at the level appropriate to the individual and should have such forms and methods as are suited to the needs of mature men regardless of the academic level. (Recommendations 50–58.) In addition to this general plan of education the Conference recommended a plan of supplementary education. This plan recognizes the special responsibility of the Nation to those who had entered upon an extended course of education which was interrupted by their military service and it also recognizes the need of the Nation for specially trained personnel. Under the program of supplementary education scholarships not to exceed four years in duration would be granted on a competitive basis in the fields of higher education and in such technical and professional fields as

* Demobilization and Readjustment—Report of the Conference on Postwar Readjustment of Civilian and Military Personnel, National Resources Planning Board, June, 1943, Superintendent of Documents, Washington, D. C.

† Congressional Record, October 27, 1943, p. 8881.

offer some likelihood of satisfactory and useful employment. (Recommendations 59–66.)

We cannot be certain that such a program of federal subsidization will be adopted. It seems very probable, however, that some very similar plan will take effect. For the purpose of discussion, I should like to assume that this is the case. And I should like to consider some aspects of the situation in college mathematics with such a program imposed on our other tasks.

4. The load in mathematics. What the Conference called the "general" plan of education provides for one year of either general education or training for vocations or professions in which there is a predictable opportunity for employment. Even before a comprehensive survey of the needs in the various trades and professions has been made, we can make some prediction of the impact of this program on the mathematical load. Certainly, if we recall the importance that has been attached by the armed forces and by industry to elementary training in mathematics for their mechanics and foremen, we cannot doubt that there will be a great deal to do in teaching what we think of as "shop mathematics." I hope that most of this teaching can be done in trade schools, some special junior colleges, and in special extension courses of the universities or in industry itself. For I am sure that collegiate staffs will be fully occupied at more nearly the collegiate level.

But what of the young men who in ordinary times would have gone directly from high school into jobs requiring no further academic training? Many thousands of them who will have received training and seen service as electricians' mates or airplane mechanics or radio technicians will be stirred to seek advancement in similar lines. Many of them will wish to spend the time of their subsidized training as students of engineering or science. To the extent to which they can meet or nearly meet the entrance requirements of our colleges, they can be absorbed into the regular engineering and scientific curricula. Since mathematics is a subject that cannot be bypassed nor easily replaced by experience in the field, mathematics staffs will be under special stress to carry this additional burden. Indeed it is not unlikely that the load of teaching elementary college mathematics will exceed our present one, including the present Army and Navy programs.

5. Advanced mathematics in engineering and scientific courses. I foresee also a considerable expansion at the upper end of the engineering and scientific curricula. Many young men will have completed the somewhat abbreviated or at least hurried courses called advanced engineering in the specialized service programs. These young men as well as many recent engineering graduates on leaving the army will see in the federal educational program an opportunity to continue their studies from an apprentice to a professional level. It would probably be idle here to try to fix the point at which preprofessional work in engineering ends and professional training begins. It is certainly true that in such professions as law, medicine, and the ministry, formal schooling does not end

after four years beyond the high school. Perhaps we are willing to entrust our rights, our health, and our souls, to the schools but not our material possessions. No more than we should expect every physician to be using basic science in research should we expect every professional engineer to employ mathematics at the research level. Yet, because of the essentially quantitative character of engineering work, we might not be too far off the mark in accepting the skill with which an engineer uses applied mathematics as one measure of his professional attainment. It has been repeatedly pointed out of late, notably by Thornton C. Fry in his report on mathematics in industry,* and by R. G. D. Richardson in this MONTHLY,† that the study of applied mathematics in America has not progressed as it has in certain foreign countries nor as has the study of pure mathematics here. It is perhaps a natural consequence of this that in many of our four- and five-year engineering schools the curricula are less mathematical and to an extent less professional in character than they might well be. Not in all but in many of them it is the tradition to consider mathematics as a tool subject taught on a relatively formal and mechanical basis. In such cases the special power of applied mathematics is largely lost. Perhaps the ideal in engineering training would be to combine the power of generalization, which implies the discernment of the essential in a problem, and which is so characteristic of modern American mathematics, with an interest in and familiarity with the materials of some field of technological application. I am inclined to believe that such considerations will ultimately lead away from the formal toward the more theoretical, that is, the ultimately more practical type of mathematics course in engineering colleges. Immediately after the war this tendency may not exist and may even be reversed for the courses in the first two years of the curriculum. This reversal will probably result from the lack of qualified teachers, the weak preparation of the mass of entering students, and from the desire of men already advanced in years and maturity of experience to pursue their training in as rapid and concentrated a form as possible.

But the war will itself have produced such advances in engineering practice that there will be a great demand for men of really superior training for the mere manufacture, installation and operation, let alone the design, of equipment.

Some days ago I talked with a young man, a recent first-year graduate student of mathematics, who is employed in one of the large government laboratories. The work he is doing is not concerned with the discovery of basic scientific facts, but is exceedingly practical in character and involves the application of familiar principles to strictly engineering problems. He was able to solve some of these problems by means of somewhat complicated differential equations, the application of potential theory, contour integration, and the use of Fourier transforms. He feels that, though he has had no engineering training,

* Research—A National Resource—II, VI, 4, pp. 268–288—A House Document, 77th Congress.

† Applied Mathematics and the Present Crisis. Vol. 50, No. 7, Aug.-Sept., 1943, pp. 415–423.

he is at an advantage as compared with others in the laboratory because he finds it natural to formulate a problem in mathematical language and to solve it mathematically. Another first-year graduate student of my acquaintance was employed by a company that manufactures intricate apparatus now used in the war. In this case approximations by means of orthogonal polynomials furnished the key to important improvements in equipment. These are merely two examples that many of you could doubtless duplicate many times. The point that I wish to emphasize is not the applicability of mathematics to engineering, which needs no emphasis. It is rather the fact that these were first-year graduate students and the techniques that they employed would be accessible to many young engineers with only a moderate increase in their mathematical background.

From these considerations I should expect a heavy increase in registration in our courses of intermediate and moderately advanced mathematics in engineering and scientific schools. I also expect a strong demand for additional or more extensive courses at this level, such as more courses in statistics with special emphasis on manufacturing and quality control, potential theory, advanced differential equations, theory of elasticity, group theory and fluid dynamics. How far and how promptly our depleted and overburdened staffs will be able to meet this demand is another story.

I should like to quote from a letter that I have just recently received from Colonel Andrews of the Reemployment Division of Selective Service, whom I have already mentioned. It seems to indicate that my own opinions are shared by those studying the problem from the point of view of employment. He writes: "Because of the unusual technical advances which are certain to occur following the war, all kinds of mathematicians, scientists, and research specialists will be in great demand. Such opportunities are already being discussed in the large industries and, eventually, the men in the services will be informed of the need for such specialists. It is believed that thousands of the young men will seek further training in these technical fields as soon as they are no longer needed in the armed forces. This program will call for a tremendous mathematics and science emphasis at the freshman and sophomore levels, especially, and also a continuation of these subjects in the junior and senior years. Perhaps the graduate work in these subjects will have the greatest challenge of their history. In order to meet these expanding demands, colleges and universities which have the proper staffs and laboratories should begin now the development of curricula to meet the needs."

6. Need of interdepartmental collaboration; liaison courses. What I have said applies equally well to the training of engineers and of other scientists. In both cases there are certain lessons to be learned from the military courses that we have taught. Some of these have required the closest collaboration between mathematicians and physicists. The pre-meteorological courses in mathematics, vectorial mechanics, and physics well illustrate the advantage of such collaboration in the organization of material. Especially in their earlier parts, a careful

ordering of topics supplied in one course the materials necessary in another in time for their utilization there, and avoided unnecessary duplications of material. I hope that with the return of peace these lessons of cooperation will not be forgotten. In our standard curricula there is usually sufficient time to allow a course in one subject to follow logically certain prerequisite courses in other fields. Thus a man-sized course in physics ordinarily follows at least a year of college mathematics. But probably we shall have many mature students seeking rapid advancement in rather narrow scientific or technical training. It may then be necessary and, in their cases, desirable to teach them their mathematics and physics concurrently instead of in sequence. Our experience with the war training courses has shown that in case of necessity this can be done with considerable success. Such procedures depend, however, on the closest and most sympathetic interdepartmental collaboration. I shall refer later to the other necessity of planning for courses that are less narrow and concentrated in design.

With the increased interest in applied mathematics after the war I expect a tendency toward what I may call "liaison" courses. By these I mean courses that do not lie wholly in one conventional field, but furnish contact between two or more departments. The pre-meteorological course in Vectors and Mechanics is an example. It formed a most useful link between mathematics and its applications. It was most interesting also in another respect. It showed the feasibility of introducing relatively advanced material early in the college course. In the "C" program it successfully presented to freshmen concurrently with the first course in calculus and after only twelve weeks of college mathematics, material of vector analysis that had usually been reserved for juniors and seniors. The early presentation of this material undoubtedly greatly strengthened the calculus course by giving greater reality to such notions as curvilinear velocity and acceleration. It may well be that other entire courses or special topics require less maturity than we have hitherto assumed and that they can be introduced advantageously earlier in the curriculum than we supposed, especially for students seeking rapid advancement in a special field.

7. Statistics. Another interesting type of liaison course is one in elementary statistics. An immense amount of statistical work will be required as a result of the war. To realize this one has to think only of such sample problems as the rehabilitation of veterans, job surveys, reconversion of industry, quality control in the manufacture of new products, distribution of food and other commodities to our own and foreign populations, taxation and debt refunding, social security, and the redistribution of populations here and abroad. And I believe that mathematics departments should give careful study to their responsibilities in training for this work.

We have probably all considered this problem; some but not all of us have reached successful solutions. I know of one university where beginning courses in statistics are taught in departments of sociology, psychology, educational psychology, economics, preventive medicine, agriculture, mathematics and probably some others. It is as if, in a college of engineering, the basic mathe-

matics were taught in separate departments such as physics, chemistry, electrical engineering, and mechanical engineering since each mistrusted the applicability of general mathematics to its own field. Yet, in statistics as elsewhere, it is the special power of mathematics to recognize the essence of a problem regardless of its immediate origin, to formulate quantitative definitions that really correspond to intuitive notions, to supply a tractable notation, and to analyze the relations existing between the elements. The inauguration of liaison courses in statistics involves not only the presentation of the theory common to all fields but also such interdepartmental cooperation that the results can really be utilized in the different departments. The problem is difficult, as we all know, but is likely to grow in importance. In relatively few institutions, I believe, has it been solved in a satisfactory way.

Not the least of the difficulties in connection with these liaison courses is that of securing instructors who have the ideal training already suggested for engineers, a thorough knowledge of fundamental mathematics combined with an interest in and real familiarity with some field of statistical or other application. In our graduate schools we should probably encourage or even require our students to become familiar with at least one field of applied mathematics as well as with their own line of specialization.

8. The need for general rather than limited courses. Before passing on to other matters I should like to make an observation suggested by these remarks on general or fundamental courses of statistics in contrast to courses limited to the applications in special fields. It has been remarkable and gratifying that in their training programs, the Army and Navy have sought mathematics. Not army mathematics nor navy mathematics, but fundamental mathematics. In their outlines of courses and in their lists of suggested materials of instruction, the services have very evidently appreciated that a general understanding of principles can more easily be applied to special situations than techniques learned only in connection with special problems can be transferred to unfamiliar areas. Of course, this does not mean that our teaching should deal only with general theorems and principles unrelated to any special problems that are familiar or exciting to the student. On the contrary, our courses should be as rich in varied and timely applications as we can possibly make them. Such applications provide the keenest motivation and the most satisfying means of clarification. Some of the books published during the emergency have made a genuine contribution just because they supplied a wealth of timely illustration. There have been relatively few, and most of them have been short-lived, of the meretricious publications designed to catch the reader with the bait of being, as they put it, "completely streamlined for the emergency," but written without regard to general principles or to the soundness that is the very strength of mathematics. Tendencies in this direction appeared early in the emergency, but received little encouragement from the armed services themselves. The most mathematically minded of navigators is not inclined to solve long problems in spherical trigonometry while navigating an airplane. But the Army and Navy

showed great appreciation of the value of a general mathematical background even for men in training for very specific tasks. They were glad to enlist the aid of the colleges in securing this background for their trainees, and were willing themselves to provide the special applications required in service. I believe that this is the case even in the Pre-Flight course. It is true that the content of this course is extremely elementary. That is a question of the level of instruction and the quality of the men selected, not of generality. In fact it is shockingly disturbing to find that at the college age it is necessary for the men to receive this type of instruction, which is so generally necessary that all should have received it as children.

How fortunate it is that the Army did not build its curricula too narrowly around a single limited objective, but preferred to supply fundamental training in the basic sciences is well illustrated in the recent developments in the pre-meteorological programs. What a tragic waste it would have been had all the mathematical and physical training been centered in a tight circle about immediate meteorological applications! As it was, once the rude shock of the suspension of the "A" course had been absorbed, the morale of the men came back again, for we could assure them in all good faith that their effort was not wasted and that what they had learned could be transferred with no loss whatever to many other interesting and useful tasks.

These thoughts also have their lessons for the period after the war. If the Army can go wrong in predicting the requirements for meteorological officers (and few of us would quarrel with it for overestimating rather than underestimating them), is it likely that errors will not be made in surveys of the requirements in various trades and professions? And can any counselling service be so perfect that it can foretell with certainty the best outlets for all of the individual men? To a great extent we must resist the pressure to make our courses too limited and specific. We must so imbue them with the content and spirit of real mathematics that their results can be transferred to any field of application or of thought.

9. The necessary and the unnecessary in mathematics. I should like to make another remark closely related to this one. Some of the persons engaged in planning for the education of returning veterans are insistent, and, in a measure, rightly so, that this education be practical and concentrated in form. They point out that a man in his twenties will be unable to afford a long time for continuing his studies and that he will demand a maximum of content in a minimum of time. I agree with such statements and I believe that for many of these returning soldiers we shall have to give highly concentrated courses, as I have said before. But I cannot follow these gentlemen quite so far as they would have us go. One of them, in referring to some of the wartime mathematics courses says, "Mathematics has had much of the unnecessary subject matter 'squeezed out,' and it is very likely that this condition in the future will maintain." If by this is meant that in the future we should scrutinize our subject

matter to see that it contributes to some really useful purpose, I agree. But I reserve the right to judge what is and what is not "unnecessary." And I should not consider subject matter unnecessary because it contributes to logical understanding and independence of thought instead of merely to formal technique. Nor should I consider a course too dilute merely because it covers the lapse of time necessary for some contemplation. By too much "squeezing" we may lose the life-giving juices and obtain indeed a desiccated product.

10. Placement problems. When our soldiers return, there will be great difficulty in placing them at suitable levels in our educational program. Our colleges should take steps at once to establish counselling systems adequate for the immense responsibility. It is to be hoped that these systems will not be entrusted to narrow groups of experts without a broad knowledge of the possibilities in the various fields, and that the basic sciences will be duly represented. Obviously, because of their maturity, the returning soldiers cannot be sent to high schools. But many of them will not have completed the prerequisites for the college courses they will wish to follow. We must share the responsibility of fitting these men into the educational scheme. In some wartime programs we have experienced the possibility of giving courses at elementary levels to prepare men for their collegiate work. Distasteful as it may be in some respects, I foresee the necessity of giving such courses for the returning veterans. I am not entirely sure that we should be wrong in making such courses available in peacetime to civilians. In foreign languages and in the natural sciences it is possible for college students to obtain beginning courses. In such courses it is possible for the students to proceed more rapidly than in high school and soon to press on the heels of those entering with one or two years of high school preparation. The chief danger in a similar procedure in mathematics would be in the fact that some high school advisers and administrators, knowing that these courses are available in college, would encourage pupils to omit mathematics from their high school programs. They would overlook the highly sequential nature of mathematics and not realize the necessity of beginning that subject as early in one's experience as possible. But the courses I have suggested may nevertheless be desirable solely as a means of rescue for the returning veterans and for the young civilians who, through poor advice or change of plans, may have omitted the high school mathematics necessary for their collegiate work.

11. Mathematics in general education. Up to this point I have spoken of mathematics chiefly in connection with its applications. Perhaps this is because these aspects have been stressed in the specialized training programs of the Army and Navy. There is a strong danger that these aspects are so obvious now that many educators will come to think of mathematics as useful only in its most crassly practical connections. There are some who have that opinion now. In some quarters there will be a strong reaction away from the technological toward general education. They ask, as people are asking in so many realms of life, of what avail will it be to win the war unless a better world emerges from it.

To help make this better world, we must educate our young people in the social sciences, in literature, in the fine arts, in philosophy. We must give them a general education. To all of this I agree, of course. The danger lies in the very fact that mathematics has become so conspicuous for its immediate practical necessity that its utility in general education will be forgotten.

In one instance, at least, the Army gave a sort of left-handed recognition to mathematics as a tool of general education. When asked why mathematics had to be included in a certain one of its training programs, the officer in charge replied, "Well, mathematics matures the men." Sometimes, I am sure, we have felt that it not merely matured the men but made them grow old before their time. What the officer meant, of course, was that mathematics is especially useful in the development of the general qualities that it values in its specialized personnel—qualities, perhaps, of clear, disciplined, hard-headed thought and clear expression.

The proposed federal program of subsidization of education that I have already outlined provides for general education as well as for vocational or professional training. If we are to do our share in carrying out the program, we must keep vividly in our own minds how mathematics can contribute to the objectives of a general education. And we must keep it in the minds of administrators in high schools and colleges and of the public as well. Of all the topics that I have mentioned this evening this one of the place of mathematics in general education lies as close as any to my heart. I had originally intended to make it the central theme of this address. It was my thought to recall the objectives of a college education and to examine the manner and the extent to which mathematics can definitely and in some respects uniquely contribute to attaining those objectives. The subject is too important and too large for the cursory treatment that I could give it now. With the hope that it will be discussed at length in some more suitable time and place, I shall assume that, as mathematicians, you share my feelings in this matter, and pass on to some of the immediate implications.

If it is true that we shall have many students returning from the service or coming to us from the high schools for a program of general education to which mathematics can make valuable contributions, then we must see to it that those contributions are made. Besides our extensive mathematical curricula, which secure many of the advantages of general education while training the student in his specialty, we must provide elementary terminal courses for this purpose. These courses, if less complete in the techniques necessary for continuation courses, should be rich in essential ideas. They should insist on precision and clarity of language. They should reveal mathematics as a logical structure arising from simple and natural assumptions, explicitly recognized, and leading to important and interesting results. They should indicate the necessity in any system of thought of exact definition of terms and provide examples of definitions that clarify and correspond to one's intuitive notions. We must not lead the student to believe that mathematics is vague or inconclusive by attempting

to include too great a range of material or any material too remote from his experience. The time was when it was believed that the world could be saved by education—any education. But we have learned that education built on authority alone can work as efficiently for evil as for good. If education is to contribute to our national welfare, it must conduce to independence of thought, by which the individual is able to examine for himself the assumptions and conclusions behind dogma. I am convinced that the courses I have in mind can aid materially in establishing desirable habits of scientific thinking. As you know, such courses already exist in many institutions. I believe that we should make still further provision for them.

12. Shortages of staff. As I have surveyed these tasks that lie ahead of us, I have been haunted by a certain fear. It is the fear that we shall not have the personnel properly trained to carry the great load. As I have indicated, there is every prospect that at the freshman and sophomore levels our burden of teaching, far from decreasing, will actually increase after the war. Added to this, we should have more students than ever before at the higher levels, where the war-time load is most conspicuous by its absence. Just who is to do this teaching? Some of our staff, away on war duty, will return. But many who have tasted the flesh-pots of industry will not wish to go back to the genteel poverty of college teaching. Many teachers lent to us from other fields will be called back to duty in their own departments. Many scientists have been sacrificing their own interests to the common good by teaching instead of devoting their energy to research. It would dry up the very sources of progress to ask them to continue this sacrifice indefinitely.

And what of the supply of replacements? Almost we can say there isn't any. The normal crop of young students has been mostly absorbed into the Army or Navy or other war service. Selective Service, it is true, has recognized mathematics as a field in which there exists a critical shortage. But so far as students are concerned this has resulted chiefly in the deferment of undergraduates. Most graduate students, under the pressure of war-training courses and the attractions of high salaries, have been teaching instead of studying. A full-time graduate student or a part-time assistant is a *rara avis*.

13. The training of mathematicians. Of all our future tasks, and I have referred to a number of them this evening, perhaps the most pressing, most important, and most difficult is this one of the training of mathematicians for our college and university staffs. The plant must be erected before munitions can be produced. We can begin even now to do a little work on the problem. Just the other day a former high school teacher whose previous maximum salary for a nine-month year was \$1620 was offered a college position at \$3600 for twelve months. Such opportunities are tempting. But I believe that our younger staff should be informed of the opportunities that will exist after the war, and should be encouraged to take every present chance to carry on their studies and their research. The present load of teaching must be carried on. However,

young people of special promise should be encouraged to do their teaching under conditions that offer some opportunity for study. So far as the slightest break in the wartime schedules permits, we should provide graduate courses and encourage our young teachers and students to take them. If we realize the prime necessity of this part of our program, we can do something to relieve the situation, but only in a piecemeal and inadequate way.

Under the present war training contracts, institutions are able to offer attractive salaries. It is not likely that the scholarships offered under any program of federal subsidization after the war will be large enough wholly to pay the costs of instruction. Colleges and universities will be hard put to it to offer salaries commensurate with the demand. It is not too early to begin to bring the prospective needs of education before the public with the hope of winning financial support. Even if such a campaign of public education were successful, it would not solve our problem. Money can attract mathematicians, but it cannot make them.

14. A possible source of teachers. There is only one large source of partly trained manpower. That is in the armed services themselves and especially in their present educational programs. The Army and Navy should recognize their responsibility to the servicemen after demobilization. If it is true, as I believe it is, that the scarcity of teachers will be as critical for these men as the scarcity of physicians, it would seem reasonable that the Army should immediately undertake a long-range program of training scientists similar to its program of medical training. If, for example, the best fifteen or twenty per cent of graduating pre-meteorology students were to be offered a year or more of concentrated but not too narrow training in mathematics and its applications, wonders could be done in creating a supply of men capable of giving at least elementary college instruction. This particular plan is fantastically improbable. But it might not at all be a waste of effort for the officers of the mathematical organizations to present to the A.S.T.D. the desperate necessity of some such scheme of long-range advanced instruction in mathematics.

15. The unity of the mathematical program at all levels. I should like to mention just one other thing before I close. I refer to the essential unity of the mathematical program at all levels from the junior high school, through the stages of high school, junior college, and college, to the graduate school. Research and graduate teaching vitalize college work. The colleges stimulate the teaching and set the standards for the high schools; in turn they are dependent on the high schools for both the number and quality of their students. Because of the extremely sequential nature of our subject this unity is even more important in mathematics than it is in most fields. Students who have been badly advised or badly taught in high school have special difficulty in repairing the damage to their mathematical training. I do not need to recall here the trends in attitude toward high school mathematics. It is enough to say that many educators expect and some of them desire that recent trends will be reversed and those of a

few years ago will be renewed. We must furnish what help we can to the high school program. Quite apart from what we are tempted to feel are malicious attitudes toward mathematics, we must educate ourselves to the very genuine difficulties in the high school mathematical curriculum. Some of these difficulties result from the individual differences in ability of students; others from their differences in purpose. Others arise from an ignorance of the number of paths barred to them by an early neglect of mathematics.

The Mathematical Association of America has always had close contacts with the other mathematical organizations. We have held joint meetings with both the American Mathematical Society and with the National Council of Teachers of Mathematics. I can only urge that in the highly critical period just after the war, as an organization and as individuals we participate even more actively in the problems of our high school colleagues. Only by considering the entire mathematical program as a unit can we achieve the strength that comes from unity.

In this survey of what might be called "War's Math and Aftermath," I have attempted to consider the general magnitude of the task ahead, with some of its opportunities and problems. At moments I have stated my own opinions while quite aware that many of you will not agree with them. If I have spoken dogmatically at certain points, I did so only for purposes of economy in presentation and not because I felt that I had direct inspiration in the matter. I do believe that it is time for the subject to be brought before the house, and I hope that my own speculations may give rise to profitable discussion.

CARDIOIDS ASSOCIATED WITH A CYCLIC QUADRANGLE

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1. Introduction. If a quadrangle is inscribed to a circle, then for any given line l in the plane of the quadrangle there are four orthopoles, one with respect to each of the four triangles formed by the vertices. S. Kantor has proved that the four orthopoles lie on a line which we shall call the Kantor line of the line l with respect to the cyclic quadrangle [1]. It was later proved that the quadrangle need not be cyclic but can be *any quadrangle* in the plane [2], the line in the general case being called "the orthopolar line of the line l."

Using circular coordinates, the equation of the Kantor line is

$$(1) \qquad 2az + 2s_4\bar{a}\bar{z} - a^2 - as_1 - \bar{a}^2s_4 - \bar{a}s_3 = 0$$

where s_i $(i=1, 2, 3, 4)$ are the elementary symmetric functions of the unit vectors t_1, t_2, t_3, t_4 to the vertices of the cyclic quadrangle, and $z/a+\bar{z}/\bar{a}=1$ is the conjugate equation of the given line l. The envelope of Kantor lines of a pencil of lines on the fixed point z_0 with respect to the cyclic quadrangle is [2]

$$z = \frac{z_0}{2} + \frac{s_1}{2} - \bar{z}_0 t - \frac{z_0 s_4}{2t^2}. \tag{2}$$

This is the map equation of a deltoid* whose size is one-half the distance from the center of the pencil to the circumcenter.

O. J. Ramler has derived the equation of the ellipse locus of orthopoles of the pencil of lines on a point z_0 with respect to a triangle [3]. It is

$$z = \frac{z_0}{2} + \frac{\alpha_1}{2} - \frac{1}{2}\left(\bar{z}_0 t + \frac{\alpha_3}{t}\right) \tag{3}$$

where α_i $(i=1, 2, 3)$ are the symmetric functions of t_1, t_2, and t_3. In all, four such ellipses can be obtained by taking three at a time of t_1, t_2, t_3, t_4. The clinant of this ellipse is

$$\frac{dz}{d\bar{z}} = \frac{\alpha_3(\alpha_3 - \bar{z}_0 t^2)}{\alpha_3 z_0 - t^2} \tag{4}$$

while the clinant of the deltoid is

$$\frac{dz}{d\bar{z}} = \frac{s_4}{t}. \tag{5}$$

By use of the cubic in t

$$t^3 - \frac{t_i}{\bar{z}_0} t^2 - \frac{s_4}{t_i \bar{z}_0} t + \frac{z_0 s_4}{\bar{z}_0} = 0 \qquad (i = 1, 2, 3, 4), \tag{6}$$

intersections of each of the ellipses (3) with the deltoid are obtained. Now if we set the clinants of the deltoid and ellipse equal we obtain the same cubic (6). If the coefficients of a cubic satisfy certain conditions, then all the roots of the cubic correspond to points of the unit circle, or at least one does. It can be shown that the coefficients of the cubic (6) do satisfy these conditions [4]. Thus, each ellipse is tangent to the deltoid at least once.

It is the purpose of this paper to investigate the number of the points of tangency of these curves and to show how the number depends upon the position of z_0 with reference to four cardioids associated with the quadrangle. We shall consider also some of the properties of these cardioids and their relation to the quadrangle.

2. The cubic and its discriminant. If in the cubic (6), with $i=1$, $z_0=T$, a point on the unit circle, then (6) becomes

$$(t - t_1 T)\left(t^2 - T\frac{s_4}{t_1}\right) = 0. \tag{7}$$

The roots of this equation are Tt_1, $\pm\sqrt{TS_4/t_1}$. If T equals S_4/t_1^3, two of the roots

* A deltoid is a three-cusped hypocycloid; its "size" is the radius of its inscribed circle.

coincide and the ellipse is tangent at a cusp of the deltoid.

In general, to find the region of z_0 where the ellipse will have one, two, or three points of tangency with the deltoid we first obtain the discriminant, Δ, of the cubic by the ordinary method.* Upon expanding and simplifying,

$$\Delta = \frac{s_4}{\bar{z}_0^4}\left(\frac{4s_4^2\bar{z}_0}{t_1^3} + s_4 - 27z_0^2\bar{z}_0^2 s_4 + 18z_0\bar{z}_0 s_4 + 4z_0 t_1^3\right). \tag{8}$$

3. The cardioid. Turning from the discriminant, let us find the conjugate equation of a cardioid with a cusp at $z=1$, when $t=1$. The map equation is [5, 6]

$$z = 2t - t^2. \tag{9}$$

Eliminating t between (9) and its conjugate we find the conjugate equation of this cardioid is

$$z^2\bar{z}^2 - 6z\bar{z} + 4z + 4\bar{z} - 3 = 0. \tag{10}$$

The second factor of the discriminant (8) resembles equation (10) except for a transformation constant which is readily found. If the discriminant is set equal to zero, however, we must account for the factor $s_4/\bar{z}_0^4$. Since $s_4 = t_1t_2t_3t_4$, $|s_4| = 1$. If $\bar{z}_0$ were zero, then the ellipse becomes

$$z = \frac{\alpha_1}{2} - \frac{\alpha_3}{2t} \tag{11}$$

where $\alpha_1 = t_2+t_3+t_4$ and $\alpha_3 = t_2t_3t_4$. The deltoid would be reduced to the point $s_1/2$. The locus (11), which is the nine-point circle of triangle $t_2t_3t_4$ passes through $z = s_1/2$ when $t = -\alpha_3/t_1$.

Thus we may say even here that the ellipse is "tangent" to the "deltoid" $z = s_1/2$. It is a well known theorem that the nine-point circles of the four triangles formed by four concyclic points are concurrent. The map equation $z = 1/2(t_1+t_2+t_3+t)$ shows that the circle passes through $s_1/2$ when $t = t_4$.

Now setting the discriminant Δ equal to zero, omitting the factor $s_4/\bar{z}_0^4$ and replacing z_0 by z, we have

$$z^2\bar{z}^2 - \frac{2}{3}z\bar{z} - \frac{4}{27}\frac{t_1^3}{s_4}z - \frac{4}{27}\frac{s_4}{t_1^3}\bar{z} - \frac{1}{27} = 0. \tag{12}$$

By inspection of (10) and (12) it is clear that the transformation $x = -3t_1^3z/s_4$ will send the latter into the former. Thus, the number of the points of tangency of ellipse and deltoid depends upon the position of z_0 relative to the cardioid given by equation (12).

4. The cusps and the cusp tangents. Solving the equation of the cardioid (12) with the equation of the circumcircle of the quadrangle, $z\bar{z} = 1$, results in the equation

* See Dickson's Elementary Theory of Equations.

$$z^2 - 2\frac{s_4}{t_1}z + \frac{s_4^2}{t_1^6} = 0 \tag{13}$$

which has a double root, $z = s_4/t_1^3$. Hence the cardioid is tangent to the circumcircle. If we call this vector τ, the equation of the diameter $O\tau$, O being the circumcenter, is

$$z - \frac{s_4^2}{t_1^6}\bar{z} = 0. \tag{14}$$

Substituting $z = s_4^2\bar{z}/t_1^6$ in equation (12) we may obtain the intersections of this diameter with the cardioid. For this we have the equation

$$z^4 - \frac{2}{3}\frac{s_4^2}{t_1^6}z^2 - \frac{8}{27}\frac{s_4^3}{t_1^9}z - \frac{1}{27}\frac{s_4^4}{t_1^{12}} = 0. \tag{15}$$

By synthetic division, $z = s_4/t_1^6$ is shown to be a root, and, furthermore, the vector $z = -s_4/3t_1^3$ is a triple root of (15). Hence it is the vector to the cusp of the cardioid. This point is at a distance from O equal to one-third the diameter of the circumcircle.

Now for every point z_0 in the plane of the quadrangle there are four cardioids, each associated with the deltoid and one of the four ellipses mentioned in Section 2. The cusps therefore lie on a circle concentric with the circumcircle, the latter having a radius three times as large as the former. Furthermore, each cardioid is tangent to the circumcircle at the points

$$\frac{s_4}{t_1^3}, \quad \frac{s_4}{t_2^3}, \quad \frac{s_4}{t_3^3}, \quad \frac{s_4}{t_4^3}.$$

Since all four cusps lie on the circle $|z| = 1/3$, then when $|z_0| < 1/3$ the point z_0 lies inside each of the four cardioids. To obtain three distinct points of tangency we let $|z_0| \geqq 1$ since then the point lies outside the cardioid which is tangent to the circumcircle.

References

1. Kantor, S., Ueber eine Gattung merkwuerdiger Geraden und Punkte bei vollstaendigen n-Ecken auf dem Kreise, Sitzungsberichte der Mathematisch-Naturwissenschaftlichen Classe, Staatsdruckerei, Wien, 78, 2–3, 1828, p. 193.

2. McBrien, V. O., On some systems of orthopolar and Kantor lines of lines referred to a quadrangle, The Catholic University of America Press, Washington, D. C., 1942.

3. Ramler, O. J., The orthopole loci of some one-parameter systems of lines referred to a fixed triangle, this MONTHLY, 27, 1930, p. 130.

4. Ramler, O. J., Quadratic and Cubic Equations, this MONTHLY, 50, 1943, pp. 507–509.

5. Morley, Frank and Morley, F. S., Inversive Geometry, Ginn and Co., London, 1933, p. 239.

6. Wagner, P. S., An extension to Clifford's Chain, Amer. Journal of Mathematics, 51, 1929, p. 473.

AFFINE GEOMETRY OF CONVEX QUARTICS

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1. Introduction. Let C be a curve defined by the equation $f(x, y)=1$, where $f(x, y)$ is a homogeneous polynomial of degree 4. Let P be a parallelogram with one vertex at the origin and the other three vertices on C. C and P will retain these meanings throughout this article. If C is convex, the area of P is a single valued continuous function of the inclination of one of its sides with the x-axis. The following note uses elementary propositions in affine geometry* to locate those parallelograms P of maximum or minimum area for convex quartics C. Mordell† has recently solved the problem by other methods in the case where the equation of C is $Ax^4+By^4=1$.

2. Preliminary area theorems. We shall now derive some theorems which will be used in the proof of our main result, which appears in Theorem 5.

THEOREM 1. *Let the points* $(1, 0)$, $(\pm 1/2, \sqrt{3}/2)$ *be on the curve* C. *Let* $m_1(=(1/r(dr/d\theta))$ *be the cotangent of the angle from the radius vector joining the points* $(0, 0)$ *and* $(1, 0)$, *to the tangent to the curve at the latter point. Let* m_2, m_3 *be analogously defined for the points* $(\pm 1/2, \sqrt{3}/2)$. *If the points,* $(0, 0)$, $(1, 0)$, $(\pm 1/2, \sqrt{3}/2)$ *be vertices giving a maximum or minimum area for* P, *then* $(m_1+m_2+m_3)/3+m_1m_2m_3=0$.

Proof. Let (r_1, θ_1), (r_2, θ_2), (r_3, θ_3), $\theta_1<\theta_2<\theta_3$, be the polar coordinates of the three points of P which lie on C. Clearly,

$$\text{Area of } P=r_2r_1\sin(\theta_2-\theta_1)=r_3r_1\sin(\theta_3-\theta_1)=r_3r_2\sin(\theta_3-\theta_2).$$

If P has a maximum or a minimum area, we have, by logarithmic differentiation,

$$\frac{1}{r_2}\frac{dr_2}{d\theta_2}d\theta_2+\frac{1}{r_1}\frac{dr_1}{d\theta_1}d\theta_1+\cot(\theta_2-\theta_1)(d\theta_2-d\theta_1)=0,$$

$$\frac{1}{r_3}\frac{dr_3}{d\theta_3}d\theta_3+\frac{1}{r_1}\frac{dr_1}{d\theta_1}d\theta_1+\cot(\theta_3-\theta_1)(d\theta_3-d\theta_1)=0,$$

$$\frac{1}{r_3}\frac{dr_3}{d\theta_3}d\theta_3+\frac{1}{r_2}\frac{dr_2}{d\theta_2}d\theta_2+\cot(\theta_3-\theta_2)(d\theta_3-d\theta_2)=0.$$

Let us specialize P so that it becomes the parallelogram $(0, 0)$, $(1, 0)$, $(\pm 1/2, \sqrt{3}/2)$. By using the fact that this parallelogram is composed of two equilateral triangles and by replacing the derivatives by their equivalent cotangent expressions, we have

* W. C. Graustein, Introduction to Higher Geometry, New York, 1930, pp. 179–186.

† L. J. Mordell, Lattice points in the region $|Ax^4+By^4|\leq 1$, Journal London Math. Soc., 16, 1941, pp. 152–156. The reader is also referred to this article for an immediate application of our problem to lattice theory.

$$\left(m_1 - \frac{\sqrt{3}}{3}\right)d\theta_1 + \left(m_2 + \frac{\sqrt{3}}{3}\right)d\theta_2 = 0,$$
$$\left(m_1 + \frac{\sqrt{3}}{3}\right)d\theta_1 + \left(m_3 - \frac{\sqrt{3}}{3}\right)d\theta_3 = 0,$$
$$\left(m_2 - \frac{\sqrt{3}}{3}\right)d\theta_2 + \left(m_3 + \frac{\sqrt{3}}{3}\right)d\theta_3 = 0.$$

We eliminate the differentials and obtain

$$\begin{vmatrix} m_1 - \frac{\sqrt{3}}{3} & m_2 + \frac{\sqrt{3}}{3} & 0 \\ m_1 + \frac{\sqrt{3}}{3} & 0 & m_3 - \frac{\sqrt{3}}{3} \\ 0 & m_2 - \frac{\sqrt{3}}{3} & m_3 + \frac{\sqrt{3}}{3} \end{vmatrix} = 0.$$

After expanding this determinant and dividing by 2, we have $m_1m_2m_3 + (m_1+m_2+m_3)/3 = 0$. Thus the theorem is proved.

We shall now use the concept of conjugate diameters. The definition of a pair of these diameters, as given in the theory of conics, may be extended immediately to any convex oval.

THEOREM 2. *If P has a maximum or a minimum area, then either one of the sides of P or one of its diagonals lies along one of a pair of conjugate diameters of C.*

Proof. If we subject a curve to an affine transformation, conjugate diameters are mapped into conjugate diameters and areas remain unchanged but for a constant multiple. It will, therefore, be sufficient to prove the theorem for C after it has been subjected to an arbitrary affine transformation. Accordingly, let C' be the affine map of C in which P is mapped into the parallelogram $(0, 0)$, $(1, 0)$, $(\pm 1/2, \sqrt{3}/2)$.

C' being a fourth order curve with equation $f(x, y) = 1$, its polar equation may be written in the form

$$\frac{1}{r^4} = A_0 + A_2 \cos 2\theta + A_4 \cos 4\theta + B_2 \sin 2\theta + B_4 \sin 4\theta.$$

If we substitute the coordinates of the points $(1, \pi/3)$, $(1, 2\pi/3)$ into the above equation, the two relationships show that $B_2 = B_4$.

By differentiating the equation of C', we have

$$-\frac{4}{r^5}\cdot\frac{dr}{d\theta} = -2A_2 \sin 2\theta - 4A_4 \sin 4\theta + 2B_2 \cos 2\theta + 4B_4 \cos 4\theta.$$

By substituting the three sets of coordinates, $(1, 0)$, $(1, \pi/3)$, $(1, 2\pi/3)$, into this relation we get for the three cotangents of Theorem 1:

$$\begin{aligned} -4m_1 &= 6B_2, \\ -4m_2 &= -\sqrt{3}A_2 + 2\sqrt{3}A_4 - 3B_2, \\ -4m_3 &= \sqrt{3}A_2 - 2\sqrt{3}A_4 - 3B_2. \end{aligned}$$

We see that $m_1+m_2+m_3=0$. Hence, by Theorem 1, at least one of the cotangents m_1, m_2, m_3 vanishes. Without restriction in generality we may assume this to be m_1. Otherwise, by a rotation about the origin, we may bring the point with the vanishing cotangent into the polar axis. Now consider the new parallelogram, $(0, 0)$, $(1, 0)$, $(\pm 1/2, \sqrt{3}/2)$. This would have the same cotangents, in different order, and would likewise have a maximum or a minimum area. Clearly, in every event, a side or diagonal of the original parallelogram now lies along the polar axis. Because of the vanishing cotangent, we have $B_2=0$ and, consequently, $B_4=0$. Hence the equation of C' has the form

$$\frac{1}{r^4} = A_0 + A_2 \cos 2\theta + A_4 \cos 4\theta.$$

This equation shows the curve to be symmetric with respect to the lines $\theta=0$ and $\theta=\pi/2$. Hence these two lines are a pair of conjugate diameters and the theorem is established.

3. Some theorems in affine geometry. The following theorems which we use in the proof of Theorem 5, may be of some interest on their own account.

Let $f(x, y)=1$ be the equation of a fourth order convex curve C. Assuming the first member to be homogeneous, we have a real factorization

$$f(x, y) \equiv (ax^2 + hxy + by^2)(cx^2 + kxy + dy^2).$$

We define this factorization to be ***normalized*** if the areas of the two ellipses, $ax^2+hxy+by^2=1$, $cx^2+kxy+dy^2=1$, are equal. A normalized factorization is unique.

Except in the trivial case in which they coincide, the two component ellipses of a normalized factorization intersect in the four vertices of a parallelogram. We define this parallelogram to be the *critical parallelogram* of the curve C. An affine transformation maps the critical parallelogram into the critical parallelogram of the image of C.

THEOREM 3. *Except in the case in which it degenerates into an ellipse, a convex fourth order curve C has exactly two pairs of conjugate diameters: a pair along the diagonals of its critical parallelogram and a pair parallel to the sides of this parallelogram.*

Proof. Let $f(x, y)=1$ be the equation of a fourth order convex curve C. The two factors of the normalized factorization of $f(x, y)$ are distinct if we exclude

the case in which the curve degenerates into an ellipse. We may map the two component ellipses of this normalized factorization by a suitable affine transformation so that they become congruent to each other and with their major and minor axes along the coordinate axes. The normalized factorization of the resulting equation takes the form

$$(ax^2 + by^2)(bx^2 + ay^2) = 1. \tag{1}$$

The critical parallelogram is now a square with sides parallel to the coordinate axes and with the lines $y = \pm x$ as diagonals. Clearly the curve is symmetric with respect to the lines $y=0$, $x=0$, $y=\pm x$. Consequently the line pairs, $(y=0, x=0)$, $(y=\pm x)$, are conjugate diameters.

We now prove that these are the only conjugate diameters. Let us map any arbitrary pair of conjugate diameters into the coordinate axes by an affine transformation and consider the normalized factorization,

$$(ax^2 + hxy + by^2)(cx^2 + kxy + dy^2) = 1,$$

of the resulting curve. The condition for the axes to be conjugate diameters is that the coefficients of the terms involving x^3y and xy^3 vanish; *i.e.*,

$$ak + ch = 0,$$
$$bk + dh = 0.$$

We distinguish two cases: 1, $ad-bc=0$; 2, $h=k=0$.

As none of a, b, c, d may vanish without introducing a real linear factor into $f(x, y)$, in which case the curve would not be convex, we have, in Case 1,

$$\frac{a}{c} = \frac{b}{d} = r \text{ (say).}$$

We deduce $a=cr$, $b=dr$ and $h=-kr$. The factorization now becomes

$$(crx^2 - krxy + dry^2)(cx^2 + kxy + dy^2) = 1.$$

By hypothesis this factorization is normalized, the component ellipses thus having equal area; therefore $r=1$. It follows easily that these ellipses intersect along the coordinate axes and so the conjugate diameters lie along the diagonals of the critical parallelogram of the curve.

In Case 2 the axes of the ellipses lie along the coordinate axes and it follows that they, and therefore the conjugate diameters, are parallel to the sides of the critical parallelogram of the curve. Thus we have located all the conjugate diameters and the theorem is proved.

THEOREM 4. *By a suitable affine transformation any finite fourth order curve, $f(x, y)=1$, with homogeneous first member, may be mapped into a curve whose equation is*

$$x^4 + kx^2y^2 + y^4 = 1, \qquad 2 \leqq k.$$

Those curves with $k \leqq 6$ are convex, those with $k > 6$ concave.

Proof. In equation (1) let $k=a^2+b^2$. If we impose the additional condition that each of the ellipses (of equation (1)) have area π, it follows that $ab=1$. Equation (1) may now be written

$$x^4 + kx^2y^2 + y^4 = 1.$$

As $a^2+b^2 \geqq 2ab=2$, we have $k \geqq 2$. Thus the first part of the theorem is established.

The above equation in polar coordinates becomes

$$\frac{1}{r^4} = \cos^4\theta + \sin^4\theta + k\sin^2\theta\cos^2\theta.$$

This is equivalent to

$$\frac{1}{r^4} = 1 + \frac{k-2}{4}\sin^2 2\theta.$$

Let $v=(k-2)/4$. To prove the remainder of the theorem it is sufficient to show the curves are convex if, and only if, $0 \leqq v \leqq 1$.

Let $h>0$. The area of the sector bounded by the curve and the radius vectors at θ and $\theta+h$ is

$$\frac{1}{2}\int_0^h \frac{dt}{\sqrt{1+v\sin^2 2(\theta+t)}}.$$

The curve will be convex if this area is not less than that of the triangle formed by joining the ends of the radius vectors for arbitrarily small h; *i.e.*, if

$$(2) \qquad \int_0^h \frac{dt}{\sqrt{1+v\sin^2 2(\theta+t)}} \geqq \frac{\sin h}{\sqrt[4]{(1+v\sin^2 2\theta)(1+v\sin^2 2(\theta+h))}}.$$

A straightforward calculation gives the Taylor expansion

$$\int_0^h \frac{dt}{\sqrt{1+v\sin^2 2(\theta+t)}} - \frac{\sin h}{\sqrt[4]{(1+v\sin^2 2\theta)(1+v\sin^2 2(\theta+h))}}$$
$$= \frac{h^3}{6}\,\frac{2v+1-(2v+v^2)\sin^2 2\theta}{(1+v\sin^2 2\theta)^{5/2}} \cdots.$$

As this is clearly positive for $v<1$, the inequality (2) holds. Hence the convexity of the curves is established for $0 \leqq v < 1$; *i.e.*, for $2 \leqq k < 6$. The inequality also proves the convexity of the curve for which $k=6$, except at the points for which $\sin^2 2\theta = 1$. As we may approach these points by convex curves with $k<6$, lying outside of the curve for which $k=6$, this curve cannot be concave at these points and so must be convex throughout. For $v>1$ the coefficient of h^3 is negative for $\sin^2 2\theta=1$. Thus these curves are concave. The theorem is now completely proved.

4. The main result. We are now in a position to establish the result indicated in the introduction.

THEOREM 5. *For a convex curve C of fourth order, the maximum value of the area of the parallelogram P is attained when one of its sides or diagonals coincides with a diagonal of the critical parallelogram of the curve, and the minimum value is attained when one of the sides or diagonals of P is parallel to a side of the critical parallelogram.*

Proof. It follows from Theorem 4 that it is sufficient to prove this for the curves,

$$x^4 + kx^2y^2 + y^4 = 1, \qquad 2 \leqq k \leqq 6.$$

The critical parallelogram of each of these curves is a square with sides parallel to the coordinate axes. Because of the symmetry of the curves about the lines $y=0$, $x=0$, $y=\pm x$, maxima and minima occur when a side or diagonal of P is along one of these lines. Theorems 2 and 3 show that these exhaust every possibility. It remains to distinguish maxima and minima. Let us calculate the area A_1 of the parallelogram P with side along the x-axis. An elementary calculation shows the square of the altitude to be $(\sqrt{k^2+60}-k)/8$. As the base of the parallelogram is 1, we obtain $A_1^2=(\sqrt{k^2+60}-k)/8$. If A_2 denote the area of the parallelogram P with side along the line $y=x$, a similar calculation, after rotating the coordinate axes through 45°, gives

$$A_2^2 = \frac{k-6+4\sqrt{k^2+3k+6}}{(2+k)^2}.$$

For $k=2$ the curve is a circle and the two values A_1, A_2 must be equal, which fact is easily verified. However they can never be equal for any other value of k, for then the maximum and minimum values would be the same; consequently the area of P would be a constant and so a maximum everywhere. It would then follow from Theorem 2 that every diameter would be a conjugate diameter in contradiction to Theorem 3.

However $A_1^2-A_2^2$ is a continuous function of k and if it changed signs within the interval $2<k\leqq 6$, a value of k would exist in this interval for which $A_1=A_2$, in contradiction to what we have proved above. Consequently one of these is a maximum, the other a minimum for all values of k. By actual calculation we find, for $k=4$,

$$A_1^2 = .58\cdots, \qquad A_2^2 = .59\cdots.$$

Hence A_2 is a maximum, A_1 a minimum for all values of k, and the problem is completely solved.

DISCUSSIONS AND NOTES

EDITED BY MARIE J. WEISS, Sophie Newcomb College, New Orleans, La.

The Department of Discussions and Notes is open to all forms of activity in college mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

THE EIGHT-POINT CIRCLE AND THE NINE-POINT CIRCLE

LOUIS BRAND, University of Cincinnati

It has apparently never been noted that the famous theorem on the *nine-point circle* is merely a special case of a very simple theorem on the *eight-point circle;* namely

When the diagonals of a plane quadrilateral are perpendicular, the mid-points of its sides, and the feet of the perpendiculars dropped from the mid-points on the opposite sides, all lie on a circle described about the mean center of the vertices.

Proof. In the quadrilateral $ABCD$ let P, Q, R, S denote the mid-points of the sides AB, BC, CD, DA (Fig. 1). Then $PQRS$ is a parallelogram whose sides are parallel to the diagonals AC, BD of the quadrilateral. When AC and BD

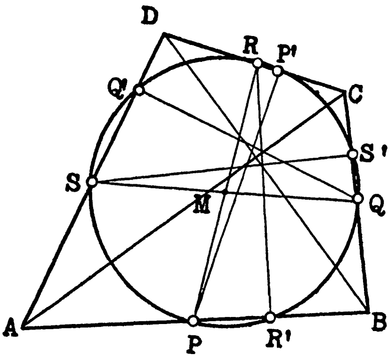

FIG. 1

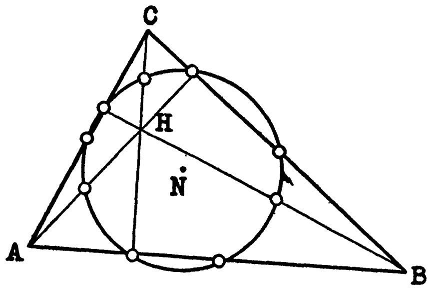

FIG. 2

are perpendicular, $PQRS$ is a rectangle; its diagonals PR, QS bisect each other at M, the mean-center of the points A, B, C, D (their centroid when each point has the weight 1). Therefore P, Q, R, S lie on a circle centered at M; and if perpendiculars from P, Q, R, S are dropped upon the sides opposite, their feet P', Q', R', S' will also lie on this circle. Thus the theorem on the eight-point circle is proved.

In the figure formed by a triangle ABC and its three altitudes meeting at the orthocenter H (Fig. 2), the three quadrilaterals $ABCHA$, $BCAHB$, $CABHC$ all have perpendicular diagonals. Their three eight-point circles (six-point circles in this case) coincide. The eighteen (3×6) points on this circle reduce to nine

as each point appears twice. The common eight-point circle of the quadrilaterals, about the mean center of A, B, C, H, is the nine-point circle of the triangle.

THEOREM. *For any triangle ABC whose orthocenter is H, a circle whose center is the mean center of A, B, C, H, passes through the nine points: the mid-points of the sides, the feet of its altitudes, and the mid-points of the segments joining H to the vertices.*

The center N of the nine-point circle is the centroid of A, B, C, H when each point has the weight 1. Now if G is the centroid of A, B, C, namely the point of intersection of the medians of the triangle ABC, we may also say that N is the centroid of G with the weight 3 and H with the weight 1. Therefore N divides the segment HG in the ratio 3/1.

NOTES ON THE GEOMETRY OF THE TRIANGLE

O. S. ADAMS, U. S. Coast and Geodetic Survey

If we use conjugate coordinates to denote the vertices of the triangle by turns t_1, t_2, t_3 of the circumcircle, the equations of the sides become

$$x + t_1t_2\bar{x} = t_1 + t_2,$$
$$x + t_1t_3\bar{x} = t_1 + t_3,$$
$$x + t_2t_3\bar{x} = t_2 + t_3.$$

The t's are complex numbers with absolute value of unity. Denote their elementary symmetric functions by S_1, S_2, S_3. If we add the three equations of the sides of the triangle we get

$$3x + S_2\bar{x} = 2S_1$$

in which x is the coordinate of a definite point in the triangle. By taking the conjugate of this equation, remembering that $\bar{S}_2 = S_1/S_3$ and $\bar{S}_1 = S_2/S_3$, we find

$$3S_3\bar{x} + S_1x = 2S_2.$$

By eliminating $\bar{x}$ from these two conjugate equations, we get

$$x = \frac{6S_1S_3 - 2S_2^2}{9S_3 - S_1S_2}.$$

This is the symmedian point of the triangle, which is usually denoted by K. At once the question arises, "why does this procedure give the symmedian point of the triangle?"

If we have an equation of a line in the self conjugate form, we can substitute in it the coordinate of a point not on the line. What does this procedure yield? If we take a point p in the plane of the triangle but not on a given side, such as $x + t_1t_2\bar{x} = t_1 + t_2$, we find that the image of p with respect to this side is

$$x = t_1 + t_2 - t_1t_2\bar{p}.$$

The foot of the perpendicular from p on this side is then equal to half the sum of the point and its image, or

$$x_3 = \tfrac{1}{2}(p + t_1 + t_2 - t_1t_2\bar{p}).$$

The directed length of this perpendicular from p is given by

$$x_3 - p = \tfrac{1}{2}(-p + t_1 + t_2 - t_1t_2\bar{p}).$$

Similarly we have

$$x_2 - p = \tfrac{1}{2}(-p + t_1 + t_3 - t_1t_3\bar{p}),$$
$$x_1 - p = \tfrac{1}{2}(-p + t_2 + t_3 - t_2t_3\bar{p}).$$

We see from these equations that the substitution of p, a point not on any side of the triangle, in the equations of the sides gives twice the directed lengths of the perpendiculars on the sides. If we take the sum of these three vectors and equate it to zero, we see that we have the same expression for p as was given for x by addition of the equations of the sides. This shows that the resultant of two of these vectors is equal in length to the third but in opposite direction. In other words, the sum of the three vectors gives a closed triangle returning to the starting point.

The symmedian point is the only point in the triangle that possesses this property with respect to the sides. This property is at once evident from the well-known fact that the symmedian point is the median point of its own pedal triangle. The feet of the perpendiculars from K upon the sides are

$$x_1 = \tfrac{1}{2}(K + S_1 - t_1 - t_2t_3\overline{K}),$$
$$x_2 = \tfrac{1}{2}(K + S_1 - t_2 - t_1t_3\overline{K}),$$
$$x_3 = \tfrac{1}{2}(K + S_1 - t_3 - t_1t_2\overline{K}).$$

The median point is equal to $1/3$ the sum of the vertices, or

$$\frac{x_1 + x_2 + x_3}{3} = \frac{1}{6}(3K + 2S_1 - S_2\overline{K}) = K,$$

since $2S_1 - S_2\overline{K} = 3K$, the known relation between the K's and S's of the triangle and a relation that can easily be verified by substitution of the values of K and $\overline{K}$ in terms of the S's. Since the median point is located at $2/3$ of the distance from the vertex to the opposite side on the median and since the distance from the median point to the side is equal to half the resultant of the vectors from the median point to the other two vertices, it follows that the vector sum of the three is zero, but these vectors in the pedal triangle of K are the perpendiculars from K upon the sides of the triangle.

When one is dealing with the incenters it is better to make use of the coordinates described by H. A. DoBell in the February, 1932 number of this MONTHLY. In this system we have

$$t_1 = \frac{T_2T_3}{T_1}, \quad t_2 = \frac{T_1T_3}{T_2}, \quad \text{and} \quad t_3 = \frac{T_1T_2}{T_3}.$$

The T's are roots of the equation

$$T^3 - \Sigma_1 T^2 + \Sigma_2 T - \Sigma_3 = 0.$$

The incenter then becomes

$$I = -\Sigma_1.$$

The relations between the sigmas and the S's then become

$$S_1 = \frac{\Sigma_2^2 - 2\Sigma_1\Sigma_3}{\Sigma_3},$$
$$S_2 = \Sigma_1^2 - 2\Sigma_2,$$
$$S_3 = \Sigma_3.$$

It is known that the incenter is the Nagel point of the pedal triangle of the circumcenter. The circumcircle of this pedal triangle is the nine point circle, or

$$x = \frac{S_1 - t}{2}.$$

To determine the Nagel point of the base circle we must employ the transformation

$$x' = S_1 - 2x = \frac{\Sigma_2^2 - 2\Sigma_1\Sigma_3}{\Sigma_3} - 2x$$

in which x now becomes $-\Sigma_1$

$$x' = \frac{\Sigma_2^2 - 2\Sigma_1\Sigma_3}{\Sigma_3} + 2\Sigma_1 = \frac{\Sigma_2^2}{\Sigma_3}.$$

The Nagel point of the base circle is thus shown to be equal to Σ_2^2/Σ_3 and this value is easily derived from other geometrical relations in the triangle. The Nagel point is the point of intersection of the lines from the vertices to the points where the opposite sides are tangent to their respective escribed circles. All of the other special points of the triangle can be determined in terms of the complex coordinates by proper manipulations or transformations.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND H. S. M. COXETER

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to H. S. M. Coxeter, 24 Strathearn Boulevard, Toronto 10, Canada.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 606. *Proposed by H. T. R. Aude, Colgate University*

An enemy ship A is proceeding on a course directly east at its top speed of 25 miles per hour when it is spied by a more powerful ship B of a speed of only 20 miles per hour. At the instant of discovery the ship B is 10 miles directly south of A. On account of the shore line the enemy ship A must continue its course, while the course of B is altered to $N\theta E$ to bring A within range. If the effective range is not more than 7 miles, find the course of B that will bring about the least distance AB, and the interval of time (while following this course) during which A will be within range.

E 607. *Proposed by V. Thébault, San Sebastián, Spain*

Consider an orthocentric tetrahedron $ABCD$, of orthocenter H. Let O, A', B', C', D' be the circumcenters of the tetrahedra $ABCD$, $BCDH$, $CDAH$, $DABH$, $ABCH$. Prove that the tetrahedra $ABCD$ and $A'B'C'D'$ are homothetic from a center which divides OH in the ratio 3:2. Show also that the lines AA', BB', CC', DD' pass through the centers of gravity of the respective tetrahedra $BCDH$, $CDAH$, $DABH$, $ABCH$.

E 608. *Proposed by C. H. Wolfe, Lakeside High School, Ohio*

A certain three-digit number yields a quotient of 26 when divided by the sum of its digits. If the digits are reversed, the quotient is 48. What is the smallest three-digit number for which this is possible?

E 609. *Proposed by Frank Hawthorne, Allegheny College*

Show that the diagonals of three faces of a parallelepiped, drawn from the same vertex and prolonged half their length, determine three points which are coplanar with the opposite vertex.

E 610. *Proposed by Howard Eves, Syracuse University*

(*a*) Show that all closed curves of the same constant diameter, d, have the same perimeter, πd.

(*b*) What is the least area that a closed curve of constant diameter d may have?

(Such a "curve of constant width" touches two parallel lines, distant d apart, drawn in any direction. See, *e.g.*, H. Steinhaus, *Mathematical Snapshots*, 1938, p. 51.)

SOLUTIONS

Conflicting Claims

E 566 [1943, 254]. *Proposed by Michael Wilensky, Cincinnati, Ohio*

Suppose there is a principle according to which a claimant gets a share of the contestable thing, proportionate to his claim; so that when one of two claimants claims the whole, while the other claims only half of it, the former gets three quarters (viz., the incontestable half, and half the contestable half) and the latter gets one quarter (which is half his claim). Find a formula for the share of each of n claimants, when the kth claimant claims $1/k$ of the entity $(k=1, 2, \cdots, n)$.

Solution by H. D. Larsen, University of New Mexico. Denote the rth claimant by C_r and his proportionate share by S_r. Then since C_n's claim of $1/n$ of the entity is contested by each of the other claimants,

$$S_n = \frac{1}{n}\cdot\frac{1}{n} = \frac{1}{n^2}\cdot$$

Further, that part of the whole claimed by C_r but not by C_{r+1} is equal to

$$\frac{1}{r} - \frac{1}{r+1}\cdot$$

Inasmuch as this part is contested by r persons, C_r's share in it is

$$\frac{1}{r}\left(\frac{1}{r} - \frac{1}{r+1}\right) = \frac{1}{r^2(r+1)}\cdot$$

But this is precisely the difference between S_r and S_{r+1}; that is,

$$S_r = S_{r+1} + \frac{1}{r^2(r+1)}\cdot$$

This recursion formula may be used to calculate the different shares, starting with $S_n = 1/n^2$.

Also solved by Frank Hawthorne, Monte Dernham, and the proposer.

Editorial Note. Larsen's recursion formula may be written in the form

$$S_r + \frac{1}{r} - \frac{1}{r^2} = S_{r+1} + \frac{1}{r+1},$$

whence

$$S_r + \frac{1}{r} - \sum_r^n \frac{1}{q^2} = S_{r+1} + \frac{1}{r+1} - \sum_{r+1}^n \frac{1}{q^2} = \cdots = S_n + \frac{1}{n} - \frac{1}{n^2} = \frac{1}{n}$$

and, finally,

$$S_r = \sum_r^n \frac{1}{q^2} - \frac{1}{r} + \frac{1}{n}.$$

A Generalization of E 467

E 568 [1943, 260]. *Proposed by P. D. Thomas, U. S. Coast and Geodetic Survey, Lucedale, Miss.*

In a given triangle show that the radical axes of the circumcircle with the respective circles whose diameters are any three concurrent Cevians meet the corresponding sides in three collinear points.

Solution by Howard Eves, Syracuse University. Let ABC be the given triangle, O the circumcenter, R the circumradius, and D the midpoint of the Cevian AA' through A. Through A draw AA'' perpendicular to OD, cutting BC in A''. Then, clearly, AA'' is the radical axis of the circumcircle and the circle on AA' as diameter.

We proceed now by vector analytic geometry. Take the origin at O, and let the respective position vectors of A, B, C, A', B', C', A'', B'', C'' be u, v, w, u', v', w', u'', v'', w''. Then we have

$$\begin{gathered} u\cdot u = v\cdot v = w\cdot w = R^2, \\ v\cdot w = R^2\cos 2A, \qquad w\cdot u = R^2 \cos 2B, \qquad u\cdot v = R^2\cos 2C. \end{gathered} \tag{1}$$

Also, by Ceva's Theorem, there exist constants k_1, k_2, k_3 such that

$$u' = \frac{k_3v + k_2w}{k_3 + k_2}, \qquad v' = \frac{k_1w + k_3u}{k_1 + k_3}, \qquad w' = \frac{k_2u + k_1v}{k_2 + k_1}.$$

Thus the position vector of D is

$$\frac{(k_3 + k_2)u + k_3v + k_2w}{2(k_3 + k_2)};$$

and since AA'' is perpendicular to OD,

$$(u'' - u)\cdot\{(k_3 + k_2)u + k_3v + k_2w\} = 0. \tag{2}$$

But since A'' is on BC, $u'' = (m_3v - m_2w)/(m_3 - m_2)$. Substituting this expression for u'' in (2), we find, by means of the relations (1),

$$\frac{m_2}{m_3}=\frac{k_2(1-\cos 2A+\cos 2B-\cos 2C)}{k_3(1-\cos 2A-\cos 2B+\cos 2C)}.$$

We may therefore take

$$m_2 = k_2(1-\cos 2A+\cos 2B-\cos 2C),$$
$$m_3 = k_3(1-\cos 2A-\cos 2B+\cos 2C).$$

Similarly, if we set

$$m_1 = k_1(1+\cos 2A-\cos 2B-\cos 2C),$$

we find $v''=(m_1w-m_3u)/(m_1-m_3)$ and $w''=(m_2u-m_1v)/(m_2-m_1)$. Hence, by Menelaus' Theorem, A'', B'', C'' are collinear, and the theorem is proved.

Also solved by the proposer.

The Butterfly Theorem

E 571 [1943, 326]. *Proposed by Sol Mitchell, University of Toronto*

Let O be the midpoint of a chord AB of a circle, and let CD, EF be any two other chords through O. Prove synthetically that CE and DF meet AB in points equidistant from O.

I. *Solution by Joseph Rosenbaum, Bloomfield, Conn.* Let $D'F'$ be the image of DF by reflection in the diameter through O, and let G, H be the points where CE, DF meet AB. In the notation of Johnson's "directed angles," taking the positive sense to be such that $\measuredangle EOB$ is acute, we have

$$\measuredangle BOF' = \measuredangle FOA = \measuredangle EOB = \tfrac{1}{2}\measuredangle EOF' = \measuredangle ECF',$$

i.e.,

$$\measuredangle GOF' = \measuredangle GCF'.$$

Thus O, G, C, F' are concyclic. Hence

$$\measuredangle OF'G = \measuredangle OCG = \measuredangle DCE = \measuredangle DFO = \measuredangle OF'D'.$$

Therefore G lies on $D'F'$, and is the reflected image of H.

II. *Solution by W. E. Buker, Pittsburgh Public Schools.* Let CE and DF intersect AB at G and H. respectively. Through H draw a line parallel to CE, intersecting CD and EF at K and L. By similar triangles.

$$\frac{LH}{EG}=\frac{OH}{OG}=\frac{HK}{GC} \quad\text{and}\quad \frac{LH}{FH}=\frac{DH}{KH}.$$

Therefore

$$\frac{OH^2}{OG^2}=\frac{LH\cdot HK}{EG\cdot GC}=\frac{FH\cdot HD}{EG\cdot GC}=\frac{AH\cdot HB}{AG\cdot GB}=\frac{OA^2-OH^2}{OB^2-OG^2}=\frac{OA^2}{OB^2}=1.$$

III. *Solution by Robert Steinberg, University of Toronto.* As this is essentially a theorem of affine geometry, we can replace the circle by a conic. Let CE and DF meet AB at G and H, and let I be the point at infinity on AB. Now, by Steiner's Theorem we have

$$D(AFCB) \overline{\wedge} E(AFCB).$$

Noting the points at which these two pencils meet the line AB, we have

$$AHOB \overline{\wedge} AOGB \overline{\wedge} BGOA.$$

Since the projectivity $AHOB\overline{\wedge}BGOA$ interchanges A and B, it must be an involution. Since it has one double point, O, it must have another which is the harmonic conjugate of O with respect to A and B, namely I. Since H and G are another pair of this involution, these likewise must be harmonic conjugates with respect to O and I. Hence O is the midpoint of GH.

IV. *Solution by E. P. Starke, Rutgers University.* The proposed theorem need not be restricted to a circle: it is true of any conic. Let CE meet DF in P, and let CF meet DE in Q. Then PQ is the polar of O with respect to the conic. But any line through O intersects the conic and PQ in points which, with O, form a harmonic range; therefore A, O, B, and the intersection I, with PQ, are harmonic. But $OA = OB$: thus I must be infinitely distant, *i.e.*, AB and PQ are parallel. Furthermore, in the complete quadrangle, the lines PF, PO, PC, PQ are a harmonic pencil. It intersects AB in points H, O, G, I, which are thus a harmonic range; but I is at infinity, so $GO = HO$, as required.

V. *Solution by J. H. Butchart, Grinnell College.* This is an example of Desargues' Theorem concerning conics passing through the vertices of a complete quadrangle. O is a double point of the involution determined by $CDEF$ on AB. Since A, B form a pair of corresponding points in this hyperbolic involution, the other double point is at infinity, and the points G, H (where CE, DF meet AB) are likewise equidistant from O.

Also solved by Mannis Charosh (*School Science and Mathematics*, October, 1941, p. 684), Howard Eves (in two ways, like IV and V), James Jenkins (in two ways, like II and IV), R. A. Johnson (*Modern Geometry*, p. 78), L. M. Kelly (like IV), and R. K. Morley (like II).

Consecutive Squares

E 572 [1943, 326]. *Proposed by V. Thébault, San Sebastián, Spain*

What scales of notation admit perfect squares ab, bb, and cd, where a, b, c, d are consecutive integers?

Solution by Free Jamison, U. S. Navy Air Navigation School. For any positive integer a, we may use $4a+3$ as radix. Then

$$ab \quad \text{is} \qquad a(4a + 3) + a + 1 = (2a + 1)^2,$$

$$bb \quad \text{is} \quad (a+1)(4a+3)+a+1=(2a+2)^2,$$

and

$$cd \quad \text{is} \quad (a+2)(4a+3)+a+3=(2a+3)^2.$$

Also solved by D. H. Browne, W. E. Buker, Walter Penney, E. P. Starke, Jerzy Szmojsz, Maud Willey, and the proposer.

Trilinear and Tetrahedral Polarities

E 573 [1943, 326]. *Proposed by N. A. Court, University of Oklahoma*

Given two (three) vertices of a triangle (tetrahedron), determine the remaining vertex so that a given point and a given line (plane) shall be harmonic for the triangle (tetrahedron).

I. *Solution by J. H. Butchart, Grinnell College.* Let the given side AB of the triangle meet the given line l in X. If X' is the harmonic conjugate of X with respect to A and B, and L is the given pole, let $X'L$ meet l in P. Let the harmonic conjugate of l with respect to PL and PB meet AL in Y'. Then BY' meets PL in the third vertex C. The proof is immediate from the harmonic properties of the figure.

For the three-dimensional problem, let the given plane μ meet the plane ABC in the line p whose trilinear polar with respect to ABC is P. The line PM, where M is the given harmonic point, is one locus of the remaining vertex D. Let AP meet BC in X' and let MX' meet μ in M'. Then the harmonic conjugate of X' with respect to M and M' is a point U' on AD. (See N. A. Court, *Modern Pure Solid Geometry*, p. 233.) Thus D is determined as the point where PM meets AU'.

II. *Solution by James Jenkins, University of Toronto.* Project A and B from the given pole L into F and G on the given line l. Then C is the trilinear pole of AB with respect to LFG.

Analogously, project A, B, C from the given pole M into F, G, H on the given plane μ. Then D is the tetrahedral pole of the plane ABC with respect to the tetrahedron $MFGH$.

Also solved by the proposer, who refers to this MONTHLY [1936, 89].

A Quadrilateral in a Semicircle

E 574 [1943, 326]. *Proposed by W. E. Buker, Pittsburgh Public Schools*

If a quadrilateral with sides a, b, c, x is inscribed in a semicircle of diameter x, show that

$$x^3-(a^2+b^2+c^2)x-2abc=0.$$

Solution by R. A. Johnson, Brooklyn College. The proposition is true, only with the interpretation that the quadrangle is convex. Though it is merely a

restatement of Ptolemy's Theorem, it presents a number of striking facets.

Let a, b, c, x be in order the sides of an inscribed quadrangle, x being a diameter of the circle. Then the diagonals, on the authority of Pythagoras, are $(x^2-a^2)^{1/2}$ and $(x^2-c^2)^{1/2}$, and Ptolemy's Theorem states that

$$(x^2 - a^2)^{1/2}(x^2 - c^2)^{1/2} = bx + ac. \tag{1}$$

Simplifying, and removing the extraneous root $x=0$, we have, as a necessary condition,

$$x^3 - (a^2 + b^2 + c^2)x - 2abc = 0. \tag{2}$$

But the steps are reversible, since all the symbols represent positive quantities; hence either (1) or (2) is a necessary and sufficient condition for the convex cyclic quadrangle.

If we start with a quadrangle in which a and c cross each other, the right side of (1) becomes $ac-bx$; if b and x cross, so that a and c are on opposite sides of x, it is $bx-ac$. In either case, the simplified equation that replaces (2) is

$$x^3 - (a^2 + b^2 + c^2)x + 2abc = 0. \tag{3}$$

This is the necessary and sufficient condition for the existence of a cross-quadrangle. For given values of a, b, c, the roots of (3) are the negatives of the roots of (2); hence we may interpret a negative root of (2) as indicating the existence of a cross-quadrangle.

The implied problem, to construct a cyclic quadrangle with three given sides a, b, c, so that the fourth side shall be a diameter, is solved by taking as diameter any root of (2). It is easy to see that the three roots are always real, and that one is greater than any of a, b, c; there are two negative roots, between which lie all of $-a$, $-b$, $-c$ when these are unequal. Hence for any three given lengths a, b, c there exists one convex quadrangle and one cross-quadrangle. (The numerically smaller negative root yields a real circle, in which however the chords a, b, c cannot be drawn in the real domain; each factor of the left side of (1) is imaginary.) It should be remarked that the order of the sides of the quadrangle is immaterial. If one type of cross-quadrangle can be drawn, so can the other.

In general, the problem requires the solution of an irreducible cubic, and is not possible with ruler and compasses. If two of the given lengths, say a and c, are equal, then $-b$ is a root of (2), and the positive root can be found with ruler and compasses. On the other hand, the related problem, to construct a cyclic quadrangle with four given sides, can always be solved with ruler and compasses. (See Johnson, *Modern Geometry*, p. 83.)

We naturally ask what rational solutions (2) may have. If we start by choosing a, c, x, and work with equation (1), we see that if the left side is rational, b will also be rational. The smallest solution of (2) in positive integers with $a<b<c$ seems to be

$$a = 2, \qquad b = 7, \qquad c = 11, \qquad x = 14,$$

There is another approach to the problem. If we assume it to be known that of all quadrangles with four given sides the cyclic one has maximum area, we may go further and ask how to enclose the greatest area with three given lengths and a fourth line of unspecified length. The formula for the area of a cyclic quadrangle is well known (*op. cit.*, p. 82), *viz.*

$$16F^2 = (-a+b+c+x)(a-b+c+x)(a+b-c+x)(a+b+c-x).$$

If we equate to zero the derivative of this expression with regard to x, we are led to equation (2). Thus we have the required maximum when x is the diameter of the circle, a result known to Legendre (Appendix to Book IV, 1794).

Also solved by F. A. Alfieri, D. H. Browne, J. H. Butchart, A. C. Cohen, Jr., William Douglas, Howard Eves, Charles Hatfield, Jr., Frank Hawthorne, Free Jamison, Elmer Latshaw, E. P. Starke, C. F. Strobel, Jerzy Szmojsz, H. B. Thornton, C. W. Topp, W. Unterberg, Alan Wayne, and Maud Willey.

Harmonic Progression

E 576 [1943, 386]. *Proposed by M. A. Dernham, San Francisco*

What well known type of sequence may be defined as one in which each term but the first may be obtained by taking the preceding term and a common constant and dividing their product by their sum?

Solution by J. H. Butchart, Grinnell College. If $a_{n+1} = ca_n/(c+a_n)$, then

$$\frac{1}{a_{n+1}} = \frac{1}{a_n} + \frac{1}{c}.$$

Hence $\{1/a_n\}$ is an arithmetic progression, and $\{a_n\}$ is a harmonic progression.

Also solved by R. K. Allen, W. E. Buker, Howard Eves, Frank Hawthorne, Irving Kaplansky, O. J. Karst, E. P. Starke, Alan Wayne, and the proposer.

Isosceles Tetrahedra

E 577 [1943, 386]. *Proposed by V. Thébault, San Sebastián, Spain*

Given an "isosceles" tetrahedron $A_1A_2A_3A_4$ (so that every two opposite edges are equal), let perpendiculars be drawn to the faces $A_2A_3A_4$, $A_3A_4A_1$, $A_4A_1A_2$, $A_1A_2A_3$ at their circumcenters O_1, O_2, O_3, O_4, to meet the hemispheres described exteriorly (or interiorly) on the respective circumcircles in P_1, P_2, P_3, P_4. Show that the tetrahedra $O_1O_2O_3O_4$ and $P_1P_2P_3P_4$ are isosceles, and that they have the same centroid as $A_1A_2A_3A_4$.

Solution by Howard Eves, Syracuse University. The vertices A_1, A_2, A_3, A_4 are interchanged in pairs by half-turns about the three bimedians. These half-turns also interchange the *faces* in pairs, and operate similarly on any four points homologously related to the faces. In particular, they interchange pairs of O_1, O_2, O_3, O_4, and of P_1, P_2, P_3, P_4. Hence the tetrahedra $O_1O_2O_3O_4$ and $P_1P_2P_3P_4$ are isosceles. Finally, the axes of the half-turns concur at an invariant

point, which is equidistant from all the A's, from all the O's, and from all the P's. This is the common circumcenter, incenter, and centroid of all the tetrahedra.

For an excellent introduction to the isosceles tetrahedron see N. A. Court, *Modern Pure Solid Geometry*, pp. 94–102. In connection with the present problem, cf. Exs. 16 and 17, p. 102.

Also solved by J. H. Butchart and the proposer.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known text-books or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4108. *Proposed by G. Pólya, Stanford University*

Let ${}_nP_r$ be the number of those permutations of n elements which are the products of exactly r cycles without common elements. For instance, ${}_4P_2=11$. Let ${}_nQ_r$ be the number of different classifications of n distinct elements into exactly r classes. For instance, ${}_4Q_2=7$. Prove that

$$(1)\qquad {}_nP_1x + {}_nP_2x^2 + \cdots + {}_nP_nx^n = x(1+x)(2+x)\cdots(n-1+x),$$

$$(2)\qquad {}_nQ_1x + {}_nQ_2x(x-1) + \cdots + {}_nQ_nx(x-1)\cdots(x-n+1) = x^n.$$

Note. In the Calculus of Finite Differences, ${}_nP_r$ and ${}_nQ_r$ are considered in various notations and are often called factorial coefficients; see, for instance, Steffensen, *Interpolation*, Baltimore, 1927, pp. 53–58.

4109. *Proposed by N. A. Court, University of Oklahoma*

Prove that the sum of the n^2 powers of n given points with respect to the n spheres having for diameters the n segments joining the given points to a variable point in space is constant.

4110. *Proposed by V. Thébault, San Sebastián, Spain*

Find the smallest integer whose square is the sum of two squares in seven different ways.

SOLUTIONS

Perfect and Nearly Perfect Squares

4055 [1942, 549]. *Proposed by V. Thébault, San Sebastián, Spain*

Find two numbers of ten digits of the form *aabbccddee* such that one is a perfect square and the other is a perfect square increased by 7.

Editorial Note. The proposer gave the following results without indicating his method of deduction:

$$5500002244 = 74162^2, \qquad 7744000000 = 88000^2, \qquad 1199998888 = 34641^2 + 7.$$

In the first part it is easily seen that $N = aabbccddee = 11(a0b0c0d0e) = 11M$, and that $a+b+c+d+e$ is a multiple of 11. The problem of finding the integer z so that $N = z^2$ reduces to finding z so that $M = 11Z^2$. We have $3000 < Z = pqrs < 9100$; and one method of determining all the answers is to find first the digits r, s so that the last two digits of M are $0e$, and then, p, q so that M ends in $0d0e$. This may be done by the method of differences as follows; for $Z = rs$, we have

$$11Z^2 = r^2(r^2 + 2rs)(2rs + s^2)s^2$$

$$\Delta_s = 2r(2r + 2s + 1)(2s + 1).$$

We carry out the work at first for the last two digits in the following tabular form and supply later the rest of the table if needed

s, Δ_s				
0 Δ_0			0 $2r+1$	0 1
1 Δ_1			$2r+1$ $2r+3$	1 3
2 Δ_2			$4r+4$ $2r+5$	4 5
3 Δ_3			$6r+9$ $2r+7$	9 7
4		1	$8r+7$ . . .	6 . . .

and this is continued up to $s = 9$ inclusive. The work may be checked by the use of $s = 10$. For $s = 2$, the entry $4r+4$ gives the digit 0 when $r = 4, 9$; and proceeding in this manner we find that the last two digits for a possible answer in the order of getting them are 42, 92, 08, 58. This gives the second answer of the proposer for the first part. We next consider in turn $pq42$, $pq92$, $pq08$, $pq58$. In the work for these we omit the known last two digits and the first part of the table which may be supplied later when needed. We proceed as follows for $pq42$

0 Δ_0			$9p$ $2p+2$	$2p+1$ 0	$4p+9$ 2	4 4		
1 Δ_1			$p+2$ $2p+4$	$2p+2$ 2	$4p+1$ 2	8 4		
2 Δ_2			$3p+6$ $2p+6$	$2p+4$ 4	$4p+4$ 2	2 4		
3			$5p+2$ . . .	$2p+8$ . . .	$4p+6$ . . .	6 . . .		

For $q=2$ we have as possible answers $p=4$, 9; and finally we have 4282, 3842, 6742, 8842 after rejection of those too large or too small. We now reexamine this last table with respect to the two next digits to the left, and we find that all but 6742 fail to give a third 0 in the desired place. On completing the necessary part of the table we find that 6742 gives the desired result. The remaining three sets are found to fail in the same way as above except 08 which gives the answer already found. Thus there are only two answers to the first part.

In the second part where $N=z^2+7$, we must have $z\equiv \pm 2 \mod 11$. We may proceed in a similar manner using differences to find the last two digits of z, then two more digits and finally a fifth digit. The last two digits of z must be among 41, 91, 02, 52, 48, 98, 09, 59. For 41 the values of pq are 12, 62, 33, 83, 04, 54, 25, 75, 46, 96, Denoting the first digit of z by n, we use the above congruence to find trial values of n, in most cases two values of n are found. Then, since $z\geqq 33160$, four of the resulting 16 values are discarded, and then from an estimate of the first two digits of the trial z^2 all the others except 34641 are discarded, and this latter is found to be a solution. The next six sets of last two digits rs do not yield solutions, failing mostly in an estimate of the first two digits for aa. Finally for $rs=59$ there are thirteen cases all of the first twelve failing as above, whereas the last and largest gives a solution

$$99559^2 + 7 = 9911994488.$$

Discarding the cases $a=0$, there are only two solutions.

Line of Images as Axis of Perspectivity

4056 [1942, 616]. *Proposed by J. R. Musselman, Western Reserve University*

Let the line of images of any point T on the circumscribe of triangle $A_1A_2A_3$ cut the sides A_jA_k in the points A_i'. The perpendiculars to the sides A_jA_k at A_i' form the triangle $B_1B_2B_3$; show that the straight lines A_iB_i meet in T.

Solution by Howard Eves, Syracuse University. Let A_i'' be the foot of the perpendicular from T on A_jA_k. Then, since $A_j''TA_k''A_i$ is similar to $A_j'B_iA_k'A_i$, it follows that T, B_i, A_i are collinear. Hence the theorem.

Note that $A_1'A_2'A_3'$ need not be the line of images for T, but any line parallel to the Simson line of T with respect to $A_1A_2A_3$.

Editorial Note. The line of images of T is the directrix of a parabola with the focus T and tangent to the sides of $A_1A_2A_3$; hence the parabola is tangent also to the sides of $B_1B_2B_3$. Thus the two triangles have with respect to T the same Simson line and the same line of images. If instead of the line of images we use any arbitrarily chosen parallel, as in the above note, the circumcircle of $B_1B_2B_3$ passes through T, and the line of images of the two triangles are parallel, and by the theorem of the note the point T and the chosen parallel are respectively the center and axis of perspectivity of the two triangles.

Euler's Constant

4046 [1942, 479]. *Proposed by Otto Dunkel, Washington University*

Show that γ, Euler's constant, is given by

$$\gamma = 2\left[1 - \log 2 - \frac{\tau_3}{3} - \frac{\tau_5}{5} - \cdots\right], \qquad \tau_r = \sum_{i=1}^{\infty} \frac{1}{(2i+1)^r}.$$

Solution by the Proposer. We prove first that

$$\sum_{i=1}^{\infty} \frac{1}{i(2i+1)} = 2(1 - \log 2). \tag{1}$$

We have

$$\sum_{i=1}^{n} \frac{1}{i(2i+1)} = 2\left[\sum_{i=1}^{n} \frac{1}{2i} - \sum_{i=1}^{n} \frac{1}{2i+1}\right] = 2\left[1 - \sum_{i=1}^{2n+1} (-1)^{i-1}\frac{1}{i}\right],$$

and, if we let $n \to \infty$, we have (1). A simple modification of the development

$$\log\left(\frac{n+1}{n}\right) = 2\left[\frac{1}{2n+1} + \frac{1}{3(2n+1)^3} + \frac{1}{5(2n+1)^5} + \cdots\right]$$

gives

$$-\left[\frac{1}{n} - \log\left(\frac{n+1}{n}\right) - \frac{1}{n(2n+1)}\right] = 2\sum_{i=1}^{\infty} \frac{1}{(2i+1)(2n+1)^{2i+1}}. \tag{2}$$

The sum of the left member of (2) from $n=1$ to $n=\infty$ is $-\gamma+2(1-\log 2)$, whereas the summation on the right gives a double series, obviously convergent. Summation by columns gives the equality of the problem.

In order to make the proof complete we shall show the existence of the number γ. From the graph of $1/(1+x)$ from $x=0$ to x, we see that $\log(1+x)$

lies between the areas of two trapezoids, the larger formed by the ordinates 1 and $1/(1+x)$ while the smaller has parallel sides cut off by the tangent at $x/2$. After a simple modification of the two trapezoid areas this gives

$$x - \frac{x^2}{2(1+x/2)} < \log(1+x) < x - \frac{x^2}{2(1+x)}, \tag{3}$$

$$\log(1+x) = x - \frac{x^2}{2(1+\theta x)}, \qquad \frac{1}{2} < \theta < 1, \tag{4}$$

$$C(n) = \sum_{i=1}^{n} \frac{1}{i} - \log(n+1) = \frac{1}{2} \sum_{i=1}^{n} \frac{1}{i^2}\left(\frac{1}{1+\theta_i/i}\right) < \frac{1}{2} \sum_{i=1}^{n} \frac{1}{i^2} < \frac{\pi^2}{12}. \tag{5}$$

Thus $C(n)$ increases with n and approaches a number $\gamma < \pi^2/12$. We may write

$$\frac{x^2}{2(1+x)} < x - \log(x+1) < \frac{x^2}{2(1+x/2)},$$

$$\frac{1}{2}\left[1 - \frac{1}{n+1}\right] < C(n) < \sum_{i=1}^{n} \frac{1}{i(2i+1)},$$

$$.5 < \gamma < 2(1 - \log 2) = .614^{-}.$$

Replacing i and n by $i+a$ and $n+a$, where a is a fixed positive constant, we can consider similar expressions $C(n, a)$ which are useful in the study of more complex infinite series of the type (1).

Editorial Note. Since the above formula is not convenient for computation, we shall give a more suitable expression for limits l similar to γ, where l is defined in the following theorem:

Given the real function $f(x)$ of the real variable x which has finite derivatives $f'(x), f''(x), \cdots, f^{\text{vi}}(x)$ for all values of $x \geqq 1$; $f^{\text{vi}}(x)$ is continuous and has always the same sign for such values; and $|f^{\text{vi}}(x)|$ decreases as x increases and approaches zero; then $F(n)$, where n is a positive integer, and

$$(1)\ F(n) = \sum_{j=1}^{n} f'(j) - \left[f(n+1) - \frac{1}{2}f'(n+1) + \frac{1}{12}f''(n+1) - \frac{1}{6!}f^{\text{iv}}(n+1)\right]$$

has a finite limit l as the integer $n \to \infty$.

In order to show this we carry out five integrations by parts to obtain

$$f'(x) = \Delta f(x) - \frac{1}{2}\Delta f'(x) + \frac{1}{12}\Delta f''(x) - \frac{1}{4!}\int_0^1 \psi_4(t) f^{\text{v}}(x+t)\,dt, \tag{2}$$

where $\psi_4(t) = t^2(t-1)^2$, $\Delta f^{(i)}(x) = f^{(i)}(x+1) - f^{(i)}(x)$, and setting $f'(x+1) + f'(x) = \Delta f'(x) + 2f'(x)$.

We now extend (2) by the use of the two integrations

$$\int_0^1 \psi_4(t) f^{\text{v}}(x+t)\,dt = \frac{1}{30} f^{\text{v}}(x+1) - \frac{1}{30}\int_0^1 [6t^5 - 15t^4 + 10t^3] f^{\text{vi}}(x+t)\,dt,$$

$$\int_0^1 tf^{\text{vi}}(x+t)dt = f^{\text{v}}(x+1) - \Delta f^{\text{iv}}(x);$$

and we have

$$(3)\quad \begin{aligned} f'(x) &= \Delta f(x) - \frac{1}{2}\Delta f'(x) + \frac{1}{12}\Delta f''(x) - \frac{1}{6!}\Delta f^{\text{iv}}(x) + \frac{1}{6!}\int_0^1 \psi_5(t) f^{\text{vi}}(x+t)dt, \\ \psi_5(t) &= 6t^5 - 15t^4 + 10t^3 - t = t(t-1)(2t-1)(3t^2-3t-1), \end{aligned}$$

where $\psi_5(t) = -\psi_5(1-t)$, $\psi_5(t) < 0$, for $0 < t < 1/2$ and $\psi_5(t) > 0$ for $1/2 < t < 1$. Summing each member of (3) for $x = 1, 2, \cdots, n$, we have finally

$$(4)\quad \begin{aligned} F(n) = &-\left[f(1) - \frac{1}{2}f'(1) + \frac{1}{12}f''(1) - \frac{1}{6!}f^{\text{iv}}(1)\right] \\ &+ \frac{1}{6!}\sum_{j=1}^{n}\int_0^1 \psi_5(t) f^{\text{vi}}(j+t)dt. \end{aligned}$$

The integral on the right is the sum of two integrals, the first from 0 to 1/2 and the second from 1/2 to 1, and we have the sum of $2n$ terms with alternating signs; and it is easily seen from the graph that the absolute values of the terms decrease. As n increases, the absolute value of the sum of the $2n$ terms increases, and hence $F(n)$ increases, or decreases, according to the sign of $f^{\text{vi}}(x)$; also for $n \to \infty$, the infinite series is convergent since the terms approach zero, and $F(n)$ approaches a finite limit l

$$(5)\quad l = -\left[f(1) - \frac{1}{2}f'(1) + \frac{1}{12}f''(1) - \frac{1}{6!}f^{\text{iv}}(1)\right] + \frac{1}{6!}\sum_{j=1}^{\infty}\int_0^1 \psi_5(t) f^{\text{vi}}(j+t)dt,$$

$$(6)\quad l = F(n) + \frac{1}{6!}\sum_{j=n+1}^{\infty}\int_0^1 \psi_5(t) f^{\text{vi}}(j+t)dt.$$

The absolute value of the last term on the right of (6) is less than $|f^{\text{vi}}(n+1)|/5!2^7$; and (6) gives an expression for approximating l. Thus for $f(x) = \log x$ we have

$$(7)\quad \begin{aligned} \gamma = \sum_{j=1}^{n}\frac{1}{j} &- \left[\log(n+1) - \frac{1}{2(n+1)} - \frac{1}{12(n+1)^2} + \frac{1}{5!(n+1)^4}\right] \\ &- \frac{1}{5!}\sum_{j=n+1}^{\infty}\int_0^1 \frac{\psi_5(t)}{(j+t)^6}dt, \end{aligned}$$

with the last term less than $1/2^7(n+1)^6$. For $n = 9$ the last term is less than .00000000782, and an easy computation of the remaining part gives the value of γ correct in the first seven decimals.

It is easily seen how to extend all of the above to derivatives of higher order than six, and the rule for obtaining the polynomials $\psi_k(t)$ will be discovered to be as follows. We integrate $\psi_5(t)$ from $t = 0$ to t and the result may be taken for $\psi_6(t)$; in the above we have discarded a least common integral denominator

simply for convenience. We then proceed similarly with $\psi_6(t)$ to find $\psi_7(t)$ but here we add to the integral a term of the first degree ct and determine the coefficient c so that the sum is zero for $t=1$. We continue in this way as far as we please, distinguishing the cases $\psi_{2m}(t)$ and $\psi_{2m+1}(t)$. It will be seen that

$$\psi_k(t) = A_k t^k - A_{k-1}t^{k-1} + A_{k-2}t^{k-2} - A_{k-4}t^{k-4} + A_{k-6}t^{k-6} - \cdots, \quad k \geqq 4,$$

where the coefficients A_{k-i} are positive; and that $\psi_{2m}(t) = \psi_{2m}(1-t)$, $\psi_{2m+1}(t) = -\psi_{2m+1}(1-t)$. From the above integration rule we infer that $\psi_k(t)/kA_k$ for t a positive integer is the sum $1+2^{k-1}+3^{k-1}+\cdots+(t-1)^{k-1}$. If we set $f(x) = x^k$, $k=1, 2, \cdots, 6$, in (5) and (6), we obtain these latter polynomials.

There are other important applications of (5) and (6); for example if we set $f(x) = -1/2x^2$, we obtain an approximate value of the infinite series $1+1/2^3+1/3^3+\cdots$, and if we take $n=9$, the result is correct in the first eight decimals using the estimate of error under (6). This gives $r_3 = .05179978$. The following evaluation, correct in the given figures, involves laborious computations

$$\sum_{i=1}^{\infty} \frac{1}{i^{1.1}} = 10.58444846 \cdots.$$

Triangles Having Integral Sides and Areas

4047 [1942, 479]. *Proposed by T. R. Running, Ann Arbor, Mich.*

Triangles have the sides $x-1$, x, $x+1$, the altitude h with x as base, and area A, where x, h, A are whole numbers. The first six possible triangles are given by the table

n	h	x	A
0	0	2	0
1	3	4	6
2	12	14	84
3	45	52	1170
4	168	194	16296
5	627	724	226974
6	.	.	.

Do the relations

$$h_{n+2} = 4h_{n+1} - h_n, \qquad x_{n+2} = 4x_{n+1} - x_n, \qquad A_{n+2} = 14A_{n+1} - A_n,$$

hold for all the triangles fulfilling the given conditions?

Note by W. B. Clarke, San Jose, Calif. In the *National Mathematics Magazine;* Nov. 1934, p. 63, a solution was given by E. P. Starke of my problem No. 65 which stated:

Considering only triangles whose sides are consecutive integers, and whose area is an integer, find the area of the triangle which is next larger than the one whose area is 16296.

In a footnote to his solution Starke referred to the solution of 3677 in this

MONTHLY which would appear later, 1935, 572. This has a direct bearing on 4047; and from these developments it appears to me that the answer to 4047 is yes.

Solution by E. P. Starke, Rutgers University. The proposed question has an affirmative answer. By familiar formula

$$A = \tfrac{1}{2}hx = \sqrt{\tfrac{1}{2}(3x)\cdot\tfrac{1}{2}(x+2)\cdot\tfrac{1}{2}x\cdot\tfrac{1}{2}(x-2)} = \tfrac{1}{4}x\sqrt{3x^2-12}.$$

Since A is integral, x must be even. Put $x=2y$ to obtain

$$A = hy, \qquad h = \sqrt{3(y^2-1)}.$$

Thus h must be a multiple of 3, and we may put $h=3z$ to obtain $A=3yz$ and

$$(1) \qquad y^2 - 3z^2 = 1.$$

The problem is thus equivalent to that of finding all pairs of positive integers (y, z) which satisfy (1). Consider

$$(2) \qquad y_{n+1} = 2y_n + 3z_n, \qquad z_{n+1} = 2z_n + y_n,$$

or, solved for y_n, z_n,

$$(3) \qquad y_n = 2y_{n+1} - 3z_{n+1}, \qquad z_n = 2z_{n+1} - y_{n+1}.$$

If (y_n, z_n) satisfy (1), it is easily verified that (y_{n+1}, z_{n+1}) do also, and conversely. Also (2) implies the equations

$$y_{n+2} = 2y_{n+1} + 3z_{n+1}, \qquad z_{n+2} = 2z_{n+1} + y_{n+1}.$$

If the three z's and the three y's, respectively, are eliminated between these equations and (2) we have

$$(4) \qquad y_{n+2} = 4y_{n+1} - y_n, \qquad z_{n+2} = 4z_{n+1} - z_n.$$

If y_n, z_n are non-negative, (2) implies $y_{n+1}>y_n$, $z_{n+1}>z_n$. Further if y_{n+1}, z_{n+1} are positive integers satisfying (1), $9z_{n+1}^2=3y_{n+1}^2-3<4y_{n+1}^2$ implies $3z_{n+1}<2y_{n+1}$ and y_n is positive by (3). Similarly $y_{n+1}^2=3z_{n+1}^2+1<4z_{n+1}^2$ unless $z_{n+1}=1$ implies $y_{n+1}<2z_{n+1}$ and z_n is positive. Thus if (y_{n+1}, z_{n+1}) is any solution of (1) and $z_{n+1}\neq 1$, there is obtained from (3) a solution (y_n, z_n) in smaller positive integers, and then from (y_n, z_n) another still smaller, *etc.* Since there cannot be an endless series of positive integral values of z each less than the preceding, the above reduction process must stop somewhere: *i.e.* at some step the value of z becomes 1. By now reversing the process we see that any solution of (1) is obtained from $y_0=1, z_0=0$; $y_1=2$, $z_1=1$ by repeated use of (2).

Multiplying the equations (4) by 2 and 3 respectively we have the proposed equations in x and h, valid for all triangles fulfilling the given conditions. The identity

$$2y_{n+1}z_{n+1} = (2y_{n+1} - 3z_{n+1})(y_{n+1} + 2z_{n+1}) + (2z_{n+1} - y_{n+1})(2y_{n+1} + 3z_{n+1})$$

by use of (2) and (3) becomes

$$(5) \qquad 2y_{n+1}z_{n+1} = y_n z_{n+2} + y_{n+2}z_n.$$

From (4) we may put

$$(4y_{n+1})(4z_{n+1}) = (y_{n+2} + y_n)(z_{n+2} + z_n) = y_{n+2}z_{n+2} + y_n z_{n+2} + y_{n+2}z_n + y_n z_n,$$

which can be simplified by (5) to the form

$$14y_{n+1}z_{n+1} = y_n z_n + y_{n+2}z_{n+2}.$$

Finally, since $3yz = A$, we have, upon multiplying by 3, the third of the proposed relations, valid also for all triangles fulfilling the given conditions.

Triangle Variable in Space

4061 [1942, 688]. *Proposed by N. A. Court, University of Oklahoma*

A variable triangle has its vertices on three skew straight lines, and two of its sides meet a given plane in points lying respectively on two straight lines. Show that the points of intersection of that plane with the third side of the variable triangle are collinear.

Solution by L. S. Sinclair, University of Toronto. Let the three skew lines be l, m, n, meeting the given plane in L, M, N; and let p, r be the two lines given in that plane. Let the variable triangle be ABC, with A on l, B on m, C on n, AB intersecting r, and BC intersecting p. It is required to prove that the locus of the point where AC meets the plane pr is a line.

The variable line AB, intersecting the three skew lines l, m, r, sweeps out a quadric; so also does BC (which intersects m, n, p). Hence the points A, B, C vary on l, m, n, in such a way that

$$A \barwedge B \barwedge C.$$

Therefore AC joins corresponding points of projective ranges on two skew lines, and sweeps out a third quadric. Since the triangle LMN is one possible position for ABC, the side LN is one generator of this last quadric; *i.e.*, LN forms part of the section of the quadric by the plane pr. Hence the rest of the section is another line, which is the desired locus.

Solved also by Paul Brock, W. W. Dolan and the proposer.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to B. W. Jones, White Hall, Cornell University, Ithaca, New York.

Sister Helen Sullivan of Mount St. Scholastica College was recently elected National Historian of Kappa Mu Epsilon.

The mathematics department of University of Cincinnati announces the following changes: Assistant Professors E. T. Miller and H. A. Sibert hold the

rank of major in the army; J. L. Baker and Assistant Professor E. F. White are lieutenants in the army and navy respectively; Dr. Gabriel Horvay is on leave of absence for war research. Associate Professor H. L. Miller has been promoted to a professorship; Assistant Professors H. A. Dangel and A. H. Knebel have been promoted to associate professorships; Dr. C. A. Ludeke, W. E. Restemeyer and E. T. Miller have been promoted to assistant professorships; Professor E. S. Smith (Major, U.S.A.) has retired from the army to resume academic status.

Dr. Claire F. Adler has been appointed to an assistant professorship at New York University.

Assistant Professor P. O. Bell of the University of Kansas has been promoted to an associate professorship.

Associate Professor Ethel B. Callahan of Shepherd State Teachers College, Shepherdstown, West Virginia, has been appointed to an associate professorship at Hartwick College, Oneonta, New York.

Sister James S. Creane has been appointed to an assistant professorship at Fontbonne College, St. Louis, Missouri.

Assistant Professor Paul Eberhart of Washburn University, Topeka, Kansas, has been promoted to a professorship.

W. E. Ekman of the University of South Dakota has been promoted to a professorship of mathematics and astronomy.

Assistant Professor H. F. Fehr of the State Teachers College at Upper Montclair, New Jersey, has been promoted to an associate professorship.

Edna M. Feltges of Woodrow Wilson Junior College, Chicago, has been appointed chairman of the department of mathematics.

M. P. Fobes of the College of Wooster, Ohio, has been promoted to an assistant professorship.

R. M. Foster of Bell Telephone Laboratories has been appointed head of the department of mathematics of the Polytechnic Institute of Brooklyn.

Assistant Professor J. W. Foust of Central Michigan College of Education has been promoted to a professorship.

W. A. Gager of the Junior College in St. Petersburg, Florida, has been appointed to an associate professorship at the University of Florida.

J. B. Greeley of the Polytechnic Institute of Brooklyn has been appointed to an assistant professorship at Lafayette College.

Assistant Professor E. H. Hadlock of Hastings College, Nebraska, has been promoted to an associate professorship.

Dr. M. C. Hartley has been appointed to an assistant professorship at the University of Illinois.

Associate Professor H. H. Hartzler of Goshen College, Indiana, has been promoted to a professorship.

Assistant Professor P. R. Hill of the University of Georgia has been promoted to an associate professorship.

Assistant Professor C. M. Howard of North Texas Agricultural College has been promoted to a professorship and is acting head of the department.

Dr. P. M. Hummel of the University of Alabama has been promoted to an assistant professorship.

Associate Professor J. W. Hurst of Montana State College has been promoted to a professorship.

Associate Professor J. A. Hyden of Vanderbilt University has been promoted to a professorship.

Assistant Professor F. B. Jones of the University of Texas has been promoted to an associate professorship. He is now on leave, serving in the Underwater Sound Laboratory at Harvard University.

Dr. H. T. Karnes of Louisiana State University has been promoted to an assistant professorship.

Dr. Fulton Koehler of the Northwest National Life Insurance Company has been appointed to an assistant professorship at the University of Minnesota.

J. E. LaFon of the University of Oklahoma has been promoted to an assistant professorship.

Assistant Professor R. G. Lubben of the University of Texas has been promoted to an associate professorship.

Assistant Professor W. T. MacCreadie of Bucknell University has been promoted to an associate professorship.

Associate Professor F. L. Manning of Ursinus College has been promoted to a professorship.

J. N. McClelland of Drake University has been promoted to an assistant professorship.

Associate Professor C. E. Melville of Clark University has been promoted to a professorship.

Professor H. A. Meyer of Hanover College is acting professor of mathematics at Indiana University.

Assistant Professor R. R. Middlemiss of Washington University has been promoted to an associate professorship.

Associate Professor C. G. Phipps of the University of Florida has been promoted to an associate professorship.

R. J. Pitts of Fort Valley State College, Georgia, has been promoted to an assistant professorship.

Associate Professor W. G. Pollard of the University of Tennessee has been promoted to a professorship.

Professor G. C. Priester of the Institute of Technology at the University of Minnesota has been made head of the department of mathematics and mechanics.

Dr. Adrienne S. Rayl of the University of Alabama, Birmingham Center, has been promoted to an assistant professorship.

Professor O. H. Rechard of the University of Wyoming has been relieved of the chairmanship of the department of mathematics so that he may serve as War Projects Administrator. Associate Professor C. F. Barr has been appointed acting chairman of the department.

Assistant Professor P. K. Rees of Southern Methodist University has been appointed to a professorship at Southwestern Louisiana Institute.

Dr. C. E. Rhodes of Union College has been promoted to an assistant professorship.

H. L. Rice of the University of Omaha has been promoted to an assistant professorship.

Assistant Professor F. A. Rickey of Louisiana State University has been promoted to an associate professorship.

Assistant Professor Robin Robinson of Dartmouth College has been promoted to a professorship and appointed chairman of the department of mathematics and astronomy.

Assistant Professor W. J. Robinson of Marshall College, Huntington, West Virginia, has been appointed to an assistant professorship at Allegheny College.

Assistant Professor E. B. Roessler of the University of California College of Agriculture has been promoted to an associate professorship.

A. L. Starrett of Georgia School of Technology has been promoted to an assistant professorship.

Dr. R. E. Street has been appointed to a visiting lectureship in physics at Dartmouth College.

I. L. Stright has been appointed to an assistant professorship at Baldwin-Wallace College.

Associate Professor W. R. Talbot of Lincoln University, Jefferson City, Missouri, has been promoted to a professorship.

J. E. Thompson of Pratt Institute, Brooklyn, has been promoted to an associate professorship.

Dr. H. C. Trimble of Iowa State Teachers College has been promoted to an assistant professorship.

Assistant Professor P. L. Trump of the University of Wisconsin has been promoted to an associate professorship in the teaching of mathematics.

R. N. Van Arnam of Lehigh University has been promoted to an assistant professorship of mathematics and astronomy.

Professor C. C. Wagner of Pennsylvania State College has been appointed assistant dean of the School of Liberal Arts.

Assistant Professor D. L. Webb of Texas Technological College has been promoted to an associate professorship.

Associate Professor P. D. Wilkins of Bates College has been promoted to a professorship.

Assistant Professor C. R. Wylie, Jr., of Ohio State University has been given leave of absence for service at Wright Field, Dayton, Ohio.

The following appointments to instructorships are announced:
Antioch College: C. B. Helms
Brooklyn College: Solomon Hurwitz, Joseph Landin
Brown University: W. M. Kincaid
Duke University: Dr. L. I. Wade
Michigan State College: Nicholas Musselman
State College of Washington: Dr. P. F. Nemenyi
University of Arizona: Dr. O. B. Ader
University of Cincinnati: Robert Fopma
University of Illinois: Dr. Theodore Bedrick
University of Nebraska: H. W. Linscheid
University of Rochester: Dr. Dorothy L. Bernstein

WAR INFORMATION

EDITED BY C. V. NEWSOM

Send news reports upon the utilization of mathematicians or mathematics in war activities to C. V. Newsom, University of New Mexico, Albuquerque, New Mexico.

NOTES ON THE NAVY V-12 PROGRAM

Qualifications of enlisted men now entering the V-12 Program. "In order to qualify for the program, an enlisted man already in active service must have had a minimum of two years of high school mathematics in order to qualify. Civilians qualify through a written examination, and the most recent examination given, namely, the one on 9 November 1943, was quite heavily weighted in mathematics."—Commander Alvin C. Eurich, Officer in Charge, Standards and Curriculum Section.

Teaching materials. "The Navy has purposefully refrained from making specific recommendations concerning text books to be used. It is felt that that is a matter which can be left to the institutions. The Navy feels that it has selected qualified colleges and universities, and that these courses are very similar to those which they have been accustomed to giving, and, therefore, they should be expected to rely upon their own experience for determining the materials which can be used most effectively."—Commander Alvin C. Eurich.

Faculty teaching loads. "The reference in Bulletin No. 101 to faculty teaching load, refers primarily to the amount of teaching which a faculty member does during a particular term. There have been instances where faculty members apparently have felt that, although student programs should be accelerated and everybody else in the country should work harder, their teaching load should remain at a normal level, which in some instances was very low, and was designed to give the faculty member considerable time for research. The Navy is contracting in the V-12 program for teaching only, and not for research—important though the research may be. Summer teaching is taken care of by allowance of proportionate extra compensation based on the regular salary of the instructor. The Navy does not stipulate that instructors have to teach during the summer; however, with the instructional shortage which there seems to be in certain fields, it is believed that most institutions would be faced with a very serious problem if faculty members were to teach the customary eight or nine months only."—Commander Alvin C. Eurich.

General engineering quotas. "It is . . . announced that present Navy V-12 upper-level students in the field of General Engineering are adequate to meet current needs of the Navy. Therefore, no students shall be recommended for assignment to this specialty at this time."—Navy V-12 Bulletin No. 155.

THE TRAINING PROGRAMS IN METEOROLOGY

In view of the large number of inquiries relative to the discontinuance of the

meteorological training programs sponsored by the Army, the following statement was solicited from Professor C. G. Rossby, University of Chicago, Chairman of the University Meteorological Committee.

"The Office of the Assistant Chief of Air Staff, Operations, Commitments and Requirements, determines from information submitted to them the number of personnel required for a particular branch of the Army Air Forces. The University Meteorological Committee, which is composed of representatives of the five universities training meteorologists at the professional level and the Army Training School at Chanute Field, Illinois, was informed by the Assistant Chief of Staff that the requirements of the Air Forces for meteorologists through the year 1944 had been met and that therefore there would not be any additional meteorologists trained unless the requirements for 1945 and 1946 indicated a need for additional meteorologists.

"The B and C programs were planned to prepare students for entrance into the meteorological programs conducted by the A schools. The curricula stressed mathematics and physics particularly.

"As the training programs in the A schools were discontinued for the present, there was not any reason for having additional B and C programs. However, the value of these preparatory training programs was considered so great by the Army Air Forces that it was decided to allow those students who were already enrolled in the B and C programs to complete their courses.

"A reclassification board from the Flying Training Command visited the schools having B and C programs and offered the students the following choices of additional training programs at the completion of their B and C courses: flight training including pilot, navigator, and bombardier; communications training; weather observing; advanced engineering training and other Army Specialized Training Program subjects.

"The men were allowed to indicate their choices in the order of their preference. At the completion of the present courses they will be assigned to one of the above enumerated programs.

"The above statement of the policy of the Army Air Forces prohibits us from accepting any further applications for the A, B, or C programs at the present time."

THE ALIEN BOOK REPUBLICATION PROGRAM

Approximately a year ago, the Office of Alien Property Custodian, Washington, D. C., announced to scientists, libraries, industrial and research organizations that it was considering a Book Republication Program in order to make certain foreign scientific materials available for use in the war effort. A thirty-nine page booklet was distributed at that time, containing titles for which one or more republication requests had already been received. Those interested in securing copies of any of the titles there listed were requested to advise the Custodian. As a result of the investigation then made, and currently continuing, licenses have been granted to certain individuals and publishers for the repro-

duction of a large number of books originally published in alien countries. The latest edition available in the United States before the war is being republished in each case.

A publisher obtaining a license for a particular book in the Custodian's list agrees to reproduce all or a substantial part of the work by photo-offset or otherwise in the original language of publication. Licenses are issued upon a competitive basis for a period of five years; they are non-exclusive, but the Custodian agrees not to grant a second license pertaining to any work for a period of six months after the date of the first license. A royalty fee of 15% is charged upon the list or retail price of any book reproduced; a publisher, however, is permitted the recoupment of the cost of publication before royalties become payable.

If a publisher suggests a title not included in the Custodian's list, the publisher suggesting the title for republication may submit a request for a license under essentially the same terms and conditions as given above. A license may then be granted at the discretion of the Custodian without exploratory work being done by the Custodian or other bids being asked.

Translations are licensed for the life of the copyright. In this case, the recoupment of the original publication, translation, and exploitation costs is allowed before any royalty will be charged. Thereafter, royalty at the rate of 10% of the list or retail sale price will be payable to the Custodian. Otherwise, licenses to translate and revise works are given under substantially the same terms as those indicated for republication in the original language. The translator and publisher will hold the copyright for the specific translation.

Among the books already announced as licensed for reproduction are many which are of interest to mathematicians. A selected list of these titles is given below. All communications with respect to the purchase of these works should be addressed to the licensee designated for each title, *not* to the Office of Alien Property Custodian.

Biezeno, Cornelius B. *Technische Dynamik*. Berlin, J. Springer, 1939. Licensee: Edwards Brothers, Ann Arbor; $27.80. (Original price: $31.20).

Bjerknes, V. F. K., and others. *Physikalische Hydrodynamik, mit Anwendung auf die dynamische Meteorologie*. Berlin, J. Springer, 1933. Licensee: Edwards Brothers, Ann Arbor; $18.00. (Original price: $27.50).

Born, Max. *Optik. Ein Lehrbuch der elektromagnetischen Lichttheorie*. Berlin, J. Springer, 1933. Licensee: Edwards Brothers, Ann Arbor; $13.00. (Original price: $14.20).

Courant, Richard, and Hilbert, David. *Methoden der mathematischen Physik*. Berlin, J. Springer. v. 1, 2 Aufl., 1931; v. 2, 1937. Licensee: Interscience Publishers, New York; $14.00, 2 vol. in 1. (Original price: v. 1, $12.32; v. 2, $15.92).

Cranz, Carl Julius. *Lehrbuch der Ballistik*. Berlin, J. Springer, 1925–1936. Licensee: Edwards Brothers, Ann Arbor. Complete work, $55.00; v. 1, $24.00; v. 2, $16.50; v. 3, $15.50; v. 4, $11.00. (Original price: $63.00).

Ertel, Hans. *Methoden und Probleme der dynamischen Meteorologie*. Berlin, J. Springer, 1938. Licensee: Edwards Brothers, Ann Arbor, $4.50. (Original price: $5.50).

Flügge, Wilhelm. *Statik und Dynamik der Schalen*. Berlin, J. Springer, 1934. Licensee: Edwards Brothers, Ann Arbor, $5.75. (Original price: $9.00).

Frank, Philipp, and Mises, R. Von. *Die Differential und Integralgleichungen der Mechanik und*

Physik. 2 Aufl. Braunschweig, F. Vieweg und Sohn, 1930–1935. Licensee: M. S. Rosenberg, New York, $27.50. (Original price: v. 1, $22.40; v. 2, $26.00).

Hilbert, David. *Grundlagen der Mathematik.* Berlin, J. Springer, 1934–1939. Licensee: Edwards Brothers, Ann Arbor. Complete work, $25.00; v. 1, $12.40; v. 2, $13.20. (Original price: $32.65).

Jahnke, Eugen, and Emde, Fritz. *Funktionentafeln mit Formeln und Kurven.* 3 Aufl. Leipzig und Berlin, B. G. Teubner, 1938. Licensee: G. E. Stechert, New York; $3.00. (Original price: $6.00). Second licensee: Dover Publications, New York; $3.50. (Note: The Dover edition also contains a 76 page addendum, "Index of tables of the elementary transcendentals" from the 1933 edition).

Jordan, Pascual. *Einführung in die Gedankenwelt der modernen Physik.* Braunschweig, F. Vieweg und Sohn, 1938. Licensee: Philosophical Library, New York; English translation, $4.00. (Original price: $2.40).

Peters, Jean. *Siebenstellige Werte der trigonometrischen Funktionen von Tausendstel zu Tausendstel des Grades.* Leipzig and Berlin, B. G. Teubner, 1938. Licensee: D. Van Nostrand, New York; $7.50. (Original price: $10.00).

Prandtl, Ludwig. *Vier Abhandlungen zur Hydrodynamik und Aerodynamik,* Göttingen, Kaiser Wilhelm Institute, 1927. Licensee: Edwards Brothers, Ann Arbor; $3.50. (Original price: $1.60).

Stumpff, Karl. *Tafeln und Aufgaben zur harmonischen Analyse und Periodogrammrechnung.* Berlin, J. Springer, 1939. Licensee: Edwards Brothers; $5.25. (Original price: $14.65).

Waerden, Bartel Leendert Van Der. *Moderne Algebra.* Berlin, J. Springer, v. 1, 2 Aufl., 1937, v. 2, 1931. Licensee: Frederich Ungar, New York; $6.00. (Original price: $12.80).

MATHEMATICIANS AND THE NEW SELECTIVE SERVICE REGULATIONS

On January 10, 1944, Selective Service Headquarters released amended Local Board Memoranda 115 and 115B, which become effective on February 1, and also an amended Activity and Occupational Bulletin No. 33-6, which becomes effective on February 15. These vitally concern all chairmen of departments of mathematics, for they deal with the occupational classification of both teachers and undergraduate students. This communication contains a summary of the provisions of these directives.

Mathematicians are still included in the List of Critical Occupations. It is specifically stated that the titles included in Part II (Professional and Scientific Occupations) of the List of Critical Occupations "shall be considered as also including persons engaged in full-time teaching of these professions. A person may be considered as engaged in full-time teaching if he devotes not less than 15 hours per week in contact with students in actual classroom or laboratory instruction." The problem of securing the proper classification of a *teacher* of mathematics differs in accordance with the age group into which he falls.

I. Deferment of Mathematicians in 18–21 Age Group

All requests for new or additional occupational deferments for registrants in this age group who are teachers of mathematics must be made on Director of Selective Service Form 42-A Special. An original and two copies of this form must be presented by the employer to the State Director in whose state is located the registrant's principal place of employment.

Effective February 1, no registrant in this group, at the time he is classified, may be considered as a "necessary man" entitled to be placed in Class II-A or Class II-B unless:

(a) the State Director of Selective Service has endorsed his Form 42-A Special with a statement that, based on the information furnished therein, he recommends that the local board except the registrant from the general restrictions against occupational deferment of registrants aged 18 through 21.

(b) he is classified as (1) belonging to the Personnel of the Merchant Marine or Army Transportation Corps or (2) as a student who qualifies under AOB 33-6.

Registrants in this group who are classified II-A or II-B on February 1, 1944, will, in general, not have deferments terminated before the expiration date. (This, however, does not apply to students covered in the new AOB 33-6.)

The President and Secretary of the Society have addressed a memorandum to State Directors of Selective Service, calling attention to the vital need for these mathematicians to handle the mathematics in the Army Specialized Training Program and the Navy College Training Program.

II. Deferment of Mathematicians in 22–37 Age Group

In making requests for occupational deferment, one copy of Form 42-A must be filed with the local board, as heretofore. The local board may, without going further, classify the registrant in II-A or II-B, as requested.

If the registrant is not classified in II-A or II-B and if reference to the United States Employment Service was not made by the local board prior to classification in I-A, I-A-O or IV-E, it must be made immediately after such classification. Upon suitable certification from the USES, the local board must reopen the case. The cases of registrants who are qualified for professional and scientific occupations will be forwarded by the USES to the National Roster. In all cases in which reference to the USES is made, the local board is directed not to issue an order to report for induction until it has received a report from the USES or until the expiration of 30 days after referral, whichever occurs first. If, during the 30-day period, the USES certifies to the local board that the registrant is qualified and that his removal would adversely affect the maintenance of his employer's required production, the local board must reopen the case and consider the new evidence as a basis for further deferment.

Even though a registrant is not employed in the area in which his local board is located, the registrant's local board must refer the case to the local USES office in the local board area and any further reference of the case to the USES office in which the registrant is employed will be undertaken and accomplished by the USES.

In view of these provisions, *department chairmen are urged to register their needs for personnel with the USES office where the college or university is located and in the USES office where the registrant's local board is located.* It is probable that the USES will be unable to supply any mathematicians but this is the best way to establish the fact that there is a shortage. If the USES does not have

calls for mathematicians, physicists, engineers, chemists, et cetera, it will have no reason to believe that there is a shortage.

The USES offices are required to refer cases involving scientific personnel to the National Roster. (Information has come to the Secretary to the effect that the USES offices are not now referring cases to the Roster.) If department chairmen believe that the advice given by the local USES office to the local board is inadequate or incorrect, the employer is advised to call the attention of the USES office to the instructions in Sections 5300–5309 of the USES Manual (which contain instructions involving the National Roster). If Roster advice is not sought, the chairman will do well to report all facts in the case directly to the Roster.

Department chairmen are urged to make the original requests for occupational deferment as strong as possible and to make all possible appeals. Despite the curtailment of certain college training programs, chairmen are advised to continue requests for deferment, for the shortage of qualified mathematicians is still critical.

III. Deferment of Undergraduates

In Activity and Occupation Bulletin 33-6 the position is taken that the Army and Navy Training Programs will provide adequately for the needs of the armed forces. Student deferment is for civilian needs in war production and in support of the war effort.

Students who will graduate on or before July 1, 1944, in a considerable number of fields (including engineering branches, bacteriology, chemistry, forestry, geophysics, mathematics, meteorology, optometry, pharmacy, and physics including astronomy) are eligible for occupational deferment on certification by the institution and the National Roster.

An undergraduate who is a full-time student of chemistry, engineering, geology, geophysics, or physics who will graduate after July 1, 1944, will be eligible for occupational deferment, if he is properly certified by his institution and the National Roster and will graduate within 24 months from the date of certification and provided the national quota of 10,000 is not exceeded. Mathematics was not included in the list mentioned above, in spite of the strong protests of our Washington representatives. For undergraduate students in the 18–21 age group, requests for deferment are made in duplicate on DSS Form 42 Special; for students over 22 years of age, Form 42 is used.

A national quota of 10,000 has been established for students who should be occupationally deferred at any one time by reason of pursuing courses of study in chemistry, engineering, geology, geophysics, and physics. Students deferred to graduate on or before July 1, 1944, and students deferred for reasons other than pursuing a course of study will not be counted against this quota.

Requests for information needed by the National Roster to set up institutional quotas, et cetera, have already gone out to college presidents. Information regarding the procedures to be followed by institutions and departments con-

cerned will be issued by the National Roster to college presidents in the near future. The departments concerned should be on the lookout for such information.

Department chairmen are requested to inform the Secretary of cases in which mathematicians are improperly classified. In this manner the War Policy Committee, through its Washington representatives, will be better able to present the needs of the mathematicians.

January 17, 1944.

J. R. KLINE,
Secretary, War Policy Committee

THE MATHEMATICAL ASSOCIATION OF AMERICA

THE TWENTY-SEVENTH ANNUAL MEETING OF THE ASSOCIATION

The twenty-seventh annual meeting of the Mathematical Association of America was held at Chicago, Illinois, on Saturday and Sunday, November 27 and 28, 1943, in conjunction with meetings of the American Mathematical Society. About two hundred and seven persons attended the meetings, including the following one hundred and thirty-six members of the Association:

R. P. AGNEW, Cornell University
A. A. ALBERT, University of Chicago
C. B. ALLENDOERFER, Haverford College
MAX ASTRACHAN, Antioch College
H. G. AYRE, Western Illinois State Teachers College
W. L. AYRES, Purdue University

R. W. BABCOCK, Kansas State College
RUTH MASON BALLARD, Wright Junior College
HARRY BATEMAN, California Institute of Technology
S. F. BIBB, Illinois Institute of Technology
B. H. BISSINGER, Michigan State College
L. M. BLUMENTHAL, University of Missouri
H. C. BOARDMAN, Chicago Bridge and Iron Company
J. M. BOOKSTEIN, Army Air Forces
FANNIE W. BOYCE, Wheaton College
ANGELINE J. BRANDT, Wheaton College
R. W. BRINK, University of Minnesota
E. L. BUELL, Northwestern University
H. E. BURNS, Indiana University Extension
HERBERT BUSEMANN, Illinois Institute of Technology

W. D. CAIRNS, University of New Mexico
W. B. CARVER, Cornell University
W. B. CATON, Illinois Institute of Technology
W. F. CHENEY, JR., University of Connecticut
E. W. CHITTENDEN, University of Iowa
LAURA E. CHRISTMAN, Senn High School, Chicago
R. V. CHURCHILL, University of Michigan
A. B. COBLE, University of Illinois
L. W. COHEN, University of Kentucky
N. B. CONKWRIGHT, University of Iowa
A. H. COPELAND, University of Michigan
J. J. CORLISS, De Paul University
D. R. CURTISS, Northwestern University

JOHN DECICCO, Illinois Institute of Technology
JESSE DOUGLAS, Brooklyn College
ARNOLD DRESDEN, Swarthmore College
D. M. DRIBIN, Signal Corps

J. P. ESPOSITO, Crane Technical High School, Chicago
H. J. ETTLINGER, University of Texas
H. P. EVANS, University of Wisconsin
H. S. EVERETT, University of Chicago

G. M. EWING, University of Missouri

J. V. FINCH, University of Chicago
N. J. FINE, Purdue University
EDWARD FLEISHER, Brooklyn College
L. R. FORD, Illinois Institute of Technology
J. S. FRAME, Michigan State College
T. C. FRY, Bell Telephone Laboratories

J. S. GEORGES, Wright Junior College
J. W. GIVENS, JR., Northwestern University
MICHAEL GOLOMB, Purdue University
CORNELIUS GOUWENS, Iowa State College
L. M. GRAVES, University of Chicago
V. G. GROVE, Michigan State College

D. W. HALL, University of Maryland
P. R. HALMOS, Syracuse University
R. W. HAMMING, University of Illinois
W. L. HART, University of Minnesota
M. H. HEINS, Illinois Institute of Technology
E. D. HELLINGER, Northwestern University
EDWARD HELLY, Illinois Institute of Technology
E. H. C. HILDEBRANDT, Northwestern University
T. H. HILDEBRANDT, University of Michigan
A. S. HOUSEHOLDER, University of Chicago
H. K. HUGHES, Purdue University
RALPH HULL, University of Nebraska
M. GWENETH HUMPHREYS, Sophie Newcomb College
MILDRED HUNT, Illinois Wesleyan University

M. H. INGRAHAM, University of Wisconsin

B. W. JONES, Cornell University

DORA E. KEARNEY, State Teachers College, Cedar Falls, Iowa
M. W. KELLER, Purdue University
J. R. KLINE, University of Pennsylvania
L. A. KNOWLER, University of Iowa
W. C. KRATHWOHL, Illinois Institute of Technology

E. P. LANE, University of Chicago
LUISE LANGE, Woodrow Wilson Junior College
R. E. LANGER, University of Wisconsin
C. G. LATIMER, University of Kentucky
D. H. LEAVENS, University of Chicago
MAYME I. LOGSDON, University of Chicago
A. T. LONSETH, Iowa State College

JANET MACDONALD, Hinds Junior College
C. C. MACDUFFEE, University of Wisconsin
MORRIS MARDEN, University of Wisconsin
W. T. MARTIN, Syracuse University
J. N. MCCLELLAND, Drake University
GLADYS B. MCCOLGIN, Indianapolis, Ind.
KARL MENGER, University of Notre Dame
E. J. MICKLE, Ohio State University
H. J. MISER, Illinois Institute of Technology
C. W. MORAN, Lane Technical School, Chicago
MARSTON MORSE, Institute for Advanced Study
E. J. MOULTON, Northwestern University

A. L. NELSON, Wayne University
C. V. NEWSOM, University of New Mexico
IVAN NIVEN, Purdue University
E. A. NORDHAUS, Michigan State College
E. P. NORTHROP, University of Chicago

ISAAC OPATOWSKI, Illinois Institute of Technology

W. V. PARKER, Louisiana State University
B. C. PATTERSON, Hamilton College

TIBOR RADÓ, Ohio State University
G. T. RAINICH, University of Michigan
RUTH B. RASMUSEN, Woodrow Wilson Junior College
C. B. READ, University of Wichita
W. T. REID, University of Chicago
HAIM REINGOLD, Illinois Institute of Technology
W. H. ROEVER, Washington University

R. G. SANGER, University of Chicago
E. W. SCHREIBER, Western Illinois State Teachers College
A. A. SCHWARTZ, Textile Evening High School, New York, N. Y.
M. E. SHANKS, University of Missouri
JACK SILBER, Illinois Institute of Technology
H. A. SIMMONS, Northwestern University
G. W. SMITH, University of Kansas
I. S. SOKOLNIKOFF, University of Wisconsin
C. F. STEPHENS, Great Lakes Naval Training Station
M. H. STONE, Harvard University
E. B. STOUFFER, University of Kansas
J. L. SYNGE, Ohio State University

R. M. THRALL, University of Michigan

W. R. UTZ, University of Notre Dame

H. S. WALL, Northwestern University
WARREN WEAVER, Rockefeller Foundation
K. W. WEGNER, Carleton College
J. V. WEHAUSEN, University of Missouri
E. T. WELMERS, Michigan State College
M. E. WESCOTT, Northwestern University
G. T. WHYBURN, University of Virginia
L. R. WILCOX, Illinois Institute of Technology
R. L. WILDER, University of Michigan
K. P. WILLIAMS, Indiana University
A. H. WILSON, Haverford College
R. S. WOLFE, Northwestern University

J. H. ZANT, Oklahoma A. and M. College

The meetings were held in the auditorium of the Museum of Science and Industry in Jackson Park at 57th Street and South Shore Drive, and the hotel headquarters were at the Hotels Windermere. Lunches were available at a convenient cafeteria in the Museum.

Sessions of the American Mathematical Society were held on Friday and on Saturday morning and afternoon. On Friday at 7:45 p.m. the seventeenth Josiah Willard Gibbs Lecture was given by Professor Harry Bateman on the topic, "The control of elastic fluids." Two addresses were given by invitation, one by Professor Reinhold Baer, at 11:30 a.m. on Friday, on "The higher commutator subgroups," and the other by Professor Marston Morse, at 2:00 p.m. Saturday, on "New settings for topology in analysis."

The Mathematical Association held sessions Saturday evening and Sunday morning, the program having been prepared by a committee consisting of A. H. Copeland, L. R. Ford, and W. T. Reid, *Chairman*. The program follows.

FIRST SESSION OF THE ASSOCIATION

1. "College mathematics during reconstruction," retiring presidential address by Professor R. W. Brink, University of Minnesota.
2. "Focal properties of optical and electromagnetic systems of revolution" by Professor J. L. Synge, Ohio State University.

SECOND SESSION OF THE ASSOCIATION

3. "Mathematics textbooks and the introductory courses in college mathematics" by Professor H. P. Evans, University of Wisconsin.
4. "The nature of mathematical proof" by Professor R. L. Wilder, University of Michigan.
5. "Mathematical biophysics and the central nervous system" by Professor A. S. Householder, University of Chicago.

MEETINGS OF THE BOARD OF GOVERNORS

The Board met Saturday afternoon at 3:30 and for a few minutes at noon on Sunday.

The following seventeen persons were elected to membership on applications duly certified:

R. F. BELDING, A.B.(Amherst) Instr., Vermont Acad., Saxtons River, Vt.

RUTH A. BRENDEL, A.B.(Buffalo) Instr., Univ. of Buffalo, Buffalo, N. Y.

SAMUEL BROYDE, J.D.(De Paul) Instr., Northwestern Univ., Evanston, Ill.
J. W. CRISPIN, JR. Student, Univ. of Michigan, Ann Arbor, Mich.
J. M. DANSKIN, JR., A.B.(U.C.L.A.) Seaman, Second Class, U.S.N.R., Wright Jr. Coll., Chicago 34, Ill.
E. S. J. GOUGH, B.A., B.Ed.(Montreal) Prof., Math. and Sci., Jacques Cartier Normal School, Montreal, P. Q., Canada
A. E. HALLERBERG, A.M.(Illinois) Instr., Illinois Coll., Jacksonville, Ill.
MARY A. LEE, A.M.(Wisconsin) Grad. Asst., Cornell Univ., Ithaca, N. Y.
C. E. MILLER, Ph.D.(Toronto) Prof., Univ. of New Brunswick, Fredericton, N. B., Canada
AGNES L. MORGAN, A.M.(Columbia) Teacher, Brackenridge High School, San Antonio, Tex.
IRMA R. MOSES, A.B.(Cornell) Grad. Asst., Cornell Univ., Ithaca, N. Y.
JAMES PALLADINO, B.S.(Long Island Univ.) Instr., Physics, Long Island Univ., Brooklyn, N. Y.
J. S. ROSEN, Ph.D.(Washington Univ.) Asso. Prof., Eastern New Mexico Coll., Portales, N. M.
F. E. ROTHCHILD, M.S.(Louisiana State) Prof., Math. and Phys. Educ., Arkansas A. and M. Coll., Monticello, Ark.
C. T. RUDDICK, Ph.D.(Pennsylvania) Asst. Prof., Mount Union Coll., Alliance, Ohio
BENJAMIN SLEPIN, LL.B.(South Jersey Law Sch.) Instr., Frankford Arsenal Trade School, Philadelphia, Pa.
ADRIAN STRUYK, M.E.(Stevens Inst. of Tech.), A.M.(Columbia) Teacher, High School, Clifton, N. J.

The Secretary reported the deaths of the following members during the year 1943:

W. M. Carruth, Professor of mathematics, Hamilton College. (January 23, 1943)
C. H. Currier, Associate Professor emeritus of mathematics, Brown University. (January 5, 1943)
H. H. Dalaker, Professor emeritus of mathematics, University of Minnesota. (May 20, 1943)
E. L. Dodd, Professor of actuarial mathematics, University of Texas. (January 9, 1943)
Guido Fubini, Institute for Advanced Study. (June 6, 1943)
Lillian Hackney, Professor emeritus of mathematics, Marshall College. (February 4, 1943)
G. A. Harter, Professor emeritus of mathematics, University of Delaware. (July 22, 1943)
H. E. Hawkes, Dean, Columbia University. (May 4, 1943)
E. R. Hedrick, recently Vice President and Provost, University of California at Los Angeles. (February 3, 1943)
L. L. Locke, Brooklyn. (August 28, 1943)
R. L. Menuet, Professor of mathematics, Tulane University. (May 9, 1943)
H. H. Mitchell, Professor of mathematics, University of Pennsylvania. (March 13, 1943)
A. G. Rau, Dean, Moravian College. (1942)
N. S. Risley, Assistant Professor of mathematics, Fenn College. (December 30, 1942)
H. M. Showman, Lecturer and Registrar, University of California at Los Angeles. (June 24, 1943)
Clara E. Smith, Professor emeritus of mathematics, Wellesley College. (May 12, 1943)
E. B. Van Vleck, Professor emeritus of mathematics, University of Wisconsin. (June 2, 1943)
J. J. Westemeier, Professor of mathematics, Dowling College. (February 1, 1943)
J. E. Williams, Dean and Professor of mathematics, Virginia Polytechnic Institute. (April 19, 1943)

The Secretary-Treasurer announced that he would prepare a complete financial report for the year after January 1, 1944, and that this report would be published in the MONTHLY. He reported that the financial operations for the

year would apparently compare favorably with the budget items as estimated at the beginning of the year.

Letters were read from several members protesting against the holding of sessions of the Association on Sunday. No action was taken on the question; but in the discussion the opinion was expressed that such Sunday sessions were necessitated by war conditions, and that there was no intention to continue the practice after the present emergency.

It was voted, on the request of the Secretary-Treasurer, to have a thorough audit of the accounts of the Association at the end of the year, the Finance Committee to arrange for such an audit.

H. M. Gehman of the University of Buffalo was elected a member of the Finance Committee for a term of four years to succeed R. E. Langer.

The Editor-in-Chief nominated and the Board elected the following associate editors of the MONTHLY for the year 1944:

E. F. BECKENBACH	H. P. EVANS	M. R. HESTENES
L. M. BLUMENTHAL	B. F. FINKEL	B. W. JONES
N. B. CONKWRIGHT	J. S. FRAME	R. E. LANGER
H. S. M. COXETER	ORRIN FRINK, JR.	C. V. NEWSOM
W. M. DAVIS	MARJORIE GROVES	VIRGIL SNYDER
OTTO DUNKEL		MARIE J. WEISS

At the meeting on Sunday, R. E. Langer was nominated to succeed T. C. Fry as the representative of the Association on the National Research Council for a term of three years beginning July 1, 1944.

ANNUAL BUSINESS MEETING

The annual business meeting of the Association was held on Sunday morning at 9:30, President Cairns presiding.

The results of the election of officers were announced as follows:

First Vice-President for the term 1944–45: W. M. Whyburn, University of California at Los Angeles.

Governors at Large for the term 1944–46: R. G. Sanger, University of Chicago, and Morgan Ward, California Institute of Technology.

W. B. CARVER, Secretary-Treasurer

THE FALL MEETING OF THE INDIANA SECTION

The twenty-first annual meeting of the Indiana Section of the Mathematical Association of America was held at Butler University, Indianapolis, Indiana, on Friday, October 29, 1943, in conjunction with the fall meeting of the Indiana Academy of Science. Professor P. M. Pepper, Chairman of the Indiana Section of the Association, presided at the morning session, and Professor W. E. Edington, Chairman of the Mathematics Section of the Academy, presided at the afternoon session.

Forty-three individuals registered at the meetings, including the following

twenty-five members of the Association: Emil Artin, W. L. Ayres, K. W. Crain, J. E. Dotterer, W. E. Edington, P. D. Edwards, Michael Golomb, G. H. Graves, Cora B. Hennel, M. W. Keller, E. L. Klinger, Gladys B. McColgin, H. A. Meyer, G. T. Miller, Paul Muehlman, P. M. Pepper, J. C. Polley, Maxwell Reade, D. A. Rothrock, C. P. Sousley, M. S. Webster, J. W. Wiley, K. P. Williams, H. E. Wolfe, A. J. Zanolar.

At the business meeting the following officers were elected for next year: Chairman, Emil Artin, Indiana University; Vice-Chairman, W. L. Ayres, Purdue University; Secretary, M. W. Keller, Purdue University. It was decided to hold the next meeting in conjunction with the annual meeting of the Indiana Academy of Science.

The following papers were presented:

1. *Isohedral polyhedra*, by Leon Alaoglu and Dr. J. H. Giese, Purdue University, introduced by Professor W. L. Ayres.

In this paper isohedral and isogonal polyhedra were defined, and it was indicated that the classical regular polyhedra are both isohedral and isogonal. Attention was then directed to polyhedra which are isohedral, but which have isogonality replaced by the weaker property that equal numbers of faces meet at the vertices. It was pointed out that for finite polyhedra of genus zero (topological spheres) the Euler polyhedron formula reduces the possibilities to the usual five, ranging from tetrahedron to icosahedron. Constructions using the greatest possible numbers of unequal edges per face were devised to show the existence of all of these five types except the icosahedron with scalene triangular faces. In the case of genus one (topological tori) the Euler formula reduces the possibilities to triangular, quadrilateral, and hexagonal faces. Constructions were devised to establish the existence of isohedral tori with $12n$ $(n \geqq 3)$ triangular faces, and of isohedral tori with $8n$ $(n \geqq 4)$ quadrilateral faces.

2. *The elementary functions*, by Professor Emil Artin, Indiana University.

Professor Artin showed how to introduce the elementary functions e^x, $\log x$, $\cos x$ and $\sin x$ in a completely rigorous manner by using only the simplest facts relating to limits. The results thus obtained cover all properties of these functions, including the infinite product for $\sin x$. It is thus possible to have all these functions available from the beginning in a course in advanced calculus.

3. *A method for the solution of algebraic or transcendental equations*, by Dr Michael Golomb, Purdue University.

The speaker pointed out that the familiar methods for the solution of equations have certain shortcomings. (Newton's and Horner's methods apply only to real roots, while Graeffe's method applies only to algebraic equations, *etc.*) He derived a new method based upon Hadamard's investigations of the singularities of functions defined by Taylor series. The symmetric functions of the zeros of smallest absolute value were given as limits of quotients of persymmetric determinants involving successive coefficients in the Maclaurin expansion of the reciprocal of the function.

4. *Some developments in the analytic theory of continued fractions*, by Dr. Marion Wetzel, Indiana University, introduced by Professor K. P. Williams.

This address dealt with certain recent contributions to the analytic theory of continued fractions. These contributions have attempted to bring together many isolated results, and fit them into a larger analytic structure. The speaker regarded the continued fraction as an infinite sequence of linear fractional transformations in the complex plane. The class of continued fractions

$$\frac{1}{b_1 + z} - \frac{a_1^2}{b_2 + z} - \frac{a_2^2}{b_3 + z} - \cdots, \quad (a_p \neq 0)$$

for which the quadratic form

$$\sum_{p=1}^{n} \mathfrak{I}(b_p + Z)X_p^2 - 2\sum_{p=1}^{n-1} \mathfrak{I}(a_p)X_pX_{p-1}$$

is positive definite for all values of $\mathfrak{I}(z) > 0$ was discussed. These continued fractions include the classical case in which $\mathfrak{I}(b_p) = \mathfrak{I}(a_p) = 0$, and also the case $\mathfrak{I}(b_p) \geqq 0$, $\mathfrak{I}(a_p) = 0$, discussed in a paper by Hellinger and Wall in the *Annals of Mathematics*, vol. 44, 1943, pp. 103–127. Necessary and sufficient conditions for positive definiteness in terms of the imaginary parts of the coefficients in the quadratic form were given. The speaker cited some applications of this characterization, including connections with theorems on convergence regions.

5. *Remarks on surfaces*, by Professor J. W. T. Youngs, Purdue University, introduced by Professor M. W. Keller.

Professor Youngs made some expository comments on classical and modern surface theory.

The afternoon session was devoted to a panel discussion of the mathematics training offered for the armed forces in Indiana colleges. The following topics were brought before the meeting:

I. *The content of A. S. T. P. courses with special attention to the relegation of theory to a place of minor importance*, by Professor W. L. Ayres, Purdue University.

II. *Standards for the retention of men in the A. S. T. P.*, by Professor K. P. Williams, Indiana University.

III. *Content of the V-12 program (excluding navigation)*, by Professor J. C. Polley, Wabash College.

IV. *Navigation in the V-12 program*, by Professor R. F. McDaid, Indiana State Teachers College.

V. *Content of courses in the advanced navy program in light of the preparation of the students enrolled*, by Professor Paul Pepper, University of Notre Dame.

VI. *Army pre-flight courses*, by Professor J. L. Beal, Butler University.

VII. *Navy pre-flight courses*, by Professor W. E. Edington, DePauw University.

VIII. *Navy primary and secondary flight courses*, by Professor P. D. Edwards, Ball State Teachers College.

M. W. KELLER, *Secretary*

CALENDAR OF FUTURE MEETINGS

Twenty-Seventh Summer Meeting, Wellesley, Mass., August 12–14, 1944.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN, Pittsburgh, Pa., April, 1944
ILLINOIS, Normal, Ill., May 12–13, 1944
INDIANA, Indianapolis, November 10, 1944
IOWA, Cedar Rapids, April 15, 1944
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK, New York, April 22, 1944
MICHIGAN, Ann Arbor, March 18, 1944
MINNESOTA
MISSOURI
NEBRASKA, Lincoln, May 6, 1944
NORTHERN CALIFORNIA
OHIO, Columbus, April 6, 1944
OKLAHOMA
PHILADELPHIA, Philadelphia, November, 1944
ROCKY MOUNTAIN, Greeley, Colo., April 14–15, 1944
SOUTHEASTERN
SOUTHERN CALIFORNIA, Los Angeles, March 11, 1944
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE
WISCONSIN, Milwaukee, May, 1944

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TYPES OF SYMMETRIES*

T. L. WADE, Florida State College for Women
R. H. BRUCK, University of Wisconsin

1. Symmetric and skew-symmetric matrices. It is well known that a matrix $A=(A_{ij})$ can be written as the sum of a symmetric and of a skew-symmetric matrix: $A=S+Q$, where

$$S = \tfrac{1}{2}(A + A')$$

is symmetric, and

$$Q = \tfrac{1}{2}(A - A')$$

is skew-symmetric. Here A' denotes the transpose of A. In index notation $A_{ij}=S_{ij}+Q_{ij}$, where

$$S_{ij} = \tfrac{1}{2}(A_{ij} + A_{ji}) \quad \text{and} \quad Q_{ij} = \tfrac{1}{2}(A_{ij} - A_{ji}). \tag{1}$$

The matrix A_{ij} is symmetric if $Q_{ij}=0$, and skew-symmetric if $S_{ij}=0$. It is significant that a symmetric matrix will transform under a linear transformation into a symmetric matrix, and a skew-symmetric matrix will transform into a skew-symmetric matrix.

2. Tensors. In the consideration of types of symmetries for ordered arrays which are invariant under general linear transformations, it is convenient to use the viewpoint and terminology of tensor algebra. If the coordinate transformation is

$$\bar{x}^k = T^k_i x^i \qquad (i = 1, 2, \cdots, n),$$

a 2-way matrix A_{ij} is a covariant tensor of order two if its components satisfy

$$A_{ij} = \left| T \right|^w T^k_i T^l_j \bar{A}_{kl},$$

for some constant w, where $|T|$ is the determinant of the transformation matrix T^k_i. The fact that the types of symmetries discussed in Section 1 are invariant under a general linear transformation may be expressed by saying that these symmetries have tensorial significance. Furthermore, for a 2-indexed tensor they are the only types of symmetries which do have tensorial significance.

More generally, if an ordered array of scalars, or point functions, $A_{i_1i_2\cdots i_p}$ transforms according to the law

$$A_{i_1i_2\cdots i_p} = \left| T \right|^w T^{k_1}_{i_1} T^{k_2}_{i_2} \cdots T^{k_p}_{i_p} \bar{A}_{k_1k_2\cdots k_p},$$

when the variable coordinate transformation is $\bar{x}^k=T^k_i x^i$, then $A_{i_1i_2\cdots i_p}$ is a covariant tensor of order p. In this paper we shall discuss general types of symmetries, of which complete symmetry and complete skew-symmetry are special cases, and shall consider some applications of these general types of symmetries

* Presented to the American Mathematical Society, Dec. 28, 1942.

to tensors of order greater than two. These general classes of symmetries were first explored by A. Young [1], and later studied by J. A. Schouten [2], by J. von Neumann [3], by H. Weyl [4], [5], and more recently by the present writers [6], [7].

3. Symmetries for third order tensors. From an arbitrary 3-indexed tensor A_{ijk} may be obtained a completely symmetric tensor

$$S_{ijk} = \frac{1}{3!}\,[A_{ijk} + A_{ikj} + A_{jki} + A_{jik} + A_{kij} + A_{kji}], \tag{2}$$

and a completely skew-symmetric tensor

$$Q_{ijk} = \frac{1}{3!}\,[A_{ijk} - A_{ikj} + A_{jki} - A_{jik} + A_{kij} - A_{kji}]. \tag{3}$$

However

$$R_{ijk} \equiv A_{ijk} - (S_{ijk} + Q_{ijk}) \tag{4}$$

is not the zero tensor. In fact

$$R_{ijk} = \frac{2}{3!}\,[2A_{ijk} - A_{jki} - A_{kij}] \tag{5}$$

The tensor R_{ijk} possesses no very obvious type of symmetry; however

$$R_{ijk} = S^{(1)}_{ijk} + S^{(2)}_{ijk}, \tag{6}$$

where

$$\begin{aligned} S^{(1)}_{ijk} &= \frac{2}{3!}\,\{(A_{ijk} - A_{kji}) + (A_{jik} - A_{kij})\}; \\ S^{(2)}_{ijk} &= \frac{2}{3!}\,\{(A_{ijk} - A_{jik}) + (A_{kji} - A_{jki})\}. \end{aligned} \tag{7}$$

The tensor $S^{(1)}_{ijk}$ is symmetric in i and j, and the tensor $S^{(2)}_{ijk}$ is symmetric in i and k. Both of these tensors clearly would vanish if A_{ijk} were completely symmetric or completely skew-symmetric. Similarly

$$R_{ijk} = Q^{(1)}_{ijk} + Q^{(2)}_{ijk} \tag{8}$$

where

$$\begin{aligned} Q^{(1)}_{ijk} &= \frac{2}{3!}\,\{(A_{ijk} + A_{jik}) - (A_{kji} + A_{jki})\}; \\ Q^{(2)}_{ijk} &= \frac{2}{3!}\,\{(A_{ijk} + A_{kji}) - (A_{jik} + A_{kij})\}. \end{aligned} \tag{9}$$

The tensor $Q^{(1)}_{ijk}$ is skew-symmetric in i and k, and the tensor $Q^{(2)}_{ijk}$ is skew-symmetric in i and j, and each of these tensors would vanish if A_{ijk} were completely symmetric or completely skew-symmetric. The tensor R_{ijk} can be written in other ways as the sum of two "partially symmetric" tensors, or of two "partially skew-symmetric" tensors; but the two relations (6) and (8) suffice to show that R_{ijk} has its peculiar type of symmetry different from the complete symmetry of S_{ijk} and different from the complete skew-symmetry of Q_{ijk}.

Analogous to (1) we may write

$$A_{ijk} = S_{ijk} + R_{ijk} + Q_{ijk}. \tag{10}$$

Thus we see that for an arbitrary 3-index tensor to be symmetric it is necessary that the skew-symmetric part Q_{ijk} be zero, but this condition is not sufficient; in addition the part we have designated by R_{ijk} must be zero.

A 4-indexed tensor may be written as a sum of five tensors, corresponding to (10), and each one of these, except for the completely symmetric and completely skew-symmetric tensors, may be written as the sum of several tensors with partial skew-symmetries (symmetries) of different natures. How this is done for a 4-indexed tensor or, more generally, for a p-indexed tensor will be evident from a discussion of A. Young's symmetry operators.

4. Young's symmetry operators. With p letters we may construct the tableau

$$[\alpha] = [\alpha_1, \alpha_2, \cdots, \alpha_h]: \quad \begin{matrix} a_{11} & a_{12} \cdots a_{1\alpha_1} \\ a_{21} & a_{22} \cdots a_{2\alpha_2} \\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ a_{h1} & a_{h2} \cdots a_{h\alpha_h} \end{matrix}$$

where $\alpha_i \geqq \alpha_{i+1} > 0$ for all i, and $\sum_{i=1}^{h} \alpha_i = p$; the letters a_{ij} stand for the numbers 1 to p in some order. There is a tableau for each partition of p; the tableau and the partition, and in fact the corresponding irreducible representation of the symmetric group on p letters, are represented by the same symbol $[\alpha]$. In this connection it is convenient to use an exponential notation when two or more adjacent α's are equal. Thus $[2, 1^2]$ denotes the partition $[2, 1, 1]$ of 4.

From the ith row of $[\alpha]$ we can construct the substitutional operator $\{a_{i1}a_{i2} \cdots a_{i\alpha_i}\}$, which represents the sum of all permutations of the symmetric group on the α_i letters exhibited between the brackets; the product of these h operators we denote by P_α. Similarly from the jth column of $[\alpha]$ we construct the operator $\{a_{1j}a_{2j} \cdots \}'$, which represents the sum of all the even permutations minus the sum of all the odd permutations of the symmetric group on the letters standing in the jth column; the product of these column operators we denote by N_α. With any arrangement of the p letters in the tableau form $[\alpha]$ we may associate the product $P_\alpha N_\alpha$ or the product $N_\alpha P_\alpha$.

With the partition $[\alpha] = [2, 1]$ of 3 we may construct the tableau form $\begin{matrix} a_1 a_2 \\ a_3 \end{matrix}$ and the 3! particular tableaux

$$(11)\qquad \begin{array}{l}(1)\\ 12\\ 3\end{array}\qquad \begin{array}{l}(2)\\ 13\\ 2\end{array}\qquad \begin{array}{l}(3)\\ 21\\ 3\end{array}\qquad \begin{array}{l}(4)\\ 23\\ 1\end{array}\qquad \begin{array}{l}(5)\\ 32\\ 1\end{array}\qquad \begin{array}{l}(6)\\ 31.\\ 2\end{array}$$

Let $P_t \equiv P_\alpha^{(t)}$ and $N_t \equiv N_\alpha^{(t)}$ be associated with the tableau numbered (t). Then $P_1 = \{12\}\{3\} = \{12\} = 1 + (12)$; $N_1 = \{13\}'\{2\}' = \{13\}' = 1 - (13)$; and

$$P_1N_1 = [1 + (12)][1 - (13)] = 1 + (12) - (13) - (123).$$

Here we use the parentheses ( ) in the conventional manner of circular permutations. Using P_1N_1 as a substitutional operator on the indices of A_{ijk} we obtain

$$\begin{aligned} A_{ijk}\cdot P_1N_1 &= A_{ijk}\cdot 1 + A_{ijk}(12) - A_{ijk}(13) - A_{ijk}(123) \\ &= A_{ijk} + A_{jik} - A_{kji} - A_{jki}. \end{aligned}$$

We note that

$$(12)\qquad {}_{[\alpha]}A_{ijk}^{(1)} \equiv A_{ijk}\cdot\frac{2}{3!}P_1N_1 = \frac{2}{3!}[(A_{ijk} + A_{jik}) - (A_{kji} + A_{jki})]$$

is the tensor $Q_{ijk}^{(1)}$ of (9). Similarly

$$P_2N_2 = [1 + (13)][1 - (12)] = 1 + (13) - (12) - (132)$$

and

$$(13)\qquad {}_{[\alpha]}A_{ijk}^{(2)} \equiv A_{ijk}\cdot\frac{2}{3!}P_2N_2 = \frac{2}{3!}[(A_{ijk} + A_{kji}) - (A_{jik} + A_{kij})]$$

is the tensor $Q_{ijk}^{(2)}$ of (9).

Reversing the order of P_1 and N_1 we get

$$N_1P_1 = [1 - (13)][1 + (12)] = 1 + (12) - (13) - (132).$$

The corresponding tensor $A_{ijk}\cdot\frac{2}{3!}N_1P_1$ is the tensor $S_{ijk}^{(1)}$ of (7); likewise $A_{ijk}\cdot\frac{2}{3!}N_2P_2$ is the tensor $S_{ijk}^{(2)}$ of (7).

Note that the tensor $A_{ijk}\cdot\frac{2}{3!}P_1N_1$ is skew-symmetric in i, k, its first and third indices; this is a consequence of the fact that we operated last with N_1. Similarly the tensor $A_{ijk}\cdot\frac{2}{3!}N_1P_1$ is symmetric in i, j, its first and third indices, since we operated last with P_1. This illustrates the following fact: *The tensor obtained by operating on $A_{i_1i_2\cdots i_p}$ with the operator $P_\alpha^{(t)}N_\alpha^{(t)}$ for a given tableau is skew-symmetric in each set of indices belonging to the same column of the tableau, and the tensor obtained by operating on $A_{i_1i_2\cdots i_p}$ with the corresponding $N_\alpha^{(t)}P_\alpha^{(t)}$ is symmetric in each set of indices belonging to the same row of the tableau.*

Taking the sum of the 3! products P_tN_t corresponding to the tableau (11) (or of the 3! products N_tP_t) we obtain

$$(14)\qquad 3[2 - (123) - (132)]$$

which, when used as an operator on A_{ijk} yields the tensor R_{ijk} of (5) except for

a constant factor. But the operator (14) may also be obtained, aside from the factor 3, by taking the sum of P_1N_1 and P_2N_2 (or the sum of N_1P_1 and N_2P_2). In general we write

$$T_\alpha = \left(\frac{f_\alpha}{P!}\right)^2 \sum_{t=1}^{P!} P_\alpha^{(t)} N_\alpha^{(t)},$$

where the constant f_α is given by

$$(15) \qquad f_\alpha = p! \frac{\prod_{r<s}^{h} (l_r - l_s)}{\prod_{r=1}^{h} l_r!}, \qquad l_r = \alpha_r + h - r(r = 1, \cdots, h).$$

For $p=2, f_{[2]}=1; f_{[1^2]}=1$. For $p=3, f_{[3]}=1; f_{[2,1]}=2, f_{[1^3]}=1$. For $p=4, f_{[4]}=1$; $f_{[3,1]}=3; f_{[2^2]}=2; f_{[2,1^2]}=3; f_{[1^4]}=1$. Alternatively, f_α is the degree of the irreducible representation $[\alpha]$, and this may be obtained as the characteristic of (1^p) from the character table of the symmetric group on p letters [8].

A tableau $[\alpha]$ is defined to be standard if the letters in each row and column appear in the order of some given sequence. Thus of the tableaux (11) those designated as (1) and (2) may be taken as standard. It can be shown that exactly f_α of the $p!$ tableaux are standard. Moreover, it can be shown that T_α may be expressed in terms of the standard tableaux, this expression being

$$(16) \qquad T_\alpha = \frac{f_\alpha}{P!}\left[P_1N_1M_1 + P_2N_2M_2 + \cdots + P_fN_fM_f\right].$$

The permutation operators M_t are introduced so that the f_α "prepared" operators $P_tN_tM_t$ are orthogonal. For all cases for $p<5$ the M's are each the identity permutation. In some cases, as for example when $p=5$, $[\alpha]=[3, 2]$ and $[\alpha] = [2^2, 1]$, some of the M's are not the identity and must be determined [9].

5. Symmetries for fourth order tensors. As we have said above, a 4-indexed tensor $A_{i_1i_2i_3i_4}$ may be written uniquely as the sum of five tensors corresponding to the five partitions of the number 4. For $[\alpha]=[2^2], f_\alpha=2$, and we may take as our two standard forms $\begin{smallmatrix}12\\34\end{smallmatrix}$ and $\begin{smallmatrix}13\\24\end{smallmatrix}$. One may obtain ${}_{[2^2]}A_{i_1i_2i_3i_4}$ from $A_{i_1i_2i_3i_4}$ by applying to the latter the operator

$$(17) \qquad \frac{2}{4!}\left[\{12\}\{34\}\{13\}'\{24\}' + \{13\}\{24\}\{12\}'\{34\}'\right].$$

This gives

$$(18) \quad {}_{[2^2]}A_{i_1i_2i_3i_4} = \frac{2}{4!}\left[2A^{(1^4)}_{i_1i_2i_3i_4} + 0\cdot A^{(1^2,2)}_{i_1i_2i_3i_4} - 1\cdot A^{(1,3)}_{i_1i_2i_3i_4} + 0\cdot A^{(4)}_{i_1i_2i_3i_4} + 2\cdot A^{(2^2)}_{i_1i_2i_3i_4}\right],$$

where $A^{(\rho)}_{i_1i_2i_3i_4}$ is the sum of all tensors obtained from $A_{i_1i_2i_3i_4}$ by subjecting the

indices to the different permutations of class (ρ). The coefficient of $A^{(\rho)}_{i_1i_2i_3i_4}$ within the brackets is the characteristic for class (ρ) of the irreducible representation $[2^2]$. (See paper [6].) Evidently we could have avoided the use of Young's diagrams, and worked with the character table instead, provided we were interested only in the tensor corresponding to the Young tableau form. However, a tensor associated with a particular Young diagram is perhaps best considered in relation to the corresponding Young operator.

6. Number of independent components. The number of independent (scalar) components of $A_{i_1i_2\cdots i_p}$ is n^p, n being the dimension of the coordinate system. If c_α denotes the number of components of ${}_{[\alpha]}A_{i_1i_2\cdots i_p}$, then

$$n^p = c_{[p]} + \cdots + c_{[\alpha]} + \cdots + c_{[1^p]}. \tag{19}$$

For $p=2$ the relation connecting the dimensionalities of the special symmetric tensors is

$$n^2 = \frac{n(n+1)}{2} + \frac{n(n-1)}{2},$$

and for $p=3$ the relation is

$$n^3 = \frac{n(n+1)(n+2)}{3!} + \frac{4n(n^2-1)}{3!} + \frac{n(n-1)(n-2)}{3!}.$$

Schouten [2] has obtained expressions for the c_α's in terms of n for $p=4$, but the difficulties of his method become great for larger values of p. From the definition of the rank r_α, of the immanent tensor ${}_{[\alpha]}I^{i_1i_2\cdots i_p}_{j_1j_2\cdots j_p}$ introduced in [6] as given in Section 4 of [10], it has been shown [7] that $c_\alpha = r_\alpha$. Since r_α may be read off from the character table for the symmetric group on p letters, it can be said that expressions for c_α are readily obtainable for those values of p for which the character tables are available, these being for $p \leqq 13$, [11], [12], [13].

Knowing the number of independent scalar components, c_α, of a tensor ${}_{[\alpha]}A_{i_1i_2\cdots i_p}$ corresponding to a Young T_α, we may obtain the number of independent scalar components, ${}_tc_\alpha$, in the tensor corresponding to a particular Young diagram. For as a consequence of Theorems VI and VIII of [10] we have

$$ {}_tc_\alpha = \frac{1}{f_\alpha}\cdot r_\alpha. \tag{20}$$

The number of independent scalar components [7] in the tensor ${}_{[2^1]}A_{i_1i_2i_3i_4}$ is

$$c_{[2^2]} = n\binom{n+1}{3},$$

and the number of such components in the tensor corresponding to the particular Young diagram $\substack{12\\34}$ is

$${}_1c_{[2^2]} = \frac{n}{2}\binom{n+1}{3}$$

References

1. A. Young, On quantitative substitutional analysis, Part I, Proceedings of the London Mathematical Society (1), vol. 33, 1901; Part II, *ibid.*, (1), vol. 34, 1902, pp. 361–397.
2. J. A. Schouten, Der Ricci-Kalkul, Berlin, 1924, Chapter VII.
3. J. von Neumann, §127, van der Waerden, Moderne Algebra II, Berlin, 1931.
4. H. Weyl, The Theory of Groups and Quantum Mechanics, New York, 1932, Ch. V.
5. ———, The Classical Groups, Princeton, 1939, pp. 96–136.
6. T. L. Wade, Tensor algebra and Young's symmetry operators, American Journal of Mathematics, vol. 63, 1941, pp. 645–657.
7. R. H. Bruck and T. L. Wade, The number of independent components of the tensors of given symmetry type, Bulletin American Mathematical Society, vol. 49, 1943, pp. 470–472.
8. F. D. Murnaghan, The Theory of Group Representations, Baltimore, 1938.
9. A. Young, On quantitative substitutional analysis, Journal of the London Mathematical Society, vol. 3, 1928, pp. 14–19.
10. R. H. Bruck and T. L. Wade, Bisymmetric tensor algebra, Part II, American Journal of Mathematics, vol. 64, 1942, pp. 734–753.
11. F. D. Murnaghan, The characters of the symmetric group, American Journal of Mathematics, vol. 59, 1937, pp. 739–753.
12. M. Zia-ud-Din, The characters of the symmetric group of degrees 12 and 13, Proceedings of the London Mathematical Society (2), vol. 42, 1937, pp. 340–355.
13. D. E. Littlewood, The Theory of Group Characters, Oxford, 1940.

BOOLEAN REPRESENTATION THEORY

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1. Introduction. Recent proofs of Stone's theorem on representations of Boolean algebras [6], [7], have been given by Frink [3] and McCoy-Montgomery [4], [5]. The simplest form of the theorem asserts that every Boolean algebra (or more generally Boolean ring) is isomorphic to a Boolean algebra (or Boolean ring) of sets or classes. The latter is called the representation of the former. Frink's proof made use of Zorn's Lemma [8] and a new set of postulates for Boolean algebras. McCoy and Montgomery used a principle formulated by them concerning the direct product of rings. Stone's original proof [7] made use of detailed transfinite induction and rather specialized algebraic arguments.

It is proposed here to give a composite proof, using methods due to all of the above authors, and providing what seems to be a simplified treatment of the main theory. At the same time, the proof will depend on standard algebraic methods in the theory of ideals. The proof to be outlined follows the general plan of Stone's proof but the crucial arguments are not so specialized as his, nor is the proof as general as that of McCoy and Montgomery. We follow Frink in using Zorn's Lemma. No consideration will be given here to the approach to the representation problem from the standpoint of lattice theory.*

* See Garrett Birkhoff, Lattice Theory, American Mathematical Society, Colloquium Publications, Vol. 25, 1940, Chapters V, VI.

2. Properties of Boolean rings. First, we recall a few elementary facts about Boolean rings as proved by Stone [7]. A Boolean ring is defined as a ring in which every element, a, is idempotent, $aa=a$. It follows that every element is its own additive inverse ($a+a=0$), and the ring is commutative. A Boolean ring with unit element e (such that $ae=a$ for every element a) becomes a Boolean algebra when Boolean addition ($\vee$) is defined by the relation

$$a \vee b = (a + b) + ab$$

and the Boolean complement a' by

$$a' = a + e.$$

It follows that $a+a'=e$, $aa'=0$. A non-trivial Boolean ring (one containing more than one element) is a field if and only if it has two elements, in which case it is abstractly identical with the field F_2 (of integers 0, 1, mod 2).* (A Boolean ring with more than two elements always has divisors of zero.) Every Boolean ring can be imbedded in a Boolean ring with unit element. Hence in order to prove that a representation exists for an arbitrary Boolean ring it is sufficient to prove the same fact for any Boolean ring with unit element. This can be done in the latter case by means of sets of homomorphisms of the ring into F_2.

3. Maximal ideals in a Boolean ring. Hereafter, we use B to denote an arbitrary non-trivial Boolean ring with unit element.

Let a be any element $\neq 0$ belonging to B, and let $J(a)$ be the collection of all ideals in B which fail to contain a (a non-empty collection since the ideal 0 belongs to $J(a)$). Consider any subsystem of ideals of $J(a)$ linearly ordered under set inclusion. The union of all ideals of the subsystem is an ideal belonging to $J(a)$. Zorn's principle† asserts that under these circumstances there exists in the collection $J(a)$ at least one ideal J_m which is maximal in $J(a)$. Thus we have:

THEOREM 1. *For any element $a\neq 0$, belonging to B, there exists in B at least one ideal J_m which is maximal with respect to the property of not containing a.*

We next prove that J_m actually is a maximal ideal in the ring, in the sense that any more inclusive ideal is the entire ring.

THEOREM 2. *The ideal J_m of Theorem 1 is a maximal ideal in the ring B.*

Proof. Let K be any ideal in B such that K contains J_m as a proper sub-ideal. J_m is maximal with respect to the property of not containing a; hence K must

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* The operation tables for F_2 are, of course:

$$0+1=1+0=1, \quad 0+0=1+1=0,$$
$$0 \cdot 1=1 \cdot 0=0 \cdot 0=0, \quad 1 \cdot 1=1.$$

† The principal provides a short-cut method of transfinite induction. It may be stated as follows: If a collection of sets $[S]$ contains the union of every linearly ordered subsystem of $[S]$ (under set inclusion), then there exists at least one set S_m belonging to $[S]$ which is not a proper subset of any other set in the collection, or, briefly, which is *maximal* in $[S]$.

contain a. The complementary element a' either (1) belongs to J_m, or (2) does not belong to J_m.

In the first case, we know at once that $e(=a+a')$ belongs to K. Thus K is the entire ring B, and J_m is a maximal ideal in B.

In the second case, consider the set C of elements of the form $J_m+a'x$ for all x belonging to B. C is an ideal which contains J_m as a proper sub-ideal. Hence, a, as well as a', belongs to C. Thus e belongs to C and we can write

$$e = j + a'y \begin{Bmatrix} \text{for some element } j \text{ of } J_m \\ \text{and some element } y \text{ of } B \end{Bmatrix}.$$

Therefore $$ae = aj + a(a'y) = aj + 0,$$

or $$a = aj.$$

But since j belongs to ideal J_m (which is closed under multiplication by all elements of B), this means that a also must belong to J_m, which is a contradiction.

Thus case (2) is impossible and the conclusion holds that J_m is maximal in B.

COROLLARY. *For any element $a \neq 0$, belonging to B, there exists at least one ideal, J_m, not containing a, which is maximal in B.*

4. The representation. We now refer to a well-known theorem of ideal theory (see Albert [1], p. 255, or Birkhoff-MacLane [2], p. 356) which may be paraphrased as follows: *Given a commutative ring, R, with unit element, and an ideal, J, in R; then the quotient ring R/J is a field if (and only if) J is a maximal ideal in R.* Thus for any element $a \neq 0$, in the Boolean ring, B, there exists a quotient ring, B/J_m, which is a field. But this quotient ring is automatically a Boolean ring so the field must be abstractly equivalent to the field F_2. Hence we can construct a homomorphism, h, of B into F_2 with the elements of J_m mapped on 0 and the complement of J_m, including a and e, mapped on 1.

We can now restate the corollary of the preceding section as follows:

THEOREM 3. *For each element, $a \neq 0$, belonging to B, there exists at least one homomorphism, h, of B into F_2 such that $h(a)$, the image of a in F_2, $=1$.*

COROLLARY. *There fails to exist a homomorphism, h, of B into F_2 such that $h(x)=1$ if and only if $x=0$ in B.*

The desired representation can now be constructed by means of the following two theorems concerning sets of homomorphisms. The operations $\cap$, Δ, are used to designate intersection and symmetric difference* of sets, respectively. The elements a, b, are now any elements of the ring B, not excluding 0.

* The intersection of two sets is the set of all elements common to both. The symmetric difference of two sets is the set of all elements belonging to one or the other of the sets but not both.

THEOREM 4. *Let $H(a)$ be the set of all homomorphisms, h, of B into F_2 such that* $h(a)=1$.

Then (1) $H(ab) \quad = H(a) \cap H(b)$,

(2) $H(a+b) = H(a) \triangle H(b)$,

(3) $H(a) \quad = H(b)$ implies $a=b$.

Proof. 1. $H(ab)=$ [set of all h such that $h(ab)=1$]. But $h(ab)=h(a)\cdot h(b)$ by definition of homomorphism and inasmuch as $h(a)$, $h(b)$ are both elements of F_2 we have

$$h(a)\cdot h(b)=1 \text{ only if } \textit{both } h(a)=1 \textit{ and } h(b)=1.$$

That is, $H(ab)=H(a)\cap H(b)$.

2. $H(a+b)=$ [set of all h such that $h(a+b)=1$]. Again $h(a+b)=h(a)+h(b)$ and

$$h(a)+h(b)=1 \text{ only if } \textit{either } h(a)=1 \textit{ or } h(b)=1, \text{ but } \textit{not both.}$$

Thus $H(a+b)=H(a) \triangle H(b)$.

3. $H(a)=H(b)$ implies that the set $H(a) \triangle H(b)$ is empty. In this case, then, $H(a+b)$ is an empty set. But, by the corollary to Theorem 3, $H(x)$ is empty only if $x=0$ in B. Thus $a+b=0$ and $a=b$ (by the Boolean ring property that every element is its own additive inverse).

THEOREM 5. *Under the set operations of $\cap$ and $\triangle$ the totality of sets $H(x)$, where x runs through all elements of B, form a Boolean ring. This ring of sets is isomorphic to B under the correspondence $x \to H(x)$, $\times \to \cap$, $+ \to \triangle$.*

References

1. A. A. Albert, Modern Higher Algebra. University of Chicago Press, 1937.

2. Garrett Birkhoff and Saunders MacLane, A Survey of Modern Algebra, New York, The Macmillan Co., 1941.

3. Orrin Frink, Jr., Representations of Boolean algebras, Bulletin of the American Mathematical Society, vol. 47, 1941, pp. 755–756.

4. N. H. McCoy and Deane Montgomery, A representation of generalized Boolean rings, Duke Mathematical Journal, vol. 3, 1937, pp. 455–459.

5. N. H. McCoy, Remarks on divisors of zero, this MONTHLY, vol. 49, 1942, pp. 286–295.

6. M. H. Stone, The representation of Boolean algebras, Bulletin of the American Mathematical Society, vol. 44, 1938, pp. 807–816.

7. M. H. Stone, The theory of representations for Boolean algebras, Transactions of the American Mathematical Society, vol. 40, 1936, pp. 37–111.

8. Max Zorn, A remark on method in transfinite algebra, Bulletin of the American Mathematical Society, vol. 41, 1935, pp. 667–670.

AN ELECTRICAL CHINESE RING PUZZLE

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1. Mechanical Chinese rings. Many of the readers of this magazine are undoubtedly familiar with the mechanical puzzle called Chinese rings. As usually displayed in the novelty shops, it consists of an elongated metal loop intertwined with a series of rings. This is shown diagrammatically in Figure 1. The puzzle consists in removing the loop from the rings. To accomplish this, the rings are dropped or lifted through the loop in a special sequence. Each time a ring is dropped or raised is called a move. The number of moves to get the loop

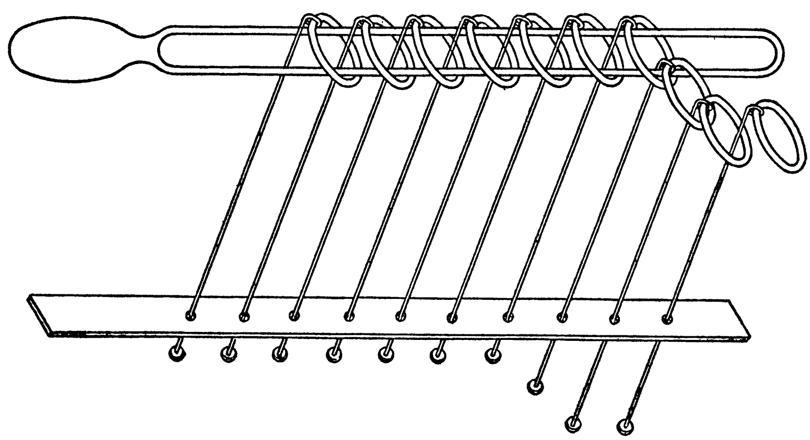

FIG. 1

clear of the rings is obviously a function of the number of rings of which the puzzle is constructed. A 7 ring puzzle requires 85 moves, a 5 ringer only 21, while a 15-ringer requires 21,845 moves.

Unlike many mechanical puzzles, the solution of this one does not depend on cut-and-try methods only. W. W. R. Ball in his "Mathematical Recreations and Essays" and numerous other writers on the subject have expounded the mathematical rules governing this puzzle. The analysis, therefore, need not be given here. The general formula for M, the number of moves to free the loop, in terms of n, the number of rings, is:

$$M = \frac{2^{n+1} - 1}{3} \text{ for an odd number of rings, and}$$

$$M = \frac{2^{n+1} - 2}{3} \text{ for an even number of rings.}$$

Thus for a 7-ringer, $M=(2^8-1)/3=85$.

The author can remove the loop of a 7 ring puzzle, requiring 85 moves, in about 100 seconds. No doubt, with practice this can be reduced to 85 seconds,

or about a move a second. On this basis a 50-ringer will take (provided no mistakes were made) about 25 million years!

2. Electrical Chinese rings. For a long time the author wondered whether an electrical analog to this puzzle could be constructed using lamps instead of rings, with electrical relays suitably interlocked to accomplish the desired purpose. The problem, once the requirements were translated into electrical terms, turned out to be much simpler than was at first supposed. Seven small lamps with red color-caps were used, each controlled from 2 push-buttons, one to light and one to extinguish the lamp. To lend effect, a master lamp with a larger green color cap was arranged to light only when all 7 of the smaller ones were lit. Space does not permit the inclusion here of a photograph showing the device as assembled. Besides the 2 push-buttons, each lamp has 2 electrical relays associated with it, an ON relay, having 2 normally-open and one normally-closed contact and an OFF relay, which has only a single, normally-closed, contact. "Normally," in electrical parlance, means the condition when the coil of the relay carries no current.

Operation parallels the ring-puzzle conditions. A lamp lit corresponds to a ring below the loop or OFF; a lamp out corresponds to a ring above the loop or ON. The lighting of the green lamp corresponds to the loop having been completely freed. Pressing any ON button at random, will in general not light the associated lamp nor will pressing its OFF button extinguish it if it is lit. A lamp can be lit or extinguished only if the correct sequence has been followed previously and then *only by its own push-buttons.* This sequence is exactly the same as that followed in removing the loop from the rings of the Chinese puzzle.

The fundamental principle underlying the ring-puzzle solution is that, in order to take a ring off or put it on the loop, the preceding one must be on and all prior to that must be off the loop. Similarly, with the relays and lamps, in order to light any lamp the preceding one must be out (corresponding to a ring *on* the loop) and all prior to that must be lit. Thus, to light Lamp 5, Lamp 4 must be out and Lamps 1, 2 and 3, lit.

3. Electrical circuit. In these days of radio wiring diagrams, Figure 2 should readily be followed by anyone who is at all conversant with electrical circuits. All of the relays and lamps are connected similarly—their ON and OFF buttons are connected through the normally-closed contact of the preceding relay and the normally-open contacts of all relays prior to that. The connections of ON Relays 1 and 2 differ slightly from the others because they do not have a number of "prior" relays. Once a relay is energized it "seals itself in" so that it stays energized (and the corresponding lamp lit) independently of the push-button. It cannot again be extinguished unless the same circuit is set up as for lighting it. The imposing of conditions under which a lamp may be *extinguished* as well as lit by the pushing of a button requires that the button *close* a circuit rather than

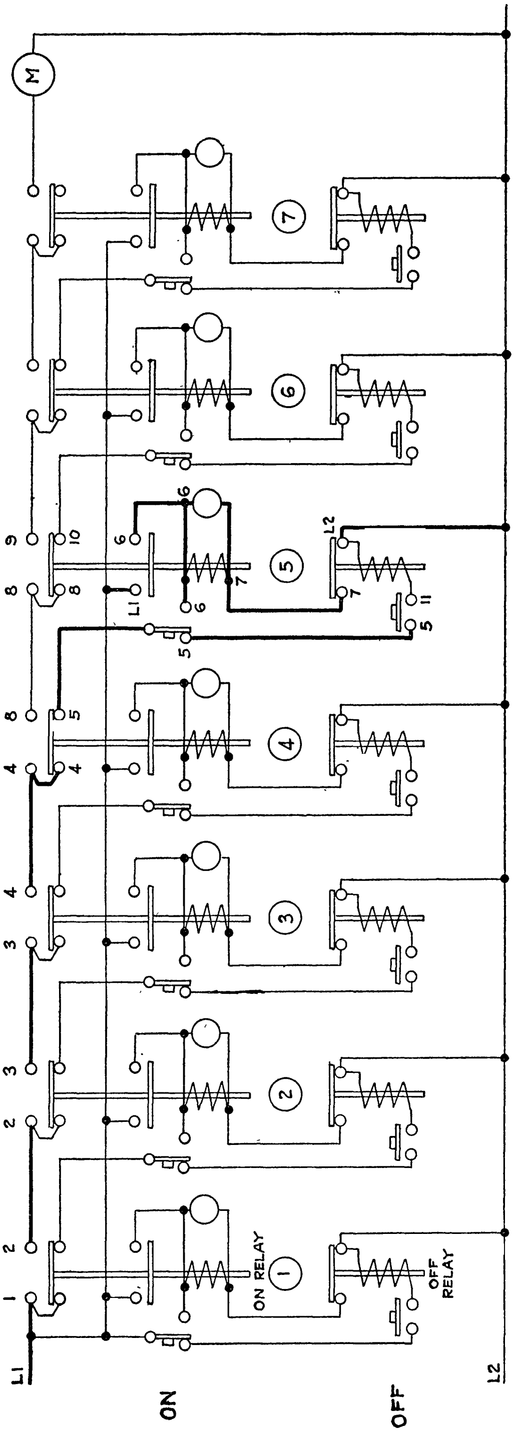

FIG. 2

open it, and since this is to *extinguish* a lamp, it is necessary to interpose an OFF relay, so that when it is *energized* by the OFF push-button, it *opens* the circuit to the lamp and ON relay.

The push-buttons associated with each relay set are shown in the diagram immediately to the left of the relays. Let us trace the circuit necessary to energize say, Relay 5 and its lamp. Starting from the incoming supply line $L2$, we trace through the "normally-closed" contact of the OFF Relay 5 to Terminal 7, through the coil of the ON Relay 5 and its lamp to Terminal 6 and to the ON push-button. Here there is no further path. Were we to press this button, we could trace across the normally-closed contact of ON Relay 4, to Terminal 4, but we could not reach $L1$ to establish a complete circuit because of the gaps 4-3, 3-2, 2-1 of the ON relays 3, 2, 1 respectively. If, however, we assume these relays are already "picked up" or energized, these gaps are closed, so that pushing ON button 5 causes its relay and lamp to operate. The relay, in turn, closes the circuit from $L1$ to 6 which now keeps it energized independently of the push-button—"sealed-in," as the electrical man says. The button can, therefore, be released and the lamp will stay lit. ON Relay 5, when energized, also opens its 8-10 contact and closes its 8-9 contact.

If, at this point, the OFF button were pushed, there would be another path from $L2$, through the coil of the OFF relay to Terminal 11 and across the OFF push-button to Terminal 5 and back to $L1$ via the same route as before. OFF Relay 5 would pick up and open the circuit across its normally-closed contact $L2$–7, thus de-energizing the ON relay and extinguishing the lamp. Releasing the button will de-energize the OFF relay and it will again close its contact, but this has no further effect on the ON relay and lamp which stay off.

If now any of the ON relays 1, 2, 3 or 4 are changed from their assumed position, there can no longer be a circuit back to $L1$ from Relay 5. Hence, pushing either No. 5 ON or OFF button can no longer light or extinguish Lamp 5. Not until the proper conditions are again established—ON Relay 4 de-energized and ON Relays 1, 2 and 3 picked up—can No. 5 relay and lamp be controlled by their push-buttons.

In tracing this circuit we assumed the proper conditions to light Lamp 5. As a matter of fact, the table shows that 5 preliminary moves would be required before Lamp 5 could be lit. Lamp 7 requires 21 preliminary moves.

This device can be built at fairly moderate cost using practically standard radio parts such as pilot lamps, midget relays and panel-mounted momentary-contact push-buttons. For a social evening, its effect is sensational and many a bridge party has been disrupted by a lady who would refuse to play until she could get the green lamp to light.

To solve the puzzle correctly, that is, to light the green lamp, the 85 moves shown in the table below must be followed. The reverse procedure is required to *extinguish* all 7 lamps—unless, of course, one were to cheat by opening the main switch.

SEQUENCE MANIPULATION OF PUSH BUTTONS TO LIGHT ALL
7 PILOT LAMPS AND THE MASTER LAMP
(85 moves required)

Move	Lamp	Move	Lamp	Move	Lamp	Move	Lamp	Move	Lamp
1.	1 on	18.	3 on	35.	1 off	52.	2 on	69.	1 on
2.	3 on	19.	1 off	36.	2 on	53.	1 on	70.	5 on*
3.	1 off	20.	2 on	37.	1 on	54.	6 on	71.	1 off
4.	2 on	21.	1 on	38.	5 off	55.	1 off	72.	2 off
5.	1 on	22.	7 on*	39.	1 off	56.	2 off	73.	1 on
6.	5 on	23.	1 off	40.	2 off	57.	1 on	74.	3 off
7.	1 off	24.	2 off	41.	1 on	58.	3 off	75.	1 off
8.	2 off	25.	1 on	42.	3 off	59.	1 off	76.	2 on
9.	1 on	26.	3 off	43.	1 off	60.	2 on	77.	1 on
10.	3 off	27.	1 off	44.	2 on	61.	1 on	78.	4 on*
11.	1 off	28.	2 on	45.	1 on	62.	4 off	79.	1 off
12.	2 on	29.	1 on	46.	4 on	63.	1 off	80.	2 off
13.	1 on	30.	4 off	47.	1 off	64.	2 off	81.	1 on
14.	4 on	31.	1 off	48.	2 off	65.	1 on	82.	3 on*
15.	1 off	32.	2 off	49.	1 on	66.	3 on	83.	1 off
16.	2 off	33.	1 on	50.	3 on	67.	1 off	84.	2 on*
17.	1 on	34.	3 on	51.	1 off	68.	2 on	85.	1 on*

* Moves marked by an asterisk indicate final ON position of the lamps.

SKEW-SYMMETRIC MATRICES AND PROJECTIVE GEOMETRY

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General identities between two n-rowed skew-symmetric matrices P, Q with elements in a field F have been noted casually for $n=3$ and $n=4$. In case one of the matrices is regular, and therefore $n=2m$, an identity will be established here for any $m=2, 3, \cdots$. According to the application of the matrices involved, these identities can be interpreted in several different ways. As systematic use of matrix calculus seems to have found little entrance into projective geometry, the application of these identities in a sketch of the projective theory of null systems seemed to be most suggestive.

1. Matrices of rank 2. Columns with three coördinates briefly called 3-columns, representing points or vectors in the 3-dimensional space over the ground field F may be denoted by small Gothic letters:

$$\mathfrak{p} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}, \qquad \mathfrak{q} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}$$

The corresponding capitals denote the associated skew-symmetric matrices

$$\mathfrak{P} = \begin{pmatrix} 0 & -p_3 & p_2 \\ p_3 & 0 & -p_1 \\ -p_2 & p_1 & 0 \end{pmatrix}, \qquad \mathfrak{Q} = \cdots .$$

Then the following identity can easily be proved (*cf.* [8], p. 12):

$$\mathfrak{Q}'\mathfrak{P}\mathfrak{Q} = \mathfrak{p}'\mathfrak{q}\cdot\mathfrak{Q}, \qquad (\mathfrak{p}'\mathfrak{q} = p_1q_1 + p_2q_2 + p_3q_3), \tag{1}$$

where $\mathfrak{Q}'$ is the transpose of $\mathfrak{Q}$ and $\mathfrak{p}'$ the row having the same coördinates as the column $\mathfrak{p}$. With regard to the fact that $\mathfrak{P}\mathfrak{q} = -\mathfrak{Q}\mathfrak{p}$ is simply the so-called (outer) vector product of the two vectors represented by the columns $\mathfrak{p}$, $\mathfrak{q}$, the formula (1) is seen to be an expression for a certain threefold vector product by a simple one.

The identity (1) can be extended immediately to n-rowed skew-symmetric matrices of rank 2. Any such matrix can be written as an alternating composition of two n-columns,

$$a = \begin{pmatrix} a_1 \\ \cdot \\ \cdot \\ a_n \end{pmatrix}, \qquad b = \begin{pmatrix} b_1 \\ \cdot \\ \cdot \\ b_n \end{pmatrix},$$

viz,

$$[a, b] = ba' - ab'.* \tag{2}$$

With a second matrix $[u, v]$ of this kind one obtains immediately from the definition (2) the identity

$$[a, b]'[u, v][a, b] = \delta(a, b; u, v)[a, b] \tag{3}$$

where the numerical factor $\delta = \delta(a, b; u, v)$ is defined by

$$\delta(a, b; u, v) = a'u\cdot b'v - a'v\cdot b'u. \tag{4}$$

By making use of the canonical representation of any n-rowed skew-symmetric matrix P of rank $2r$ ($r \leqq m$) as a sum of r skew-symmetric matrices of rank 2, say

$$P = \sum_{\rho=1}^{r} [u^{(\rho)}, v^{(\rho)}] \tag{5}$$

with $2r$ linearly independent n-columns $u^{(1)}, v^{(1)}, \cdots, u^{(r)}, v^{(r)}$, one obtains from

* In the usual matrix notations ba' is the expression for the square n-matrix (of rank one if $a \neq 0$ and $b \neq 0$) with the elements $b_\mu a_\nu$ ($\mu, \nu = 1, 2, \cdots, n$). The matrix $[a, b]$, a representative of Grassmann's "combinatorial product," is of rank 2 if and only if a and b are linearly independent; otherwise it is O. This product as well as the outer products of higher order have been introduced by Grassmann in his Ausdehnungslehre as abstract symbols with certain properties fixed by axioms. Their matricial interpretation is carried through systematically in the theory of compound matrices (*cf.* [11], Chap. V). The abstract theory in modern form, with many geometrical applications, is given in the recent book of Forder [4], Chap. VII.

(3) the following more general identity which is of the same type:

$$(6) \qquad [a, b]'P[a, b] = \left(\sum_{\rho=1}^{r} \delta_\rho\right)\cdot [a, b]$$

where

$$(7) \qquad \delta_\rho = \delta(a, b; u^{(\rho)}, v^{(\rho)}) \qquad (cf.\ (4)).$$

These simple results may lead to the conjecture that a non-trivial identity exists between any two skew-symmetric n-matrices; it seems, however, to be difficult to obtain further results by proceeding on this purely formal line. However, the following general relation which turns out to be useful later on, may be noted here: For any n-rowed square matrix C one has

$$(8) \qquad [Ca, Cb] = C[a, b]C'.$$

2. Four-rowed matrices. To find out what type of identity could be expected in general, a preliminary investigation of the case $n=4$ seemed to be useful. In view of the decomposition

$$(9) \quad P = \begin{bmatrix} 0 & -\overset{(1)}{p_3} & \overset{(1)}{p_2} & \overset{(2)}{p_1} \\ \overset{(1)}{p_3} & 0 & -\overset{(1)}{p_1} & -\overset{(2)}{p_2} \\ -\overset{(1)}{p_2} & \overset{(1)}{p_1} & 0 & -\overset{(2)}{p_3} \\ \overset{(2)}{p_1} & \overset{(2)}{p_2} & \overset{(2)}{p_3} & 0 \end{bmatrix} = \begin{pmatrix} \mathfrak{P}^{(1)} & -\mathfrak{p}^{(2)} \\ \mathfrak{p}^{(2)\prime} & 0 \end{pmatrix} = \langle \mathfrak{p}^{(1)}, \mathfrak{p}^{(2)} \rangle,$$

thè reckoning with general skew-symmetric 4-matrices is reduced to the reckoning with 3-matrices and 3-columns. Here the distinction of the last column and row is a purely technical devices which naturally disappears in all final formulas.

With Minkowski [7]* we introduce the "dual matrix"

$$(10) \qquad P^* = \langle \mathfrak{p}^{(2)}, \mathfrak{p}^{(1)} \rangle'$$

of P. Then $(P^*)^*=P$ and

$$(11) \qquad P^*P = PP^* = k(P)\cdot E \qquad (E = \text{unit matrix})$$

where

$$(12) \qquad k(P) = \mathfrak{p}^{(1)\prime}\mathfrak{p}^{(2)}$$

is the invariant (Pfaffian) parameter of P (and of P^*) for which

$$(13) \qquad k(P^2) = |P|.$$

Thus, for a regular P ($|P| \neq 0$) one has $P^*=k(P)\cdot P^{-1}$.

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* In this famous work as well as in the subsequent papers of Sommerfeld [9] and Weatherburn [10] the notations are adapted to the originally intended applications of the formulae to Relativity. We leave aside this point of view here. Thus while Minkowski's calculus is invariant (or covariant) with respect to the Lorentz group, the one discussed here is orthogonal invariant (or covariant) whereby a better symmetry of the formulae is obtained. The translation of one system into the other offers no difficulty.

Let $Q=\langle \mathfrak{q}^{(1)}, \mathfrak{q}^{(2)}\rangle$ be a second skew-symmetric 4-matrix. Applying (11) to the skew-symmetric matrix $P+Q$, one can verify the identity

$$P^*Q + Q^*P = k(P, Q)\cdot E \tag{14}$$

where

$$k(P, Q) = \mathfrak{p}^{(1)\prime}\mathfrak{q}^{(2)} + \mathfrak{p}^{(2)\prime}\mathfrak{q}^{(1)} = \tfrac{1}{2}\operatorname{tr}(P^*Q) \tag{15}$$

is the common invariant ("polarized parameter") of P and Q.* In this form the identity was first established by C. E. Weatherburn ([10], p. 176, formula (28)). It can also be written in the form

$$Q'P^*Q = k(Q)\cdot P - k(P, Q)\cdot Q \tag{16}$$

which is more suitable for the purpose in hand. For $n=4$ and a singular Q we recognize in (16) the identity (6) ($k(Q)=0$).

The identity (16) involves no restrictions as to the rank of the matrices concerned. If P (and therefore P^*) is singular, say $P^*=[u, v]$, one has

$$k(P, Q) = u'Qv. \tag{17}$$

From (16) further identities can be derived, involving more than two skew-symmetric matrices. Thus we have for three, *viz.*, P, Q, R:

$$\begin{aligned} R'(Q'P^*Q)^*R = {} & (k(P, Q)\cdot k(Q, R) - k(Q)\cdot k(P, R))R \\ & - k(P, Q)\cdot k(R)\cdot Q + k(Q)\cdot k(R)P. \end{aligned} \tag{18}$$

Generally these identities show that a certain product of skew-symmetric matrices and their duals equals a linear combination of these matrices with numerical coefficients.

A dual of a skew-symmetric n-matrix cannot be defined as an n-matrix if $n>4$. Hence for the purpose of generalization of (16) the dual has to be eliminated. In fact, for a regular P this identity is equivalent to

$$k(P)\cdot QP^{-1}Q - k(P, Q)\cdot Q + k(Q)\cdot P = 0. \tag{19}$$

This relation shows the way for further generalization.

3. The general regular case. Now let P, Q be two skew-symmetric n-matrices P regular: $|P| = k_0^2 \neq 0$, $n=2m$. We consider the matrix

$$A = P^{-1}Q. \tag{20}$$

Every congruence transformation of P and Q (cogrediently) induces a similarity transformation of A, and conversely. Hence the similarity invariants of A are simultaneous congruence invariants of P and Q, and conversely.

We consider the skew-symmetric linear matrix polynomial $\lambda P - Q$ with indeterminate λ. Let $\kappa(\lambda)$ denote its invariant parameter, *i.e.*, a polynomial in λ the square of which equals the determinant

* By tr A we denote the trace (sum of the elements in the principal diagonal) of the matrix A.

$$(21) \qquad |\lambda P - Q| = |P||\lambda E - A| = k_0^2 d_A(\lambda)$$

where $d_A(\lambda)$ is the characteristic polynomial of the matrix A. If

$$(22) \qquad \kappa(\lambda) = k_0\lambda^m - k_1\lambda^{m-1} + \cdots - (-1)^m k_{m-1}\lambda + (-1)^m k_m,$$

then $k_0 = k(P)$, $k_m = k(Q)$ are the invariant parameters of P and Q, and $k_1, \cdots, k_{m-1}$ are the rational simultaneous invariants of P and Q. These numbers can be calculated explicitly by means of the similarity invariants of A because of the relation $\kappa(\lambda)^2 = k_0^2 d_A(\lambda)$.

From Cayley's well-known identity if follows immediately that $\kappa(A)^2 = 0$. This is a polynomial relation of degree $2n$ in P^{-1} and Q which in the present connection may be considered as trivial. We shall show, however, that any two skew-symmetric matrices P, Q ($|P| \neq 0$) satisfy the relation

$$(23) \qquad \kappa(A) = 0,$$

or, by (22),

$$(24) \qquad k_0Q(P^{-1}Q)^{m-1} - k_1Q(P^{-1}Q)^{m-2} + \cdots - (-1)^m k_{m-1}Q + (-1)^m k_m P = 0$$

which, evidently, for $m = 2$ gives the identity (19).

The identity (23) follows immediately from some known theorems concerning the congruence reduction and the invariant factors of the linear matrix polynomial $\lambda P - Q$ (*cf.* MacDuffee [6], Theorems 32.2 and 32.3) whereby every invariant factor occurs at least twice as a factor of the polynomial $|\lambda P - Q|$, and thus of $d_A(\lambda)$; hence at least once as a factor of $\kappa(\lambda)$. We need this fact only for the highest invariant factor which, apart from a numerical coefficient, is the minimum polynomial $h(\lambda)$ of the matrix A. (Therefore $h(A) = 0$ from which (23) follows.*

The singular case ($k_0 = 0$) which is unimportant for the geometrical applications to be given below, requires some additional discussion of the linear matrix $\lambda P - Q$; this will be carried through in a subsequent paper.

4. Remark on vector algebra. The identity (16) proves to be the essence of what could be called "Minkowski-Grassmann calculus," *i.e.*, the natural formulation of the 4-dimensional orthogonal-invariant vector algebra (*cf.* [9] and [10]). Thus the identity (24) can be expected to yield a suitable basis for an extension

* The proof can also be based upon some statements contained in the paper of Bennet [1] where, however, irrational invariants (elementary divisors) are introduced; this would entail an unnecessary restriction of the field of the matrix elements.

Added on June 4, 1943: In a letter dated April 8, 1943 which I received on June 3, 1943, N. Jacobson informs me that after having read the abstract of a part of the present paper in the Bulletin of the American Mathematical Society, April, 1943, he noticed that the identity (23) also follows easily from a theorem obtained by him in his paper: "An application of E. H. Moore's determinant of a Hermitian matrix," Bulletin of the American Mathematical Society 45, 1939, 745–748. He mentions further that another more direct proof of his theorem has been pointed out to him by J. Williamson and that a sketch of this proof is published in the paper of N. Jacobson, "Classes of restricted Lie Algebras of characteristic p. I," American Journal of Mathematics 63, 1941, 481–515. in particular pp. 496–497. Both proofs are different from the one indicated above.

of this sort of algebra to any number n of dimensions. This has been attempted occasionally. While, however, for $n=4$ all possible cases, regular as well as singular ones, could be dealt with comprehendingly in the one relation (16) this will no longer be possible for $n>4$ when the several different singular cases will have to be considered separately. This reveals at least one of the intricacies which, for $n>4$, cause an invariant vector calculus to be unsuitable for practical application. In addition it must be noted that a complete system of relations (identities) can be stated only in terms of compound matrices.

5. The $(n-1)$-dimensional projective space. All numbers (matrix elements) occurring in the following sections will be supposed to be real. By $\mathbf{P}_{n-1}$ we denote the projective space of $n-1$ dimensions over the real field ($n>2$). A point in $\mathbf{P}_{n-1}$ is defined by a real n-column $x\neq 0$. Two points x, y are said to be equal if, and only if, the columns x, y are linear dependent or "*projectively equal*" which we indicate by writing $y\simeq x$. Thus all λx ($\lambda\neq 0$) define the same point in $\mathbf{P}_{n-1}$. Similarly two matrices A, B are projectively equal: $B\simeq A$, if, and only if $B=\lambda A$.

Linear submanifolds of $\mathbf{P}_{n-1}$ are defined in the well-known way by means of systems of linear homogeneous equations, *viz.*,

$$U_r z = 0 \tag{25}$$

where U_r is a (not necessarily square) matrix of rank r with n columns ($r<n$); or by the solution

$$z \simeq V_{n-r}t \tag{26}$$

of (25) where V_{n-r} is an n-rowed matrix of rank $n-r$ and t an arbitrary l-column if V_{n-r} has l columns (parameter-representation). If $r=n-1$ the submanifold is a point; if $r=n-2$, a straight line; here there are two linearly independent solutions of (25), say x, y. Hence the general solution can be written either in the form $z\simeq\lambda x+\mu y$ or in the form

$$z \simeq [x, y]t \tag{27}$$

where t is an arbitrary n-column [see (2)]. The matrix $[x, y]$ will be called the "*first Plücker matrix*" of the straight line. From the definition (2) it can easily be shown that it is (projectively) independent of the choice of x, y on the line.

For $r=1$ the equation (25) is equivalent to a single scalar equation

$$u'z = 0 \tag{28}$$

where u is a certain n-column $\neq 0$. This equation represents, in point coordinates z, a certain $(n-2)$-dimensional hyperplane, shortly called "the plane u." If $r=2$ the equation (25) can be replaced by two equations

$$u'z = 0, \qquad v'z = 0 \tag{29}$$

with linearly independent u, v. They represent an $(n-3)$-dimensional linear manifold in $\mathbf{P}_{n-1}$, which can be characterized by its "*second Plücker matrix*" $[u, v]$. In fact $[u, v]t$ represents the most general plane through this manifold.

If $n=4$ the latter manifold is again a straight line. What relation exists between the first and the second Plucker matrices of this line? Every pair of planes u, v having as intersection the line with the first Plücker matrix $[x, y]$, must contain the points x, y and therefore, by (29), satisfy the equations $x'u=0$, $y'u=0$, $x'v=0$, $y'v=0$ which are solved by

$$u = [x, y]^*s, \qquad v = [x, y]^*t$$

where s, t are arbitrary 4-columns. This follows from the relation

$$a[b, c]^*d = | a \ b \ c \ d | \tag{30}$$

valid for any four 4-columns a, b, c, d, since the symbol $a[b, c]^*d$ is linear and homogeneous with respect to each of the four columns, and for the unit-columns $e^{(r)}$ one has $e^{(1)}[e^{(2)}, e^{(3)}]^*e^{(4)}=1$, and so it has the characteristic properties of the determinant.

By making use of the general relation (8) we obtain

$$[u, v] = - [x, y]^*[s, t][x, y]^*.$$

Hence we conclude from the identity (3) that

$$[u, v] \simeq [x, y]^*. \tag{31}$$

Thus the two Plücker matrices of a given line are mutually dual. (This simple expression of an elementary fact (Jessop, [5] §4, p. 18) is mentioned here only as a typical example of the application of the identity.)

6. **Collineations and correlations.** A one-valued invertible point transformation $y\simeq\phi(x)$ of the space $\mathbf{P}_{n-1}$ into itself which induces a one-one correspondence $v\simeq\psi(u)$ between the planes in $\mathbf{P}_{n-1}$ such that if x runs through the plane u then y runs through the plane v, is called a *projective transformation* or *collineation*. According to the fundamental theorem of projective geometry *every collineation can be represented by a linear homogeneous transformation*

$$y \simeq Ax \tag{32}$$

where A is a certain regular n-matrix. For a proof we may refer to the paper of R. Brauer [2].* The induced plane transformation of (32) is readily seen to be

$$v \simeq A'^{-1}u. \tag{33}$$

A one-one correspondence between points and planes in $\mathbf{P}_{n-1}$ which with every point x associates a plane $v\simeq\chi(x)$ and with every plane u a plane $y\simeq\omega(u)$ is said to be a *correlation* if, when x runs over all points of a plane u, the plane $v\simeq\chi(x)$ runs through all planes passing through the point $y\simeq\omega(u)$. By carrying out two correlations in succession one obtains a collineation. This means that if $u^{(1)}\simeq\chi^{(1)}(y)$, $x^{(1)}\simeq\omega^{(1)}(v)$ is a second correlation, then

* Here a somewhat more general theorem is proved which, under our present suppositions, gives exactly the above statement because the real field admits no automorphism different from the identity. For another proof see for instance [3], p. 27–32.

$$x^{(1)} \simeq \omega^{(1)}(\chi(x)), \qquad u^{(1)} \simeq \chi^{(1)}(\omega(u))$$

is a collineation. If as the second correlation we take in particular $u^{(1)} \simeq y$, $x^{(1)} \simeq v$ we see that $x^{(1)} \simeq \chi(x)$, $u^{(1)} \simeq \omega(u)$ is a collineation .Thus the fundamental theorem shows that *for every correlation* $v \simeq \chi(x)$, $y \simeq \omega(u)$ *a regular n-matrix C can be found such that it appears in the form*

$$v \simeq Cx, \qquad y \simeq C'^{-1}u. \tag{34}$$

Any correlation is seen to be invertible; the inverse of (34) is

$$u \simeq C'y, \qquad x \simeq C^{-1}v. \tag{35}$$

Now let B be the matrix of another correlation; the product of both is the collineation with the matrix

$$A \simeq B'^{-1}C. \tag{36}$$

Therefore two correlations are said to be commutative if

$$C'^{-1}B \simeq B'^{-1}C. \tag{37}$$

The square of the correlation (34) is the collineation with the matrix $C'^{-1}C$. The correlation C (*i.e.* (34) will be said to be *involutory* (or "of period 2") if its square is the identity, *i.e.*, if $C'^{-1}C \simeq E$. This means $C' \simeq C$ or $C' = \lambda_0 C$. By taking the determinants of both members of this equation one sees that $\lambda_0 = +1$ if n is odd, $\lambda_0 = \pm 1$ if n is even. Hence we see that *the matrix C of an involutory correlation in* $\mathbf{P}_{n-1}$ *is always symmetric if n is odd*; *it is symmetric or skew-symmetric if n is even.* A correlation with skew-symmetric matrix (which is naturally always involutory) is called a *null system.** Evidently null systems can exist only in odd-dimensional spaces $\mathbf{P}_{n-1}$.

Often it is useful to present a correlation in another algebraic appearance. In "running coördinates" y the equation of the plane v (*i.e.*, $v'y = 0$) that is associated with the point x by the correlation (34), is

$$x'C'y = 0. \tag{38}$$

In running plane coördinates v the equation of the point associated to the plane u is

$$u'C^{-1}v = 0. \tag{39}$$

This form of the definition of a correlation is particularly convenient for the study of its behaviour by means of projective coördinate transformations or correlations. When new coördinates are introduced by the substitution with the matrix S, so that $x \simeq S\bar{x}$, $y \simeq S\bar{y}$ the equation (38) is changed into $\bar{x}'S'C'S\bar{y} = 0$ whence we see that in the new coördinates the correlation has the matrix

$$\overline{C} \simeq S'CS. \tag{40}$$

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* R. Brauer [2] has given another, more geometric, characterization of null systems.

Similarly if we apply in $\mathbf{P}_{n-1}$ the correlation

$$x \simeq R'^{-1}\bar{u}, \qquad y \simeq R'^{-1}\bar{v}$$

the equation (38) is turned into $\bar{u}'R^{-1}C'R'^{-1}\bar{v}=0$ which defines a correlation in plane coördinates. With respect to (39) we see that, in point coördinates, this correlation would have the equation $\bar{x}'\overline{C}'\bar{y}=0$, its matrix being

$$\overline{C} \simeq (R^{-1}CR'^{-1})'^{-1} = R'C'^{-1}R. \tag{41}$$

By comparison with (37) we obtain the following theorem:

A correlation C is invariant under the correlation R of the coördinates (i.e. $\overline{C}\simeq C$ in (41)) if, and only if, these two correlations are commutative.

7. Null systems. After these preliminaries we now turn to a more detailed consideration of the correlations with skew-symmetric regular matrix P. Accordingly we suppose $\mathbf{P}_{n-1}$ to be of odd dimension $\geqq 3$, hence $n=2m$. In view of (38) and (39) the null system P is defined by the equation

$$x'Py = 0 \tag{42}$$

or by its dual

$$u'P^{-1}v = 0. \tag{43}$$

It constitutes a dual relation between point z and plane $w\simeq Pz$. Both are always coincident since, because of the skew-symmetry of P, one has for all x

$$x'Px = 0. \tag{44}$$

This property is characteristic of a null system. In fact, if the correlation with the matrix C satisfies the condition $x'Cx=0$ for all x, then C is necessarily skew-symmetric. To prove this, let $C=A+P$ where A is symmetric, P skew-symmetric. By (44) one has $x'Ax=0$ for all x, whence $A=0$ and thus $C=P$.

Now let z move through the $(n-3)$-dimensional manifold defined by its second Plücker matrix $[u, v]$. The corresponding two-dimensional plane $w\simeq Pz$ then runs over all planes containing the straight line through the two points $\bar{x}\simeq P^{-1}u$, $\bar{y}\simeq P^{-1}v$; hence the first Plücker matrix of this line is by (8)

$$[\bar{x}, \bar{y}] \simeq P^{-1}[u, v]P^{-1}. \tag{45}$$

The result of the dual consideration is contained in this formula.

If $n=4$ the manifold $[u, v]$ is likewise a straight line. Here the identity (16) leads to a very simple transformation formula for the Plücker matrices of the straight lines related by the duality of the null system. By applying the identity to the right-hand member of (45) (after having replaced P^{-1} by $P^*\simeq P^{-1}$) and making use of (17) we obtain

$$[\bar{x}, \bar{y}] \simeq u'P^*v\cdot P^* - k_0[u, v]^* \tag{46}$$

where k_0 is the invariant parameter of P. By dualization of (46) we get the second matrix of the same line

$$(47) \qquad [\bar{u}, \bar{v}] \simeq u'P^*v \cdot P - k_0[u, v]$$

which is said to be *conjugate* (with respect to the null system P) to the line with the second matrix $[u, v]$. Similarly a transformation law can be established for the first Plücker matrices of the straight lines conjugate with respect to P:

$$(48) \qquad [\bar{x}, \bar{y}] \simeq x'Py \cdot P^* - k_0[x, y].$$

From these formulae we see immediately that a line $[x, y]$ (or $[u, v] \simeq [x, y]^*$) coincides with its conjugate if, and only if

$$(49) \qquad x'Py = 0, \qquad u'P^*v = 0.$$

These lines are called self-conjugate or *null lines* for the null system P. From (49) we conclude that a straight line defined by two of its points (or as the intersection of two planes) is a null line for a given null system if and only if both of these points (or planes) coincide with the two planes (or points) which are associated with them by the null system.

Now let

$$(50) \qquad P = [u^{(1)}, v^{(1)}] + [u^{(2)}, v^{(2)}]$$

where $u^{(1)}, v^{(1)}, u^{(2)}, v^{(2)}$ are four linearly independent 4-columns. *Then the matrices $[u^{(1)}, v^{(1)}]$, $[u^{(2)}, v^{(2)}]$ are necessarily the second Plücker matrices of a pair of conjugate non-null lines of the null system P.* To prove this we consider the two points $x \simeq P^*u^{(1)}$, $y \simeq P^*v^{(1)}$ on the line conjugate to the line with the second Plücker matrix $[u^{(1)}, v^{(1)}]$. By (30) we have

$$x \simeq [u^{(2)}, v^{(2)}]^*u^{(1)}, \qquad y \simeq [u^{(2)}, v^{(2)}]^*v^{(1)}$$

and therefore the first Plücker matrix of the conjugate line is

$$[x, y] \simeq [u^{(2)}, v^{(2)}]^*[u^{(1)}, v^{(1)}][u^{(2)}, v^{(2)}]^* \simeq [u^{(2)}, v^{(2)}]^* \qquad \text{(by (8) and (3))}$$

whence by (31) we conclude that $[u^{(2)}, v^{(2)}]$ is its second Plücker matrix.

From (47) it follows that a null system P is completely defined by its invariant k_0 and a pair of conjugate non-null lines.

8. Null systems in involution. Two null systems P, Q in P_{n-1} are said to be "in involution" if they are

1. different, *i.e.*, the matrices P, Q are linearly independent;
2. commutative, which by (37) means

$$(51) \qquad P^{-1}Q \simeq Q^{-1}P, \quad i.e., \quad P^{-1}Q = \alpha Q^{-1}P$$

with a certain real number α. If we introduce again the matrix A of (20), this condition means that the matrix A, being not a scalar multiple of E itself, is (projectively) involutory:

$$(52) \qquad A^2 = \alpha E.$$

Hence α can be calculated by the rational similarity invariants of A; let

$$d_A(\lambda) = |\lambda E - A| = \lambda^n - a_1\lambda^{n-1} + \cdots + (-1)^n a_n = \prod_{\nu=1}^{n} (\lambda - \alpha_\nu)$$

where $\alpha_1, \cdots, \alpha_n$ are the characteristic roots of A. Then

$$a_1 = \operatorname{tr} A = \sum_{\nu=1}^{n} \alpha_\nu, \qquad a_2 = \sum_{\mu<\nu} \alpha_\mu\alpha_\nu$$

and

$$\alpha = \frac{1}{n} \operatorname{tr} (A^2) = \frac{1}{n} \sum_{\nu=1}^{n} \alpha_\nu^2 = \frac{1}{n} (a_1^2 - 2a_2).$$

Therefore α can be expressed by means of the rational simultaneous invariants k_0, k_1, k_2 of P and Q (in the notations of (22)). Because of

$$a_1 = 2\frac{k_1}{k_0}, \qquad a_2 = \frac{1}{k_0^2}(k_1^2 + 2k_0k_2)$$

one has

(53) $$\alpha = \frac{1}{mk_0^2}(k_1^2 - 2k_0k_2).$$

Suppose P, Q to be two null systems in involution. It follows from (52) that

$$(P^{-1}Q)^{2i} = \alpha^i E, \qquad (P^{-1}Q)^{2i+1} = \alpha^i P^{-1}Q = \alpha^{i+1}Q^{-1}P.$$

By putting these expressions into the general identity (24), all matrix products herein are reduced to scalar multiples of P or Q. We get a relation of the form

(54) $$f(\alpha)Q - g(\alpha)P = 0$$

where $f(\zeta)$ and $g(\zeta)$ are two polynomials in ζ: For $m=2l+1$ (l any positive integer)

$$f(\zeta) = k_0\zeta^l + k_2\zeta^{l-1} + \cdots + k_{2l},$$
$$g(\zeta) = k_1\zeta^l + k_3\zeta^{l-1} + \cdots + k_{2l+1};$$

for $m=2l$

$$f(\zeta) = \qquad k_1\zeta^{l-1} + \cdots + k_{2l-1},$$
$$g(\zeta) = k_0\zeta^l + k_2\zeta^{l-1} + \cdots + k_{2l}.$$

From (54) and the linear independent of P and Q we infer that *the number* α *given by* (53) *must be a root of both of these polynomials* $f(\zeta)$, $g(\zeta)$:

(55) $$f(\alpha) = 0, \qquad g(\alpha) = 0.$$

These equations are necessary conditions for two null systems to be in involution.*

* The apparent difference between the conditions for even and for odd values of m may only be noted here. It emerges again in the theory of the symplectic groups associated with the null systems which will be dealt with in another paper.

For $n=4$ $(m=2)$ the polynomial $f(\zeta)$ is the constant k_1. We have the condition

$$k_1 = k(P, Q) = 0 \tag{56}$$

which proves to be the only one since $\alpha = -k_2/k_0$ satisfies the equation $g(\alpha)=0$. From (19) it follows that the condition (56) is likewise sufficient. Supposing P in the form (50) and correspondingly $Q=[a^{(1)}, b^{(1)}]+[a^{(2)}, b^{(2)}]$, one can write the simultaneous invariant of P and Q in the form

$$k_1 = \left| u^{(1)}v^{(1)}a^{(1)}b^{(1)} \right| + \left| u^{(1)}v^{(1)}a^{(2)}b^{(2)} \right| + \left| u^{(2)}v^{(2)}a^{(1)}b^{(1)} + \right| u^{(2)}v^{(2)}a^{(2)}b^{(2)} \right|$$

whence because of (30) follows the usual geometric characterization of two $\mathbf{P}_3$-null-systems in involution: Two null systems P, Q in $\mathbf{P}_3$ are in involution if and only if the conjugate with respect to P of every non-null line of P which is a null line of Q is also a null line of Q.

For $n=6$ $(m=3)$ the two conditions (55) are

$$f(\alpha) = k_0\alpha + k_2 = 0, \qquad g(\alpha) = k_2\alpha + k_3 = 0$$

whence by means of (53)

$$k_2 = -\frac{k_1^2}{k_0}, \qquad k_3 = -\frac{k_1^3}{k_0^2}, \qquad \alpha = \frac{k_1^2}{k_0^2}$$

and

$$k_0^2\kappa(\lambda) = (k_0\lambda - k_1)^2(k_0\lambda + k_1).$$

Thus here the characteristic roots of $A=P^{-1}Q$ are $\pm k_1/k_0$ with multiplicities 4 and 2 respectively.

Bibliography

1. A. A. Bennett, Products of skew-symmetric matrices, Bulletin of the American Mathematical Society, vol. 25, 1919, pp. 455–458.

2. R. Brauer, A characterization of null systems in projective space, Bulletin of the American Mathematical Society, vol. 42, 1936, pp. 247–254.

3. G. Darboux, Principes de Géométrie Analytique, Paris 1917.

4. H. G. Forder, The Calculus of Extensions, Cambridge 1941.

5. C. M. Jessop, A Treatise on the Line Complex, Cambridge 1903.

6. C. C. MacDuffee, The Theory of Matrices, Ergebnisse der Mathematik, vol. II, 5, Berlin 1933.

7. H. Minkowski, Die Grundgleichungen für die elektromagnetischen Vorgänge in bewegten Körpern, Nachrichten der Kgl. Gesellschaft der Wissenschaften zu Göttingen, Math.-Phys. Kl., 1908, pp. 53–111; = Mathematische Annalen 68, 1910, pp. 472–525; = Gesammelte Abhandlungen von H. Minkowski, vol. II, no. XXX; = separate publication "Zwei Abhandlungen über die Grundgleichungen der Elektrodynamik" Berlin und Leipzig 1910.

8. H. Schwerdtfeger, Les fonctions de matrices I, Actualités Scientifiques et Industrielles, vol. 649, Paris 1938.

9. A. Sommerfeld, Zur Relativitätstheorie I, Vierdimensionale Vektoralgebra, Annalen der Physik, 4te Folge, vol. 32, 1910, pp. 749–776.

10. C. E. Weatherburn, Four-vector algebra and analysis, Messenger of Mathematics, vol. 49, 1920, pp. 155–176.

11. J. H. M. Wedderburn, Lectures on Matrices, American Mathematical Society Colloquium Publications, vol. XVII, New York 1934.

MATHEMATICS, 200 B.C.–600 A.D.

MAX DEHN, St. John's College

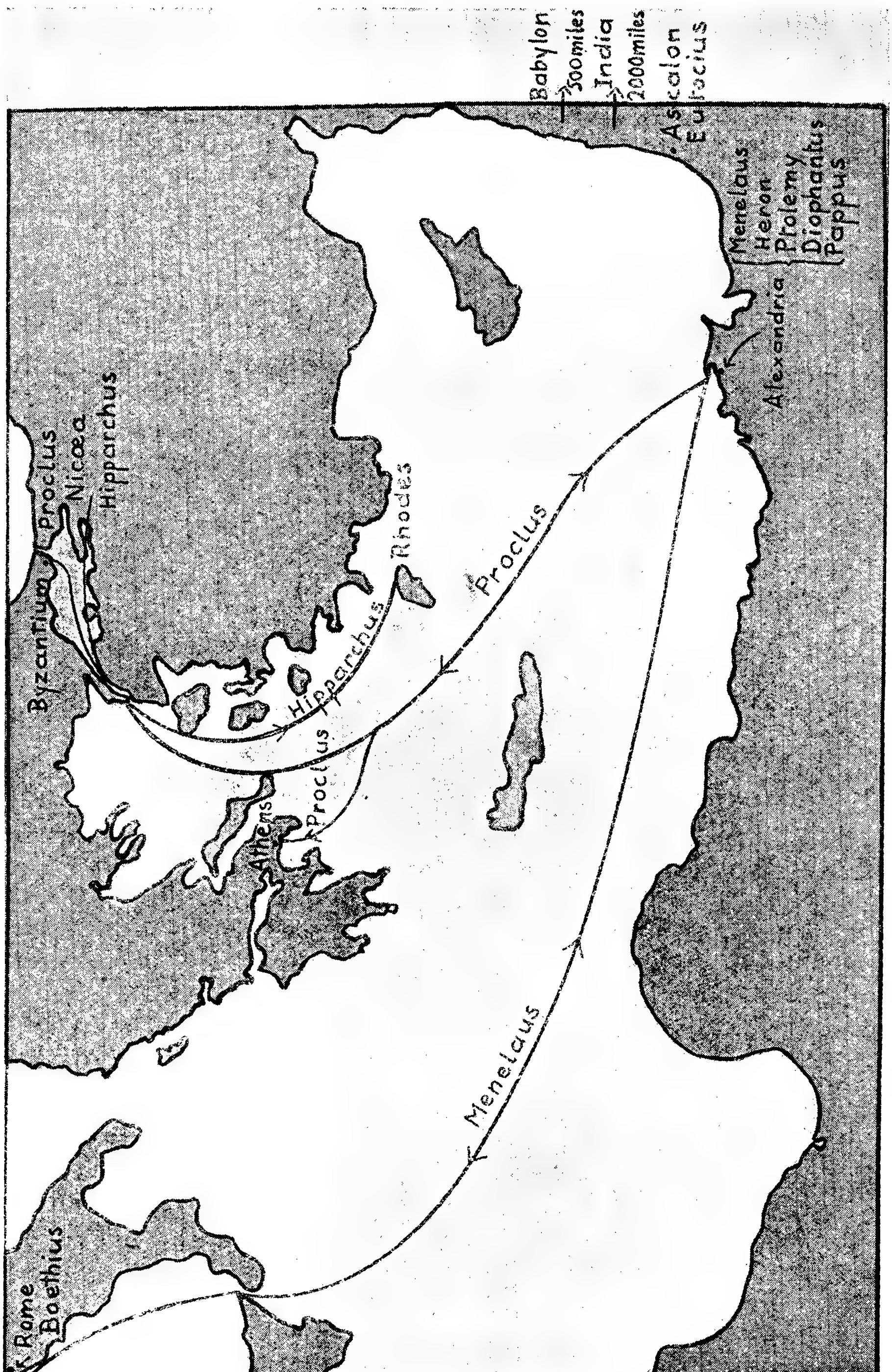
1. Trigonometry. It is in this period that we find Greek mathematics strongly influenced for the first time by phenomena outside the world of mathematical ideal entities. The astronomer observes the positions of the sun, moon, planets and fixed stars at different places and at different times. He measures a number of angles which are not independent variables. To establish their relations, to determine other, not observable, angles, for instance the angle measuring the arc between the annual circular path of the sun and the pole of the daily circles of the fixed stars, requires new mathematical investigations. All these problems pertain to the geometry of the sphere and the solution of these problems constitute what is now called spherical trigonometry.

Furthermore, we find plane trigonometry developed. The investigation of relations between the sides and the angles of a plane triangle was perhaps inspired by spherical trigonometry.

Now, in the case of spherical trigonometry, as in that of plane trigonometry, the relations are not algebraic if one measures the angles as fractional parts of a full angle, which is always the case in astronomical observations. One needs to introduce non-algebraic functions of the fractions determining the angles in order to obtain algebraic relations. It is sufficient to introduce one function of this kind: One may take the angle as an angle between the radii of a circle, then this function can be chosen as the ratio of the chord subtending the angle to the diameter of the circle.

Thus we have two problems for spherical and plane trigonometry: (1) One has to find the (algebraic) relations between the chord function of the angles and the arcs in a spherical triangle and the (algebraic) relations between the chord function of the angles and the sides of a plane triangle. (2) One has to investigate the chord function, which again may be divided into three parts: (a) the determination of the algebraic functional relations; (b) the numerical computations of the function, which in turn implies (c) the finding of certain inequalities determining the behavior of the function in the neighborhood of certain points. We note in passing that the chord function of an angle is double the sine function of the half of the angle.

2. The seeds of the notion of function and of transformation. In trigonometry the mathematicians came closer to the notion of function than in the older theory of "locus." But the general idea of function did not appear at all, still less the idea of transformation. A transformation is, one may say, nothing else than a materialized function: the transformed geometric element, for instance a new point, is a function of the old one. But this embodiment of a function is not so easily visualized as the locus. We shall see below that the main difficulties confronting the foundation of projective geometry are overcome in this period. But still we do not find the beginning of a systematic development of projective


Babylon
500 miles
India
2000 miles
Ascalon
Eutocius
Menelaus
Heron
Ptolemy
Diophantus
Pappus
Alexandria
Proclus
Nicoea
Hipparchus
Byzantium
Rhodes
Hipparchus
Proclus
Proclus
Athens
Menelaus
Rome
Boethius

geometry. We do not even find a systematic use of the fact that the conics are generated by the projective transformation of a circle.

3. Commentaries. In this period we find several valuable commentaries on the works of the mathematicians Euclid, Archimedes, and Apollonius.

4. Other peoples. The disturbances of the old world by the expeditions of Alexander had also an effect on the state of mathematics. The Greeks came into closer contact with the Egyptians and with the people of Mesopotamia. The inhabitants of India came in contact with Greek art and Greek mathematics, also in closer contact with the culture of Asia Minor. Although Greek mathematics during its first period was certainly under Babylonian influence, its peculiar and vigorous development obscured that influence for many centuries. At this time, however, the characteristics of Babylonian mathematics became quite apparent. Mathematical exercises, to be seen in cuneiform texts from 2000 B.C., appear in Greek textbooks. In India, we see a flourishing of mathematics under Greek influence, especially arithmetic which, perhaps, was reflected back to the Greeks. For the first time after Euclid we find in this period new arithmetical problems and new methods for their solution. These were again seeds which, 1500 years later, were developed into full growth.

Important new symbols for numbers were introduced in India and were transmitted to the Near East and Europe where we shall encounter them in the next period.

5. General significance of this period. The time covered by this article is very long, 800 years, in comparison with the periods covered by the former reports. In it no mathematician of such glorious fame as Euclid, Archimedes, or Apollonius appears. But it is not easy to call it a period of decadence if one recognizes the many seeds to be developed later. One mathematician of this time, Pappus (about 300 A.D.) even expressed his feeling that mathematics up to his time was only in the beginning of its development. He says: "I saw that all ⟨mathematicians⟩ move about only in the beginnings of pure and applied mathematics; and I had a feeling of awe ⟨because I was aware⟩ that I could show much better and much more useful things." Perhaps Pappus was afraid to enter alone into the vast and unknown realm of that science the existence of which he divined. Already in Pappus' time the best minds were more interested in mystical or theological problems than in scientific ones.

The great treatise of Ptolemy of Alexandria (about 150 A.D.) on astronomy stood for more than 1500 years by the side of Euclid's Elements as a book of indisputable authority.

6. The foundations of trigonometry. This treatise of Ptolemy is commonly called *Almagest* into which word the Arabs changed the original title, *ἡ μεγάλη σύνταξις*, The Great Composition. In this book Ptolemy collected, enlarged and systematized the results of preceding astronomical investigations, both practical and theoretical, especially those of the great Greek astronomer Hipparchus

(about 200 B.C.). Most famous is his systematization of the movement of the planets by using combinations of circular movements. But this part of his work is of minor interest for the history of mathematics.

The trigonometry of the *Almagest* is based on two theorems. The first is called the theorem of Menelaus, who lived about fifty years before Ptolemy in Alexandria. His work is extant only in Arabic and Hebrew translations. The figure represents the theorem of Menelaus for plane figures where we have the relation

$$\frac{A'B}{A'C} \cdot \frac{B'C}{B'A} \cdot \frac{C'A}{C'B} = 1.$$

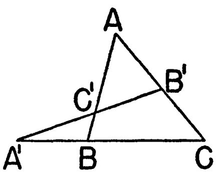


FIG. 1

This theorem is also valid for the sine function of the arcs on the sides of a spherical triangle and in this form is used by Ptolemy to prove the relations between the arcs and angles of a spherical triangle. It is interesting to see why the theorem for the plane figures is so easily changed into a theorem for spherical figures: the theorem in the plane is easy to generalize for the plane of projective geometry by taking cross ratios instead of ratios. Then we obtain immediately the corresponding theorem about planes and lines through a point which we take as center of the sphere. The cross ratio of points on a line in the plane corresponds to the cross ratio of lines in a plane through the central point. And this latter cross ratio is expressed by the sine function of the angles between the lines; hence the advantage of our sine function in comparison with the original chord function, Menelaus, of course, as well as Ptolemy, does not go the way of projective geometry to prove the spherical theorem by the plane theorem.

The second theorem is used to find the functional relations for the chord function. This theorem is called after Ptolemy and states a relation between the six distances of four points on a circle. The proof of Ptolemy is probably the shortest possible, and is to be found in all textbooks on geometry. The analysis of the theorem shows the following elements: first, an identity between two cross ratios, determined by the same four points, an identity known as Euler's identity for four points on a line; second, the algebraic identity expressing the invariance of the cross ratio under a linear transformation; third, the geometric fact that a linear transformation in the plane of a complex variable transforms lines into circles. The theorem is used in the special case where two of the four points are on one diameter. To go, by way of this theorem, to the addition theorem for the sine function is certainly not the simplest possible way.

For the numerical evaluation of the chord function Ptolemy needs the addition theorem and further the fact that the function $(\sin x)/x$ is decreasing with increasing x for $0<x<\pi/2$. The fact that $(\tan x)/x$ is increasing with increasing x for $0<x<\pi/2$ is already proved in a very simple way in Euclid's Optics. One may prove in a similar and quite as simple way that $(\sin x)/x$ is decreasing with increasing x for the same range. Ptolemy gives a rather complicated proof. The fact itself had already been used by Aristarchus of Samos more than three hundred years before Ptolemy when he discussed the appearance of the sphere of the moon. Tables for the chord function were given long before Ptolemy by Hipparchus but these have not been handed down to us.

7. Achievements in geometry. The most remarkable achievements of this period in geometry are due to Pappus. In the seventh book of his "Mathematical Collections" he proved the theorem that the cross ratio of four points on a line $(AC/AD):(BC/BD)$ is not altered by perspective projection. With the help of this theorem he proves several other propositions. By far the most important is the following: Let A, B, C, be three points on one line, A', B', C', three points on another line, then the lines AB' and BA', BC' and CB', CA' and AC', respectively, meet in three points lying on one line.

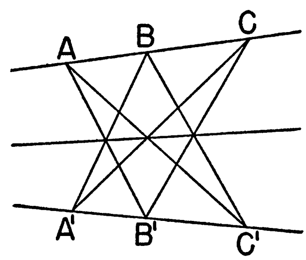


FIG. 2

This theorem marks an event in the history of geometry. From the beginning geometry was concerned with measures: lengths of lines, areas of plane figures, volumes of bodies. Here we have for the first time a theorem which is established by the ordinary theory of measures but is itself free of all elements of measurement; it states the existence of a figure which is determined through the incidence of lines and points only. It is the first "configuration" of projective geometry, and it was shown more than 1500 years later that this configuration alone is sufficient to build up projective geometry in the plane.—The mathematicians who pointed out the important role of this theorem were unjust toward Pappus in naming the theorem after Pascal.

8. Practical mathematics. After Euclid's Elements and Ptolemy's Almagest there is perhaps no ancient work on mathematical methods and natural science which had such a lasting, uninterrupted influence as that of Heron. About Heron's life we know scarcely anything. For a long time scholars tried to find out when he lived through a study of his works, and made guesses ranging from

150 B.C. to 200 A.D. For the moment it seems to them most probable that he lived in Alexandria in the first century A.D. There are books bearing his name which probably have not been written by him. In this whole collection we find little pure mathematics. It is only incidentally that, teaching all sorts of practical methods of measuring, he states and proves rigorously how to express the area of a triangle in a symmetrical way in terms of the sides. He also shows himself as a resourceful mathematician in the proof of the theorem that the three auxiliary lines in Euclid's proof of the Pythagorean theorem meet in one point. But the main tendency of Heron is to make mathematics bear fruit in the treatment of practical problems. He shows many ways of finding approximate measures.

He also treats problems of surveying, for instance the problem of finding the direction of the line joining two points, if one of the points is not visible from the other point. He solved this problem by measuring the rectangular coordinates of auxiliary points between the two given points. This is in symbols: x_0, y_0 and x_n, y_n may be the coordinates of the two given points, x_i, y_i $(i=1, \cdots, n-1)$ the coordinates of the auxiliary points; then the direction is given by

$$\frac{y_n - y_0}{x_n - x_0} = \frac{\sum_{i=1,\cdots n} (y_i - y_{i-1})}{\sum_{i=1,\cdots n} (x_i - x_{i-1})}.$$

The idea of determining the position of different points on a surface, especially the earth, by two coordinates is much older than its appearance in Heron's work. Coordinates were quite indispensable to the astronomer in determining the relative positions of the places of observation. Thus we find already Hipparchus determining the points on the globe through the two angles of latitude and longitude.

In many of the problems of Heron concerned with measuring we find similarities with Egyptian methods. There are other aspects of his work which are obviously connected with the mathematics of the Babylonians. Thus we find an example of a quadratic equation for the radius of a circle given through the sum of the number measuring the area of the circle and of the number measuring the circumference. Such problems as this have scarcely any practical value; furthermore, they are very remote from the problems of classical Greek mathematics. Similar problems, however, are to be found in very old collections of Babylonian exercises.

Heron also tries to compute the volume of a truncated pyramid whose linear measures as they are given are impossible. Out of these data he gets for the volume an expression corresponding to the square root of a negative number. He takes instead of this expression the square root of the number with positive sign, which magnitude, within the given problem, has no significance whatsoever.

We may say that there are in Heron's works characteristics of the Greek, Egyptian, and Babylonian mathematics, and even a premonition of developments of much later times.

Heron was not only an ingenious mathematician. He observed the forces of nature and used them to build all sorts of machines, most of them of very little practical use. He was probably the first to use the power of expanding steam to set heavy bodies in motion, for instance to make a sphere rotate about an axis. His manifold inventions were well known to the Renaissance physicists, among others, to Galileo.

9. Algebra and theory of numbers. There is a fourth mathematician in this period whose name still lives in the work of modern mathematicians, Diophantus. In the first book of his Arithmetic we find many problems not very different from old Babylonian problems. But the form of his treatment of these problems is very important: as the problems never lead to an irrational quantity, Diophantus is able to present a theory of equations in a seemingly modern form. All the difficulties in operations with irrational quantities, which can only be overcome by an at least partially developed theory of limits, do not appear here. Thus we find here symbols and methods of solution of equations quite similar or equivalent to modern symbols and methods. The form of Diophantus' work has undoubtedly influenced the further development of algebra. However, he was probably not the first to give this form to arithmetical operations.

We know nothing of the life of Diophantus. Those scholars may be right who suppose that Diophantus lived at Alexandria in the time of Ptolemy and Heron.

Beginning with the second book of Diophantus' Arithmetic we find a new type of problem, belonging to the theory of numbers. In the first problem of this type one has to find two rational numbers which squared and added are equal to a given square, a^2. The method of the solution is quite general. He puts one number equal to x, the other equal to $rx-a$. Then he finds $x=2ar/(r^2+1)$. He indicates the general solution but takes special values for a and r. This problem is, of course, not different from the old problem of finding two rational numbers x and y so that $x^2+y^2=1$. But already the next problem is something new: to divide the sum of two squares a^2+b^2 into two other squares. He puts one number equal to $x+a$ and the other number equal to $rx-b$ and finds $x=(2rb-2a)/(r^2+1)$. Again he indicates a general solution but takes special values for a, b, and the parameter r. This procedure is nothing else than the rationalization of the equation $u^2+v^2=a^2+b^2$.

In the same way the next example is to be considered as a rationalization of the algebraic relation $y^2-z^2=d$. In general we may characterize the problems of Diophantus as problems of rationalization of algebraic relations, *i.e.*, of the finding of a representation of an algebraic relation through rational functions of a parameter. Diophantus does not try to find solutions in integers.

These problems and their solutions made a great impression on mathematicians of the sixteenth and the seventeenth centuries and were certainly one of the reasons for a new flowering of that noble science, the theory of numbers.

10. Commentaries. The most famous of the ancient commentators is doubtless Proclus, who lived in the middle of the fifth century A.D. But he was much more a philosopher than a mathematician. He was head of the Neo-Platonic school at Athens. His contributions to mathematics are certainly very slight, but his commentary on the first book of Euclid is an invaluable source for the history of Greek mathematics. The commentary is also typical of his time, which considered metaphysical speculations, mostly of a mystical character, as the most important task for lovers of wisdom. These people looked down on mathematics because it made use of hypotheses whereas pure speculation was non-hypothetical.

Heron, too, made additions to, and commentaries on, the work of Euclid. Pappus wrote about Euclid, Apollonius and other mathematicians. Of the later commentators we mention Eutocius, who lived in the sixth century A.D. and came from Ascalon in Syria. He showed himself a very able mathematician and gave us an excellent commentary on Archimedes, where we find the most valuable report on the different solutions of problems of the third degree. He was even able to restore an old corrupted manuscript, probably of Archimedes, where we find the solution of a maximum problem, mentioned in our third report.

Here we make an end to our necessarily incomplete report on Greek mathematics.

11. The Romans. We mention only one Roman, Boethius, who was executed by Theodoric, King of the Goths, in 524. His mathematical importance lies in his role as translator. In his time Greek was no longer known by all who were interested in science. Boethius translated, along with Plato and Aristotle, also Euclid, Ptolemy, and Archimedes, but these translations are not extant. He is the first author where we find the quadruple of the four sciences, arithmetic, music, geometry, and astronomy, as constituting the mathematical branch of the liberal arts, the quadrivium. We do not owe to the old Romans any significant contribution to mathematical science.

12. Mathematics in India. We can give only a very short review of mathematical activities in India during this period. The outstanding mathematicians of this country were primarily astronomers; we mention only Aryabhatta (about 500 A.D.) and Brahmagupta (about 600 A.D.). Whereas Diophantus treated problems of finding rational values for algebraic relations of second and higher degree, the mathematicians of India treated the problem of finding integers satisfying linear relations. One recognizes that such problems are indeed of interest in astronomical investigations. It is probable that the Indians were depending on the research work of Babylonian astronomers.

Probably the Indians, in their development of a new symbolism for the writing of integers, were also indebted to the Babylonians. They determined the integer by expanding it into a power series with 10 as basic number: $n=\sum_{i=0}^{i=m} a_i 10^i$, $(a_i<10)$. Then they represented the integer by the symbol $a_m \cdots a_2 a_1 a_0$. The old Babylonians had already represented integers in this way,

using 60 as the basic number. But their representation was ambiguous because they did not use symbols for those coefficients which are equal to zero.

We find our sign 0 already in the astronomical tables of Ptolemy, and we also find it used as abbreviation for "nothing" (οὐθέν) in Heron's writings. But the systematic use of the symbol 0 came from India to the Arabs and through them to Europe, and had an inestimable influence upon all kinds of scientific and practical computations.

CLUBS AND ALLIED ACTIVITIES

EDITED BY J. S. FRAME

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to J. S. Frame, Michigan State College, East Lansing, Michigan.

CLUB REPORTS 1942-43

Kappa Mu Epsilon, Mount St. Scholastic College

The Kansas Gamma Chapter held regular monthly meetings throughout the year. Among the most interesting programs presented at the meetings were the following:

Mathematics and its place in national defense, a panel discussion in which special emphasis was placed on specific mathematical requirements for various branches of the service and how collegians can best promote the war effort.

Quiz Kids, a program to which the faculty and student body were invited, and at which an intensive study of cultural, disciplinary and practical aspects of mathematics enabled the "Quiz Kids" to answer questions proposed by an interested audience.

Famous women in mathematics, a round table discussion at which important mathematicians from various European countries were discussed in chronological order.

The two major social events of the year were the annual Christmas party and the formal tea in May which followed the initiation of new members. Officers were: President, Margaret Molloy; Vice-President, Mary Margaret Downs; Secretary, Virginia Meyers; Treasurer, Mary Margaret Walters; Faculty Sponsor, Sister Helen Sullivan, O. S. B. The officers elect for 1943–44 are: President, Virginia Meyers; Vice-President, Rosemary Solas; Secretary, Rosa Garcia; Treasurer, Mary Margaret Walters; Faculty Sponsor, Sister Helen Sullivan, O. S. B.

Aphesteon, Honorary Mathematics Association of the New Jersey State Teachers College at Montclair

Due to the accelerated program of education and the fact that many members left during the year for various branches of the armed services, the meetings of *Aphesteon* have been irregular. Among the papers presented were the following.

Complex roots, by Dr. Howard Fehr

Non-Euclidean geometry, by Ruth Wheeler

A cryptographic application to linear equations, by Dr. Albert Meder, New Jersey College for Women

Linkages, by George Kays.

Another program, devoted to the subject of aeronautics, was presented with the aid of films on map projections and celestial navigation. Strip films on pilotage were also shown. The introductory and between-film comments were provided by Dorothy De Witt. *Aphesteon* also took part in making the annual spring picnic of the Mathematics Department a success. Prizes for highest scholastic records in mathematics were awarded to John Macchi and Philip Egeth who graduated in June 1942 and are now with the armed forces. Officers for the first and second terms were: Presidents, Ruth Wheeler (I) and Dorothy De Witt (II); Secretary-Treasurers, Mary Casbarro (I) and Laurel Neild (II); Adviser, Dr. Virgil Mallory.

Delta X Mathematics Club, University of Toledo

Topics considered by undergraduate speakers included:

Geometrical applications of complex numbers, by William Landry

The life of Newton and the development of the calculus, by Ralph McBee

Time, by Julian Bulley

Dimensional analysis, by Philip O'Neill

Mil measure, by Edward Faneuf

Lineal elements, by T. R. Hunter

Continued fractions and their use, by John Mason

Geometrical constructions with rule alone and with compass alone, by Ernest and John Weaver.

Each regular meeting was followed by a social hour with mathematical games for entertainment and with refreshments. The club held also two picnics, several special parties and the annual banquet, at which a talk was given on the subject:

Discontinuous functions, by Professor Fern Welker.

Officers for the year were Presidents, Charles Delbecq and then William Landry; Vice-President, Julian Bulley; Secretary-Treasurer, Frederick Racker. Professor Wayne Dancer has been faculty adviser to the Club since its founding in 1929. The membership this year was eighty-five.

DISCUSSIONS AND NOTES

EDITED BY MARIE J. WEISS, Sophie Newcomb College, New Orleans, La.

The Department of Discussions and Notes is open to all forms of activity in college mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

A SLIDE RULE SOLUTION OF OBLIQUE SPHERICAL TRIANGLES

F. L. DENNIS, Ursinus College

The following method of solving oblique spherical triangles is well adapted to a short and fairly accurate slide rule computation.

Assuming the law of cosines in the form

$$\cos a = \cos b \cos c + \sin b \sin c \cos \alpha, \tag{1}$$

substitute

$$\alpha_b = \arctan (\cot b/\cos \alpha), \qquad \left(-\frac{\pi}{2} < \alpha_b < \frac{\pi}{2}\right), \tag{2}$$

$$r = \sqrt{\cos^2 b + \sin^2 b \cos^2 \alpha} = \cos b/\sin \alpha_b. \tag{3}$$

Dividing (1) by r and substituting from (3), we obtain

$$\frac{\cos a}{\cos b} = \frac{\sin (c + \alpha_b)}{\sin \alpha_b}. \tag{4}$$

This can be written in the form

$$\frac{\sin (90 - a)}{\sin (90 - b)} = \frac{\sin (c + \alpha_b)}{\sin \alpha_b}, \tag{5}$$

where

$$\tan \alpha_b = \frac{\tan (90 - b)}{\sin (90 - \alpha)},$$

which may be referred to as the auxiliary law of sines, since it involves the auxiliary angle α_b.

There are six forms of the auxiliary law of sines obtained by permuting the sides in all possible ways. There are six corresponding forms which can be derived from the law of cosines for angles, of the type

$$\frac{\sin (90 - \alpha)}{\sin (90 - \beta)} = \frac{\sin (\gamma - a_\beta)}{\sin a_\beta}, \tag{6}$$

where

$$\tan a_\beta = \frac{\tan (90 - \beta)}{\sin (90 - a)}, \qquad \left(-\frac{\pi}{2} < a_\beta < \frac{\pi}{2}\right).$$

These twelve formulas together with the law of sines can be used to solve all oblique spherical triangles on the slide rule shown in the diagram. The rule consists of five scales: A and B are log tan scales; E is an inverted log tan scale; and C and D are log sin scales. A 44-inch classroom demonstration rule gives accuracy corresponding to four-place logarithms for the sides and angles in a normal range.

Example 1. To find the distance between two points whose latitudes and longitudes are known, the formula becomes

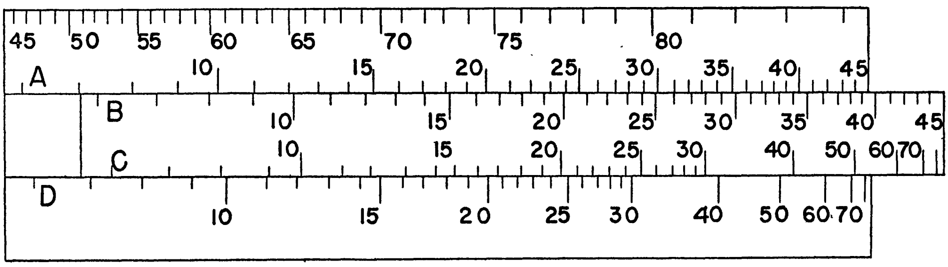
$$\frac{\sin \text{co-distance}}{\sin \text{lat } A} = \frac{\sin (\text{co-lat } B + \gamma_a)}{\sin \gamma_a}, \tag{7}$$

where

$$\tan \gamma_a = \frac{\tan \text{lat } A}{\sin (90 - \text{diff of long})}.$$

Assume A is Seattle (47°36′ $N.$, 122°20′ $W.$) and B is Manila (14°36′ $N.$, 120°57′ $E.$). To compute γ_a, set 47°36′ on scale E opposite $-26°43'$ on scale C and read the auxiliary angle $\gamma_a = -67°40'$ on scale E. Then set 7°44′ on scale C opposite $-67°40'$ on scale D and read the co-distance $= -6°8'$ on scale C opposite 47°36′ on scale D. Hence the distance is 96°8′ or 5768 nautical miles. The distance computed by the use of five-place logarithms is 5769.3 miles.

The four cases involving two like parts and one unlike part can be solved by a similar procedure.



Example 2. Given $a = 78°10'$; $b = 109°20'$; $c = 99°40'$. To compute α_b from (5), set 11°50′ on scale C opposite $-19°20'$ on scale D; move the indicator along until the sum of two numbers on the C and D scales is equal to c or 99°40′. This determines $\alpha_b = -65°25'$ on scale D since $65°25' + 34°15' = 99°40'$. To determine $90 - \alpha$, set $-19°20'$ on scale B opposite $-65°25'$ on scale E and read $(90 - \alpha) = +9°15'$ on scale C opposite the left-hand end of scale D. The other two angles can be found by using the law of sines on the C and D scales. The angle α by the use of five-place logarithms is 80°45.2′ which differs from the slide rule computation by only .2′.

It is obvious that all right spherical triangles can be solved on the slide rule since Napier's rule is reducible to sines and tangents.

By inserting two log N scales, one on the slide between the B and C scales and the other on the base below the D scale, it is possible to solve all cases of plane triangles, although the accuracy is diminished to three places. This follows because the law of sines, the law of tangents, and the law of tangents of the half-angles can be solved on the slide rule as shown in the diagram with the addition of the two log N scales.

A NOTE ON THE PRODUCT OF LINEAR FORMS

RICHARD BELLMAN, University of Wisconsin

G. Pall in the March, 1943 issue of this MONTHLY published an elegant proof of Minkowski's

THEOREM. *If* $a_{ij}(i, j=1, 2)$ *and* b_1, b_2 *are real numbers, and if* $D=|a_{ij}|\neq 0$, *then there exist integers* x_i, *such that*

$$|(a_{11}x_1 + a_{12}x_2) - b_1| \cdot |(a_{21}x_1 + a_{22}x_2) - b_2| \leqq \frac{|D|}{2}. \tag{1}$$

The conjecture is that for any n, if $L_1, L_2, \cdots, L_n$ are n real homogeneous linear forms in $x_1, x_2, \cdots, x_n$ with determinant D not zero, and $b_1, b_2, \cdots, b_n$ are n real numbers, then there exist integers x_i so that

$$\prod_i |L_i - b_i| \leqq 2^{-n}|D|. \tag{2}$$

It is perhaps interesting to prove that in the very special case where the a_{ij} represent an orthogonal transformation the conjecture is true.

Let $X_1, X_2, \cdots, X_n$ be the solutions of

$$b_i = L_i(X) = \sum_{j=1}^{n} a_{ij}X_j, \qquad i = 1, \cdots, n. \tag{3}$$

Then

$$\prod |L_i(x) - b_i| = \prod_i |L_i(x) - L_i(X)| = \prod_i |L_i(x - X)|. \tag{4}$$

An application of the arithmetic-geometric mean inequality yields

$$\prod |L_i(x - X)| \leqq \left[\frac{\sum_1^n (L_i(x - X))^2}{n}\right]^{n/2}. \tag{5}$$

Since the transformation is orthogonal

$$\sum_1^n [L_i(x - X)]^2 = \sum_1^n (x_i - X_i)^2, \tag{6}$$

and so

$$\prod_i |L_i(x - X)| \leqq \left[\frac{\sum_1^n (x_i - X_i)^2}{n}\right]^{n/2}. \tag{7}$$

We can choose the x_i, integers, so that $|x_i - X_i| \leqq \frac{1}{2}$, and then

$$\prod | L_i(x - X) | \leqq \frac{1}{2^n} \cdot \tag{8}$$

Since the determinant of an orthogonal transformation is ± 1, this gives the desired result. Orthogonality can be replaced by any condition yielding (6) with the right-hand side greater than or equal to the left.

PROBLEMS AND SOLUTIONS

ELEMENTARY PROBLEMS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND H. S. M. COXETER

Send all communications concerning Elementary Problems and Solutions to H. S. M. Coxeter, 24 Strathearn Boulevard, Toronto 10, Canada.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 611. *Proposed by D. F. Barrow, University of Georgia*

Suppose n students be standing an examination in a row of seats with a aisle at either end. If they finish in random order, find the average number of disturbances caused by students passing over one another. (If a student passes over 3 others, he causes 3 disturbances. Assume that they go out so as to cause the smallest number of disturbances. Cf. E 531 [1943, 202 and 513].)

E 612. *Proposed by V. Thébault, San Sebastián, Spain*

Show that $1110 \cdot 1111 \cdot 1112 \cdot 1113 = 1235431^2 - 1$ for any radix greater than 5. Generalize this result.

E 613. *Proposed by L. M. Kelly, U. S. Coast Guard Academy*

Can a triangle have two equal symmedians without being isosceles?

E 614. *Proposed by Joseph Rosenbaum, Bloomfield, Conn.*

Prove that the polynomial $(x+y)^n - x^n - y^n$ is divisible by x^2+xy+y^2 when $n \equiv 5 \pmod 6$, and by $(x^2+xy+y^2)^2$ when $n \equiv 1 \pmod 6$.

E 615. *Proposed by H. S. Wall, Northwestern University*

Let $g_1, g_2, g_3, \cdots$ be real numbers such that $0 < g_p < 1$ and such that the series

$$g_1 + (1 - g_1)g_2 + (1 - g_2)g_3 + \cdots$$

converges to a sum not exceeding unity. Establish the convergence of

$$1+\sum_{p=1}^{\infty}\frac{g_1g_2\cdots g_p}{(1-g_1)(1-g_2)\cdots(1-g_p)}.$$

SOLUTIONS

Sums of Residues of Powers

E 575 [1943, 326]. *Proposed by P. S. Donchian, Hartford, Connecticut*

Let (n^r) denote the standard residue of n^r modulo p (an odd prime), so that $(n^r)\equiv n^r \pmod p$ and $0\leqq(n^r)<p$. Prove that the $p-2$ numbers

$$\sum_{n=1}^{p-1}(n^r) \qquad (r=1,\cdots,p-2)$$

are a permutation of the $p-2$ numbers

$$\sum_{r=0}^{p-2}(n^r) \qquad (n=2,\cdots,p-1),$$

and that all these numbers are divisible by p.

Solution by E. P. Starke, Rutgers University. Let d be a primitive root modulo p. This implies that the numbers

$$d,\ (d^2),\ \cdots,\ (d^{p-1})=d^0=1$$

are a permutation of $1, 2, \cdots, p-1$. Let us put $n=d^g$ $(0\leqq g\leqq p-2)$.

Consider a particular value of r, $1\leqq r\leqq p-2$; suppose $(r, p-1)$, the greatest common divisor of r and $p-1$, is equal to f; and put $u=r/f$, $v=(p-1)/f$, so that $v>1$. Since

$$(d^{guf})^v\equiv(d^{p-1})^{gu}\equiv 1,$$

we see that $(d^r)^g$ is a root of the congruence

$$x^v\equiv 1 \pmod p \tag{1}$$

for any choice of g. Further, (1) has just v incongruent roots, namely

$$1,\ (d^r),\ (d^{2r}),\ \cdots,\ (d^{(v-1)r}). \tag{2}$$

Thus there are only v distinct values possible for (d^{rg}). Each is assumed f times; for, the relation

$$(d^r)^{g+v}\equiv d^{rg}d^{rv}\equiv d^{rg}d^{(p-1)u}\equiv(d^r)^g$$

implies that the f numbers

$$(d^r)^g,\ (d^r)^{g+v},\ (d^r)^{g+2v},\ \cdots,\ (d^r)^{g+(f-1)v}$$

are congruent each to each. (In this and similar results, any exponent not less than $p-1$ should be diminished by $p-1$.) Hence the $p-1=vf$ numbers (d^{rg}) may be separated into f sets each consisting of the v numbers (2). Hence

$$\sum_{n=1}^{p-1}(n^r)=\sum_{g=0}^{p-2}(d^{rg})=f\{1+(d^r)+(d^{2r})+\cdots+(d^{(v-1)r})\}\qquad (r=1,2,\cdots,p-2). \tag{3}$$

Corresponding to each f it is known that there are $\phi(v)$ distinct values of r such that $(r, vf)=f$. Thus, of the $p-2$ sums (3), each occurs $\phi(v)$ times.

Consider now a particular value of $n=d^g(1\leqq g\leqq p-2)$; suppose $(g, p-1)=k$; and put $s=(p-1)/k$, $t=g/k$. The relation

$$(n^{r+s})\equiv d^{gr+sg}\equiv d^{gr}d^{(p-1)t}\equiv d^{gr}\equiv (n^r)$$

implies that the k numbers

$$(n^r),\ (n^{r+s}),\ (n^{r+2s}),\ \cdots,\ (n^{r+(k-1)s})$$

are congruent each to each. Hence the $p-1=ks$ numbers

$$n^0=1,\ n=(d^g),\ (n^2)=(d^{2g}),\ \cdots,\ (n^{p-2})=(d^{(p-2)g})$$

separate into k sets, each set consisting of the s numbers

$$1,\ (d^g),\ (d^{2g}),\ \cdots,\ (d^{(s-1)g})$$

which, as in (2), are the incongruent roots of the congruence

$$x^s\equiv 1(\bmod p), \tag{4}$$

where $s>1$. Thus we have

$$\sum_{r=0}^{p-2}(n^r)=k\{1+(d^g)+(d^{2g})+\cdots+(d^{(s-1)g})\}\qquad (g=1,2,\cdots,p-2). \tag{5}$$

Corresponding to each k, there are $\phi(s)$ distinct values of g, and hence of n, such that $(g, ks)=k$. Thus, of the $p-2$ sums (5), each occurs $\phi(s)$ times.

If now k is taken equal to f, so that $s=v$, the congruences (1) and (4) are identical and have identical sets of roots, so that the sums (3) and (5) are identical. Furthermore each occurs the same number $\phi(s)=\phi(v)$ of times. Finally, the sum of all roots (2) of congruence (1) is congruent to minus the coefficient of x^{v-1}. Since this is zero, each of the sums (3) and (5) is divisible by p. Since $g=1, 2, \cdots, p-2$ correspond in some order to $n=2, 3, \cdots, p-1$, the proof is complete.

Seven Consecutive Digits forming a Cube

E 578 [1943, 386]. *Proposed by R. V. Heath, Wall St., New York City*

Find a perfect cube whose digits form a permutation of consecutive digits. (Cf. E 538.)

Solution by Frank Hawthorne, Allegheny College. Since the number is restricted to ten digits, the root cannot exceed 2154. Roots ending in 0, 14, 42, 53, 64, 71, 77, 92, and 99 may be eliminated because the cubes end in repeated digits. If the number contains fewer than ten digits, the root cannot end in 69; if fewer than nine digits, in 02, 31, 39, 52; if fewer than eight digits, in 09, 43, 48, 61, 98; for in each case the last two digits of the cube differ too greatly. Examining the remaining numbers with the aid of a table, we find that the only perfect cube satisfying the condition is

$$8365427 = 203^3.$$

The only higher powers satisfying the same condition are the fifth powers 32 and 243.

Also solved by W. E. Buker, M. L. Constable, Howard Eves, Irving Kaplansky, E. P. Starke, and the proposer.

Submatrices of a Matrix

E 579 [1943, 386]. *Proposed by Howard Eves, Syracuse University*

Show that a matrix of m rows and n columns contains $(2^m-1)(2^n-1)$ submatrices.

Solution by Alan Wayne, Flushing, L. I. The number of different combinations of rows from m rows is

$$\binom{m}{1}+\binom{m}{2}+\cdots+\binom{m}{m} = 2^m - 1.$$

Similarly we can choose 2^n-1 different sets of columns. Hence we get

$$(2^m - 1)(2^n - 1)$$

selections of rows and columns, or submatrices.

Also solved by W. E. Buker, Orrin Frink, Jr., Irving Kaplansky, E. P. Starke, C. W. Topp, and the proposer. The proposer remarks that the problem was suggested by exercise 3 on page 22 of Albert's *Introduction to Algebraic Theories*, where the reader is asked to write out all the submatrices of a given 4×5 matrix—a task of listing $15\cdot 31=465$ submatrices!

A Hot Water Tank

E 580 [1943, 386]. *Proposed by Albert Furman, Infantry School, Fort Benning*

A closed tank containing V gallons of water at temperature T has an inlet pipe which supplies water at temperature t. Assuming an ideal situation where there is no loss of heat and an instantaneous diffusion in the mixture, show that the temperature of v gallons of water drawn from the tank into an open container is

$$t + (T - t)(1 - e^{-v/V})V/v.$$

Solution by O. J. Karst, Newark College of Engineering. Let q be the temperature of the water in the closed tank after the passage of p gallons through the

system. The physical set-up of the problem leads to the differential equation

$$Vdq = (t - q)dp$$

with the initial condition that $q = T$ when $p = 0$. The solution is

$$q = t + (T - t)e^{-p/V}.$$

The temperature of the water in the open container is obtained as the weighted average of the temperatures of the elements of water which flow into it, the temperature q of an element being weighted by its volume dp. After the passage of v gallons, this weighted average is

$$\int_0^v q\,dp \Big/ \int_0^v dp = \int_0^v \{t + (T - t)e^{-p/V}\}\,dp/v$$
$$= t + (T - t)(1 - e^{-v/V})V/v.$$

Also solved by R. K. Allen, W. B. Campbell, Howard Eves, C. W. Johnson, S. Karlin, E. P. Starke, and the proposer.

The Farmer's Two Sons

E 581 [1943, 386]. *Proposed by D. K. Kazarinoff, University of Michigan*

A farmer died and left his two sons a herd of cattle which they sold. The number of dollars received per head was the same as the number of heads. With the proceeds of the sale the sons bought sheep at \$10 each and one lamb for less than \$10. The sheep and the lamb were divided between the two brothers so that each received the same number of animals. How much should the son who received only sheep pay to the son who received the lamb in order that the division should be equitable?

I. *Solution by Irving Kaplansky, Harvard University.* If x is the number of cattle, y the number of sheep, z the price of the lamb, we have $x^2 = 10y + z$, with y odd and $z < 10$. But it is readily shown that the penultimate digit of a square is odd if and only if the last digit is 6. Thus $z = 6$, and the luckier son should hand over \$2.

II. *Solution by L. R. Ford, Illinois Institute of Technology.* Let $10n$ be the multiple of 10 nearest the total number of cows. Then the sum available for the purchase of sheep is

$$(10n + p)^2 = 100n^2 + 20np + p^2,$$

where $p = 0, \pm 1, \pm 2, \pm 3, \pm 4$, or ± 5. Now $100n^2 + 20np$ will purchase an even number of sheep, which can be divided. The sum that remains is

$$p^2 = 0, 1, 4, 9, 16 \text{ or } 25.$$

The case $p^2 = 16$ will permit the purchase of a sheep and a lamb, while the other cases are impossible. The recipient of the \$6 lamb should be paid \$2 by his brother.

Also solved by R. K. Allen, D. H. Browne, W. E. Buker, M. L. Constable, A. H. Copeland, J. S. Cromelin, M. A. Dernham, William Douglas, C. W. Emmons, Howard Eves, N. G. Gunderson, F. A. Haight, Frank Hawthorne, Norbert Kaufman, Victoria Kloch, Elmer Latshaw, Walter Penney, W. C. Rufus, E. D. Schell, Jesse Silverman, E. P. Starke, Michael Wilensky, and C. H. Wolfe. Starke remarks that a duplicate of this problem (except that each sheep cost \$12) appeared fourteen years ago as No. 3379 [1930, 162].

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known textbooks or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4111. *Proposed by G. Pólya, Stanford University*

The axes of two unlimited circular cylinders intersect. Let a and b be the radii of the cylinders, $a>b$; γ the angle between the axes; and V the volume of the intersection (the volume within both cylinders). Writing $b/a=k$, prove that

$$\frac{16ab^2}{3\sin\gamma} < V < \frac{2\pi ab^2}{\sin\gamma},$$

$$\frac{2\pi ab^2}{\sin\gamma}\left[1-\left(1-\frac{8}{3\pi}\right)k^2\right] < V < \frac{16ab^2}{3\sin\gamma}\,\frac{5-k^2}{4},$$

and generally for $n=0, 1, 2, 3, \cdots$

$$0<\frac{2\pi ab^2}{\sin\gamma}\left[1-\sum_{\nu=1}^{n}\left(\frac{1}{2}\,\frac{3}{4}\cdots\frac{2\nu-1}{2\nu}\right)^2\frac{k^{2\nu}}{(\nu+1)(2\nu-1)}\right]-V$$

$$<\frac{2\pi ab^2}{\sin\gamma}\left[1-\frac{8}{2\pi}-\sum_{\nu=1}^{n}\left(\frac{1}{2}\,\frac{3}{4}\cdots\frac{2\nu-1}{2\nu}\right)^2\frac{1}{(\nu+1)(2\nu-1)}\right]k^{2n+2}.$$

Note. The proposer is indebted to F. R. Morris for the problem of finding approximations to V.

4112. *Proposed by N. A. Court, University of Oklahoma*

If three rulers, chosen arbitrarily, of the same system of a given hyperboloid are taken for the edges of a parallelopiped, the diagonals of the parallelopiped meet in a fixed point.

4113. *Proposed by J. H. Butchart, Grinnell College*

If every member of one family of developables cuts every member of another family of developable surfaces orthogonally, the edges of regression of the two families form a system of orthogonal geodesics on a single developable surface.

4114. *Proposed by V. Thébault, San Sebastián, Spain*

The digits 1, 2, 3, 4, 5, 6, 7, 8, 9 are taken each once and only once to form three numbers N, n, r so that $N=n^3-r$. Find these three numbers.

SOLUTIONS

Euler Line

4057 [1942, 616]. *Proposed by J. R. Musselman, Western Reserve University*

Let B_1, B_2, B_3 be the points symmetric to the vertices of triangle $A_1A_2A_3$ in its circumcenter O, and let C_1, C_2, C_3 be the reflections of A_i in the perpendicular bisector of the sides of $A_1A_2A_3$. It is known that the circles OB_1C_1, OB_2C_2, OB_3C_3 meet at a point P. Show that P lies on the Euler line of $A_1A_2A_3$ and that O is the midpoint of PD, where D is the inverse in the circumcircle of the orthocenter H of $A_1A_2A_3$.

Solution by Hüseyin Demir, Columbia University. Let $G_1G_2G_3$ be the triangle formed by the straight lines A_iC_i so that $A_1A_2A_3$ is its medial triangle, the circumcircle (O) of the latter is its ninepoint circle, G_iC_i are its altitudes, its orthocenter H' is the symmetric of H with respect to O. and the straight lines C_iB_i are concurrent in H'. Let P be the point where the circle (OB_1C_1) cuts OH', *i.e.*, OH. We have $H'O\cdot H'P=H'C_1\cdot H'B_1=H'C_1\cdot H'B_i$; hence the circles (OB_iC_i) intersect again in P. The inverse of (OB_1C_1) with respect to (O) is B_1C_1, and hence $OH'\cdot OP=\overline{OC}_1^2=R^2$. Since $OH\cdot OD=R^2$ and $OH=H'O$, we must have $OD=PO$.

Solved also by H. Eves using inversion with respect to O and power $-R^2$ which gives a concise proof.

Tetrahedron, Spheres, Sums of Powers

4064 [1942, 689]. *Proposed by V. Thébault, San Sebastián, Spain*

Given a tetrahedron $ABCD$: (1) Find the locus of points M such that the sum of the powers of the vertex A with respect to the spheres with diameters MB, MC, MD is constant. (2) Find the point M such that for the spheres with the diameters MA, MB, MC, MD the sum of the powers of a vertex with respect to the three spheres not passing through that vertex is the same for the four vertices. Show that the point M in this case is the symmetric of the centroid with respect to the circumcenter of the tetrahedron.

Solution by Howard Eves, Syracuse University. (1) We use vector algebra, denoting the points A, B, C, D, M by the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$, $\mathbf{m}$. The sum of the powers of A with respect to the spheres on MB, MC, MD as diameters is

$$(\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}-\mathbf{m})+(\mathbf{a}-\mathbf{c})\cdot(\mathbf{a}-\mathbf{m})+(\mathbf{a}-\mathbf{d})\cdot(\mathbf{a}-\mathbf{m}).$$

Therefore we have

$$(3\mathbf{a} - \mathbf{b} - \mathbf{c} - \mathbf{d})\cdot(\mathbf{a} - \mathbf{m}) = k, \text{ a constant,}$$

or, setting $\mathbf{g} = \frac{1}{4}(\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d})$, (for the centroid G),

$$(\mathbf{a} - \mathbf{g})\cdot(\mathbf{a} - \mathbf{m}) = k/4.$$

This last result shows that the locus of M is a plane perpendicular to AG.

(2) Now take the circumcenter as origin, so that $\mathbf{a}\cdot\mathbf{a} = \mathbf{b}\cdot\mathbf{b} = \mathbf{c}\cdot\mathbf{c} = \mathbf{d}\cdot\mathbf{d} = R^2$, where R is the circumradius. Then, for some k, we have

$$(\mathbf{a} - \mathbf{g})\cdot(\mathbf{a} - \mathbf{m}) = (\mathbf{b} - \mathbf{g})\cdot(\mathbf{b} - \mathbf{m}) = (\mathbf{c} - \mathbf{g})\cdot(\mathbf{c} - \mathbf{m})$$
$$= (\mathbf{d} - \mathbf{g})\cdot(\mathbf{d} - \mathbf{m}) = k/4.$$

Adding we have

$$(\mathbf{a} - \mathbf{g})\cdot\mathbf{a} + (\mathbf{b} - \mathbf{g})\cdot\mathbf{b} + (\mathbf{c} - \mathbf{g})\cdot\mathbf{c} + (\mathbf{d} - \mathbf{g})\cdot\mathbf{d} = k, \quad \text{or} \quad 4R^2 - \mathbf{g}\cdot\mathbf{g} = k,$$

whence

$$(\mathbf{a} - \mathbf{g})\cdot(\mathbf{a} - \mathbf{m}) = R^2 - \mathbf{g}\cdot\mathbf{g}.$$

Expanding and collecting we see that

$$(\mathbf{a} - \mathbf{g})\cdot(\mathbf{m} + \mathbf{g}) = 0.$$

Since we similarly have

$$(\mathbf{b} - \mathbf{g})\cdot(\mathbf{m} + \mathbf{g}) = (\mathbf{c} - \mathbf{g})\cdot(\mathbf{m} + \mathbf{g}) = (\mathbf{d} - \mathbf{g})\cdot(\mathbf{m} + \mathbf{g}) = 0,$$

it follows that $\mathbf{m} = -\mathbf{g}$, or M is the symmetric of G in O. We have incidentally shown that the common power is $k = 4(R^2 - OG^2)$.

Solved also by P. D. Thomas.

Three Point Collinearity

4065 [1943, 65]. *Proposed by P. Erdös, Princeton, N. J.*

(1) Let n given points have the property that the straight line joining any two of them passes through a third point of the set. Show that the n points lie on a straight line.

(2) Given n points which do not all lie on the same straight line, prove that if we join every two of them we obtain at least n distinct straight lines.

Solution by Robert Steinberg, Student, University of Toronto. (1) We shall suppose that the n points do not all lie on one line, and show that this leads to a contradiction. Let A, B, C be three noncollinear points among the n, and let a be any line through A, in the plane ABC, which does not contain any other point of the set. Let those joins of pairs of the n points which meet a do so at points $P_1, P_2, \cdots$. There will be at least one of the points $P_1, P_2, \cdots$ which does not coincide with A, *viz.* the one that lies on BC; thus the points $A, P_1,$

$P_2, \cdots$ decompose the line a into two or more segments. Hence we can find a point P among $P_1, P_2, \cdots$, such that one of the two segments AP contains none of the points $P_1, P_2, \cdots$ within it.

By our hypotheses, P is not one of the set of n points, but lies on a line containing at least three of them. Let these three points be named Q, R, S in such an order that P is separated from R by Q and S. Since A and R are two points of the set, the line AR will contain a third point of the set, say O. Let a meet QO at P_1, and SO at P_2. Then, by perspectivity from O, P_1 and P_2 separate P from A; *i.e.*, the points P_1 and P_2 lie *one* in *each* of the two segments AP. Thus we come to a contradiction; so in fact all the n points must be collinear.

(2) Now, suppose we have n points not all collinear, and let m be the number of distinct joins of pairs of these points. By the theorem just proved, at least one of these m lines contains only two points of the set, say A_1 and A_2. Consider the set of $n-1$ points obtained by excluding A_1. This set will contain at most $m-1$ distinct joins of pairs of points, as the line A_1A_2 is not one of these joins. We can repeat this process, excluding one point from the set in the manner described, as long as the remaining points are not all collinear. Suppose that after r such steps the remaining $n-r$ points are all collinear. Then, before the rth step we had $n-r+1$ points, of which just $n-r$ were collinear. Thus there were exactly $n-r+1$ distinct joins. But, since $r-1$ points have been subtracted from the original set, we have

$$n - r + 1 \leqq m - (r - 1),$$

i.e.,

$$n \leqq m.$$

Thus there are at least n distinct joins.

Solved also by R. C. Buck, T. Grünwald and N. E. Steenrod.

Editorial Note. The proposer enclosed with the problem an outline of Grünwald's solution of part (1) and remarked that part (2) could be easily proved by induction using (1). This proof of (1) may be put in the following form. The assumption that the n points do not lie on a straight line leads to a contradiction as follows.

Consider only those points A_i which lie in a plane determined by three noncollinear points, and project them into the points B_i, from a center O not in the plane, on a second plane parallel to OA_1 but not to any other OA_i. In the second plane we have a system of parallels each containing at least three points including the point at infinity B_1. Let l be the join of two points meeting the system at the least positive angle. Then on l there are three finite points B_i, B_j, B_k, where we suppose that B_j lies between B_i and B_k, and there are three parallels through these points. On the parallel through B_j there is at least one finite point B' distinct from B_j, and we now have the contradiction that one of the two joins B_iB', B_kB' must make a smaller positive angle with the parallels. Hence the original n points lie on a single straight line.

The proof by Buck of part (1) is similar to Grünwald's proof, and he stated that this part of his proof is not original; he was uncertain about the origin of the earlier proof. Steenrod likewise reduced part (1) to its two dimensional case; he then derived the dual theorem in the projective plane (where we have n distinct lines such that the intersection of any two is on a third) from the impossibility of constructing a map whose regions, occurring six or more at each vertex, have each three or more sides.

L. M. Kelly remarked that part (1) was proposed by Sylvester in the *Educational Times*, Mathematical Question 11851, vol. 59, 1893, p. 98; but it was not satisfactorily solved there. Erdös stated that, at Oslo, Karamata asked him about this problem which he had seen stated without proof in an old book about mechanics.

Multiplication by Addition and Reciprocation

4062 [1942, 689]. *Proposed by N. S. Mendelsohn, University of Toronto*

Show how the operation of multiplying by a real number may be expressed in terms of the operations of adding a real number and of reciprocating.

Solution by the Proposer. Multiplication by a negative number $-n^2$:

$$-n^2x = n + \cfrac{1}{-\cfrac{1}{n} + \cfrac{1}{n + \cfrac{1}{x}}}\cdot$$

Multiplication by a positive number n^2:

$$n^2x = n + \cfrac{1}{-\cfrac{n+1}{n} + \cfrac{1}{1 + \cfrac{1}{-(n+1) + \cfrac{1}{x}}}}\cdot$$

Self Polar Triangle Inscribed in Another Conic

4063 [1942, 689]. *Proposed by H. S. M. Coxeter, University of Toronto*

In projective geometry the porism of triangles inscribed in one conic and self-polar for another is commonly proved by showing that if one such triangle exists, we can find another with one vertex at *any* given point on the first conic. This statement is easily seen to be valid in complex geometry. Discuss its possible failure in real geometry.

Partial Solution by the Prosposer. Consider the conics $yz+zx+xy=0$ and $x^2+y^2=z^2$. The polar of $(2, 2, -1)$ for the latter fails to meet the former.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to B. W. Jones, White Hall, Cornell University, Ithaca, New York.

Flora Dinkines of the University of South Carolina has been promoted to an adjunct professorship.

Edwin Eagle of Taft Junior College, California, has been appointed to an assistant professorship at Oregon State College.

Professor J. M. Howie of Nebraska Wesleyan University has retired.

Assistant Professor W. J. Kirkham of Oregon State College is on leave of absence. He holds the rank of lieutenant in the United States Navy.

Associate Professor A. E. Meder, Jr., of New Jersey College for Women has been appointed secretary of Rutgers University.

Professor Richard Morris will retire as head of the department of mathematics of the New Jersey College for Women on June 30, 1944. Associate Professor C. A. Nelson will then become head of the department. Professor Morris will continue to teach in Rutger's University.

Associate Professor W. S. Slauch of New York University has been appointed professor emeritus.

Associate Professor Raleigh Schorling of the University High School of the University of Michigan has been appointed to a professorship in the department of education at the University of Michigan.

Professor E. R. Smith of the Florida State College for Women has retired.

S. Grace Smyth of Knox College has been promoted to an associate professorship.

Assistant Professor G. J. Stigler of the University of Minnesota has been promoted to an associate professorship in economics.

Fern Welker of the University of Toledo has been promoted to an assistant professorship.

Associate Professor Marie Johnson Yeaton of Oberlin College has resigned.

The following appointments to instructorships are announced:

Oregon State College: Florence Bakkum, L. R. Foote, Fred Young
Syracuse War Training Program at Auburn: Dr. S. Helen Taylor
University of Pennsylvania Physics Department: G. H. Wilson

Associate Professor G. A. Pfeiffer of Columbia University died on January 4, 1944.

Professor H. L. Rietz of the State University of Iowa died on December 7, 1943. He was a charter member of the Mathematical Association.

The Editor-in-Chief wishes to express his indebtedness to the following persons who have served as referees of papers during 1943.

R. P. Agnew, A. A. Albert, Emil Artin, E. F. Beckenbach, L. M. Blumenthal, J. W. Bradshaw, Louis Brand, H. E. Bray, R. W. Brink, H. E. Buchanan, W. B. Carver, W. B. Caton, R. V. Churchill, C. J. Coe, N. B. Conkwright, A. H. Copeland, H. S. M. Coxeter, D. R. Curtiss, H. T. Davis, W. M. Davis, John DeCicco, Max Dehn, J. S. Frame, Orrin Frink, Jr., M. G. Gaba, Edward Helly, M. R. Hestenes, T. J. Higgins, Dunham Jackson, R. A. Johnson, A. J. Kempner, Mayme I. Logsdon, C. C. MacDuffee, S. B. Meech, E. J. Moulton, J. R. Musselman, C. V. Newsom, W. C. Randels, Haim Reingold, M. A. Sadowsky, Peter Scherk, O. F. G. Schilling, Harold Shapiro, I. M. Sheffer, G. E. F. Sherwood, N. E. Steenrod, E. B. Stouffer, T. Y. Thomas, G. E. Wahlin, L. E. Ward, Marie J. Weiss, L. R. Wilcox, F. E. Wood, and R. C. Yates.

WAR INFORMATION

EDITED BY C. V. NEWSOM

Send news reports upon the utilization of mathematicians or mathematics in war activities to C. V. Newsom, University of New Mexico, Albuquerque, New Mexico.

THE REPUBLICATION OF FOREIGN MATHEMATICAL TABLES

The Office of Alien Property Custodian has licensed, during the past several months, the reprinting of scientific and technical books, of enemy origin, which are not available in a quantity sufficient to meet the demands of the wartime operations of science and industry.

In this connection the custodian has received several queries concerning the possibility of licensing the republication of additional Mathematical Tables. Licensed for republication and now available for purchase are Jahnke and Emde, "Funktionentafeln mit Formeln und Kurven," 1938; Jean Peters, "Siebenstellige Werte der Trigonometrischen Funktionen," 1938, and his "Achstellige Tafel," 1939.

Before a definite decision can be made regarding the licensing of additional Mathematical Tables for republication, it is necessary for the custodian to be informed about the extent of the need of such tables and to receive suggestions of specific titles for consideration. This can be accomplished if suggestions of specific significant tables are sent by individuals to the Office of Alien Property Custodian, Washington, D. C. These suggestions or any inquiries should be addressed to the undersigned.

HOWLAND H. SARGEANT,
Chief, Division of Patent Administration
Office of Alien Property Custodian,
Washington, D. C.

QUOTAS IN MILITARY TRAINING PROGRAMS

The American Council on Education has made a study of the number of military personnel receiving instruction in the Army and the Navy College Training Programs. The data indicate that on October 1, 1943, there were 276,821 men in the four major programs. The distribution follows.

	Number of Institutions	*Number of Trainees*
Army Specialized Training	216	129,080
Army Air Forces	151	66,512
Navy College Training	244	73,486
Navy Air Force	17	7,743

When duplications are eliminated from the total of institutions, it is found that there are 440 participating colleges and universities.

Since the compilation of the figures above, there has been a slight decrease in the total quota allotted to the Army Specialized Training Program. In fact, the rumor has been circulated that the Program would shortly be liquidated. Consequently, upon December 13, the War Department released the following statement.

"In answer to inquiries, the War Department announced today that the Army Specialized Training Program is not in process of liquidation. The number of soldiers in the Program will depend in the future, as in the past, on the actual needs of the Arms and Services.

"In this connection the Secretary of War said: 'The number of soldiers assigned for training under the ASTP will be changed from time to time so as to accord with the needs of the Army and the available manpower. It is now being reduced—but may later be either increased or still further reduced as the exigencies of the military situation or military training make advisable.'"

At the present time no essential change is contemplated in the total Navy quota. It is interesting to note, however, that the number of men from active duty being returned for study in the Navy V-12 Program has recently been doubled.

CENTRAL CLEARING AGENCY FOR ACCREDITATION

The following memorandum was issued from the Headquarters of the United States Armed Forces Institute, Madison 3, Wisconsin.

1. The United States Armed Forces Institute (originally the Army Institute) was established on 1 April, 1942. The Institute offers to men and women in the Armed Services formal off-duty correspondence instruction as an opportunity to increase their military efficiency, to prepare for a civilian occupation after the war, or to complete requirements for a high school diploma or college degree.

2. The Institute has established a Central Clearing Agency for Accreditation to help service personnel secure academic credit for their in-service training and experience. The following regional accrediting associations have endorsed the procedure of the Central Clearing Agency for Accreditation:

The New England Association of Colleges and Secondary Schools
The North Central Association of Colleges and Secondary Schools
The Middle States Association of Colleges and Secondary Schools
The Southern Association of Colleges and Secondary Schools
The Northwest Association of Secondary and Higher Schools

3. The Institute does not grant nor recommend academic credit, but will assemble all available information concerning the in-service training and experience of service personnel and forward a complete, official report of the data collected to the designated high school or college for evaluation.

4. The Institute will also assemble an official report and forward it to a prospective employer to assist the serviceman in his future civilian job plans. This report, however, will be furnished only to service personnel who are about to receive their discharge from the armed forces.

5. In order to initiate the accreditation service, it is necessary that each serviceman secure and fill out the application form "Request for Report of Educational Achievement." This form can be secured by writing to the Institute. It is suggested that high schools and colleges advise any of their former students now in the Armed Forces who desire an evaluation of their in-service training and experience to write to the Institute requesting this application form.

CARL W. HANSEN,
Lt. Col., A.G.D., Commandant

ENLISTED MEN ENTERING THE NAVY V-12 PROGRAM

Due to general dissatisfaction with the procedure followed by the Navy in selecting enlisted men for the V-12 Program, more rigorous academic criteria have recently been instituted. No complete announcement of the new standards has been officially released for publication, but available reports indicate that the following academic qualifications are now necessary.

1. Enlisted men will not be recommended for the V-12 Program unless they have had a minimum of two years of high school mathematics. Preference will be given to those who have had algebra, geometry, and trigonometry.

2. All enlisted men entering the Program must have passed a general classification test with a high score. In the case of candidates outside the continental United States who may have had no opportunity to take the classification test, an oral examination over algebra, geometry, English, and history will be given to determine whether the men are qualified for college work. Men satisfactorily passing the oral examination must still be given the written test upon their arrival at Navy stations designated as assembly points.

It is the present intention also to give enlisted men gathered at the assembly points some review work in mathematics before assignment to institutions participating in the V-12 Program.

THE NAVY V-5 PROGRAM

The Navy V-5 Program is for the purpose of training aviation cadets for the Navy Air Force. Candidates selected by the Naval Aviation Cadet Selection Boards follow the training schedule given below.

First three months—Naval Flight Preparatory School

No flight training. The entire time is spent in ground school on the following subjects: review of mathematical processes, practical arithmetic, trigonometry and geometry, fundamentals of physics, aerial navigation using plotting boards, aerology, aircraft structures, aircraft engines, Naval indoctrination, radio code, aircraft and ship recognition, physical training and drill.

Second three months—CAA-WTS Flight Training School

Initial stage of flight training with a maximum of sixty hours of flight. The ground school consists of a review of navigation, review of aerology, review of aircraft structures and engines, continued practice in radio code and semaphore, recognition, civil air regulations, physical training and drill.

Third three months—Naval Pre-Flight School

No flight training. The emphasis is on physical training and bodily contact sports. The ground school consists of a review of navigation with an introduction to celestial navigation, aerology, code, semaphore, and blinker, recognition, and Naval history.

Next three months or less—Naval Primary Flight Training Base

Advanced flight training with formation and night flying. The ground school is secondary and consists of further study of navigation, aerology, recognition, code and blinker, physical training and drill.

Next three to four months—Naval Air Station

Final transition to combat types of aircraft with emphasis on combat tactics, aerial gunnery, instrument flying, and precision air work. The ground school consists of precision work in navigation, aerology, recognition, instrument flying, aerial gunnery, and combat tactics. Information on other courses is restricted. At the end of this phase of the training the cadets are commissioned as ensigns in the USNR.

Next one to two months—Transition Training Base Naval Air Station

Transition to the type of plane to be used in combat, and the organization of combat squadrons. The ground school is of an individualized type having to do with individual squadron problems and acquainting the new pilots with combat routine.

Many aviation candidates are now taking the first two semesters of the Navy V-12 Program, being designated V-12A. At the end of this period of training, they will go to a Naval Flight Preparatory School and start the regular program as outlined above.

The mathematics now being offered in the Naval Flight Preparatory Schools is taught for four weeks, twelve hours per week. The outline of the course with the hours allotted to each topic is given below.

First week: Fundamental operations and conversion factors (4), fractions (3), percentage (1), review (1), test (1), powers and roots (2).

Second week: Powers and roots (3), angular measurement (2), vectors (3), review (1), test (1), equations (2).

Third week: Equations (4), variation (4), graphs (2), review (1), test (1).

Fourth week: Geometry (4), trigonometry (4), test (1), review of entire course (2), final examination (1).

THE MATHEMATICAL ASSOCIATION OF AMERICA

REPORT OF THE TREASURER FOR THE YEAR 1943

The following report of the Secretary-Treasurer as Treasurer for the year 1943 has been approved by the Finance Committee and accepted by vote of the Board of Governors.

I. Total Funds of the Association on January 15, 1943

(See Cairns' report, pp. 208–209 of the Monthly for March, 1943)

Current Fund (checking account)		$ 1,172.47
Savings Account, Cleveland Trust Company		1,713.66
Savings Account, Oberlin Savings Bank		811.20
Invested Funds, Cleveland Trust Company		
Carus Fund	$ 7,144.41	
Chace Fund	8,499.04	
Houck Fund	8,405.98	
Chauvenet Fund	614.40	
Life Membership Fund	824.30	
General Fund	20,438.19	45,926.32
		$49,623.65

II. Current Fund Account for 1943

Receipts		Expenditures	
Balance, Jan. 15, 1943	$1,172.47	Monthly	
Individual dues	8,650.80	Publications	$6,299.83
Institutional dues	799.90	Reprints	59.89
Initiation fees	204.00	Editor-in-Chief's office	278.65
Subscriptions	1,922.33	Secretary-Treasurer's office	
Sale of back numbers Monthly	101.95	Clerical help	3,428.88
Advertising	518.00	Printing	270.74
Sale of Archibald's Outline	72.50	Postage	574.93
Sale of the library	1,661.64	Supplies, etc.	381.79
Interest on General Fund	538.44	"Smith list"	101.82
Sale of Carus Monographs	1,036.25	Bank charges	93.93
Interest on Carus Fund	188.19	President's office	103.23
Income from Hardy Fund	120.00	Exec. and Finance committees	82.85
Sale of Papyrus (for Chace Fund)	40.00	Regional Governors	108.93
Interest on Chace Fund	223.90	Subventions	
Interest on Chauvenet Fund	16.19	A.M.S.	100.00
Interest on Houck Fund	221.44	Mathematical Reviews	500.00
Interest on Life Membership Fund	21.71	Duke Journal (from 1942)	50.00
Transferred from Carus Fund	305.14	Expense of Sections	132.72
Transferred from Houck Fund	178.56	Meetings	230.21
Transferred from Life Membership Fund	88.96	Purchase of back numbers	68.64
		To B. F. Finkel (Hardy Fund)	120.00
		Subscriptions to Annals	7.50
		Expense, sale of library	60.32
		Paid out for Carus Fund	1,529.58
		Paid out for Houck Fund	400.00
		To Clev. Trust Co. for General Fund	262.80
		Transferred to Chace Fund	263.90
		Transferred to Chauvenet Fund	16.19
		Transferred to Life Membership Fund	21.71
		Transferred to General Fund	270.86
		Balance, Dec. 31, 1943	2,262.47
	$18,082.37		$18,082.37

III. Savings Account, Cleveland Trust Company

Balance, Jan. 15, 1943	$1,713.66		
Interest (half-year)	8.31	Balance, Dec. 31, 1943	$1,721.97
	$1,721.97		$1,721.97

IV. Invested Funds, Cleveland Trust Company

Cash balance, Jan. 15, 1943	$ 27.82		
Market Value of Securities, Jan. 15, 1943	45,898.50		
Increase in value of securities	1,140.88	Market value of securities, Dec. 31, 1943	$47,330.00
Cash from Current Fund	262.80		
	$47,330.00		$47,330.00

List of Securities

	Par Value	Market Value Dec. 31, 1943
U. S. Savings Bonds	$1,700.00	$ 1,550.00
U. S. Treasury Note, 1%, Ser. A, 1946	3,000.00	3,000.00
U. S. Treasury Bond, $2\frac{3}{4}\%$, 1947	1,000.00	1,030.00
U. S. Treasury Bonds, 2%, 1950	3,000.00	3,060.00
U. S. Treasury Bonds, $1\frac{3}{4}\%$, 1948	2,000.00	2,020.00
U. S. Savings Bonds, $2\frac{1}{2}\%$, Ser. G, 1953	3,000.00	3,000.00
U. S. Savings Bonds, $2\frac{1}{2}\%$, Ser. G, 1954	8,200.00	8,200.00
Canadian Nat. Ry. Co. Bonds, $4\frac{1}{2}\%$, 1956	2,000.00	2,300.00
Gatineau Power Co. 1st Mort. Bonds, $3\frac{3}{4}\%$, Ser. A, 1969	2,000.00	1,960.00
Shawinigan W. and P. Co. 1st Mort. Bonds, $4\frac{1}{2}\%$, 1970	2,000.00	2,080.00
C. and O. Ry. Co. Ref. Mort. Bonds, $3\frac{1}{2}\%$, Ser. D. 1996	3,000.00	3,210.00
Penn. R. R. Co. Gen. Mort. Bonds, $3\frac{3}{4}\%$, Ser. C, 1970	2,000.00	1,980.00
Amer. Tel. and Tel. Co. Conv. Deb. Bonds, 3%, 1956	2,000.00	2,320.00
Columbus and So. Ohio Elec. Co. 1st Mort. Bonds, $3\frac{1}{4}\%$, 1970	2,000.00	2,180.00
Commonwealth Edison Co. Conv. Deb. Bonds, $3\frac{1}{2}\%$, 1958	2,000.00	2,240.00
Montana Power Co. 1st Ref. Mort. Bonds, $3\frac{3}{4}\%$, 1966	3,000.00	3,180.00
New York Steam Corp. 1st Mort. Bond, $3\frac{1}{2}\%$, 1963	1,000.00	1,070.00
Texas Power and Light Co. 1st Mort. Bond, 5%, 1956	1,000.00	1,080.00
Phelps Dodge Corp. Conv. Deb. Bond, $3\frac{1}{2}\%$, 1952	1,000.00	1,070.00
Hotel Cleveland Site, Land Trust Certifs.	700.00	800,00
		$47,330.00

V. Carus Fund

Balance, Jan. 15, 1943	$7,144.41	Publication of 7th Monograph	$1,224.58
Sale of Monographs	1,036.25	Honorarium, 7th Monograph	300.00
Interest	188.19	Expenses of committee	5.00
Increase in value of securities	177.46	Balance, Dec. 31, 1943	7,016.73
	$8,546.31		$8,546.31

VI. Chace Fund

Balance, Jan. 15, 1943	$8,499.04		
Sale of Papyrus	40.00		
Interest	223.90		
Increase in value of securities	211.13	Balance, Dec. 31, 1943	$8,974.07
	$8,974.07		$8,974.07

VII. Houck Fund

Balance, Jan. 15, 1943	$8,405.98		
Interest	221.44	Subvention Nat. Math. Magazine	$ 400.00
Increase in value of securities	208.82	Balance, Dec. 31, 1943	8,436.24
	$8,836.24		$8,836.24

VIII. Chauvenet Fund

Balance, Jan. 15, 1943	$ 614.40		
Interest	16.19		
Increase in value of securities	15.26	Balance, Dec. 31, 1943	$ 645.85
	$ 645.85		$ 645.85

IX. Life Membership Fund

Balance, Jan. 15, 1943	$ 824.30	Transferred to Current Fund	$ 88.96
Interest	21.71	Balance, liability as of Dec. 31,	
Increase in value of securities	20.47	1943	777.52
	$ 866.48		$ 866.48

X. General Invested Fund

Balance, Jan. 15, 1943	$20,438.19		
Transferred from Current Fund	262.80		
Transferred from Current Fund	270.86		
Increase in value of securities	507.74	Balance, Dec. 31, 1943	$21,479.59
	$21,479.59		$21,479.59

XI. Total Funds of the Association on December 31, 1943

Current Fund (checking account)		$ 2,262.47
Savings Account, Cleveland Trust Company		1,721.97
Savings Account, Oberlin Savings Bank		811.20
Invested Funds, Cleveland Trust Company		
Carus Fund	$ 7,016.73	
Chace Fund	8,974.07	
Houck Fund	8,436.24	
Chauvenet Fund	645.85	
Life Membership Fund	777.52	
General Fund	21,479.59	47,330.00
		$52,125.64

W. B. Carver, *Secretary-Treasurer*

THE TWENTY-NINTH ANNUAL MEETING OF THE KANSAS SECTION

A joint meeting of the Kansas Section of the Mathematical Association of America and the Kansas Association of Teachers of Mathematics was held at the University of Kansas, Lawrence, Kansas, on Saturday, April 10, 1943. Professor C. F. Lewis, Chairman of the Section, presided at both the morning and afternoon sessions.

The attendance was fifty-six, including the following twenty-five members of the Association: R. W. Babcock, Wealthy Babcock, W. D. Bemmels, Florence L. Black, R. D. Daugherty, Lucy T. Dougherty, Paul Eberhart, W. H. Garrett, Edison Greer, J. R. Hanna, W. C. Janes, H. E. Jordan, C. F. Lewis, Thirza A. Mossman, G. B. Price, C. B. Read, J. A. G. Shirk, D. T. Sigley, G. W. Smith, E. B. Stouffer, W. T. Stratton, Gilbert Ulmer, E. B. Wedel, J. J. Wheeler, Ferna E. Wrestler.

The following officers were elected for the coming year: Chairman, Paul Eberhart, Washburn Municipal University of Topeka; Vice-Chairman, Edison Greer, Beech Aircraft Corporation, Wichita; Secretary, Anna Marm, Bethany College. The executive committee was requested to select the time and place for the next meeting.

Professor Gilbert Ulmer, chairman of the committee on the fourth mathematics placement test, reported on the tests given at the beginning of the 1942–1943 school year to entering students in most of the colleges and junior colleges in the state. After some discussion, the committee was asked to continue the test work for the year 1943–1944.

The following papers were presented:

1. *A correspondence refresher course for mathematics teachers*, by Professor Florence L. Black, University of Kansas.

The speaker gave a discussion of the refresher course for mathematics teachers given by the University of Kansas Bureau of Correspondence Study under the sponsorship of the Federal Government.

2. *Mathematics in war*, by Professor R. W. Babcock, Kansas State College.

3. *Mathematics in air navigation*, by Professor Paul Eberhart, Washburn Municipal University.

It was pointed out by Professor Eberhart that very little mathematics is used in the actual practice of air navigation, but that a considerable amount of elementary mathematics is desirable in the training of the navigator. He stated that the most important topics in mathematics for the navigator are simple arithmetic, significant figures, reading of tables, reading of graphs, and the slide rule. For an actual understanding of celestial navigation, solid geometry and spherical trigonometry are needed. In practice, mathematical results are obtained by means of tables and mechanical computers.

4. *Cryptography-secret writing*, by Professor G. W. Smith, University of Kansas.

Professor Smith pointed out that ciphers are usually of three main types, namely concealment ciphers, transposition ciphers, and substitution ciphers. Several slides were shown illustrating each of these types. Certain methods used in deciphering were illustrated. The speaker showed how the theory of finite fields is used to construct ciphers which might well be considered "unbreakable."

5. *Adjustments in mathematics to the impact of war*, by Professor G. B. Price, University of Kansas.

Professor Price's paper was published in this MONTHLY, vol. 59, 1943, pp. 31–34.

ANNA MARM, *Secretary*

THE EIGHTEENTH ANNUAL MEETING OF THE PHILADELPHIA SECTION

The eighteenth annual meeting of the Philadelphia Section of the Mathematical Association of America was held at the University of Pennsylvania, Philadelphia, Pa., on Saturday, November 27, 1943. Professor J. E. Davis, Chairman of the Section, presided at the morning and afternoon sessions.

The attendance was forty-one, including the following twenty-seven members of the Association: E. F. Allen, A. A. Bennett, H. W. Brinkmann, W. B. Campbell, P. A. Caris, J. W. Clawson, H. B. Curry, J. E. Davis, F. L. Dennis, Margaret C. Eide, Benjamin Epstein, C. E. Heilman, R. T. Luginbuhl, F. L. Manning, A. E. Meder, Jr., Lillian Moore, Richard Morris, W. R. Murray, C. A. Nelson, C. W. Pflaum, C. J. Rees, J. B. Rosser, L. L. Smail, G. L. Walker, A. D. Wallace, Anna Pell Wheeler, P. M. Whitman.

At the business meeting the following officers were elected for the next year: Chairman, Anna Pell Wheeler, Bryn Mawr College; Secretary, P. M. Whitman, University of Pennsylvania; Program Committee, H. W. Brinkmann, Swarthmore College (Chairman), Tomlinson Fort, Lehigh University, and F. L. Manning, Ursinus College. It was voted to hold the next meeting at the University of Pennsylvania, Philadelphia, Pa., on the Saturday following Thanksgiving, 1944. The date was made subject to change if necessary.

The following papers were presented:

1. *Fixed point theorems*, by Professor A. D. Wallace, University of Pennsylvania.

The Brouwer fixed point theorem was proved and applied to the theory of matrices.

2. *Some modern viewpoints on euclidian geometry*, by Major A. A. Bennett, Aberdeen Proving Ground.

The topic arose from discussion of an elementary Hermetian geometry of the general simplex. Special aspects concerned: (1) the use of analytic as con-

trasted with synthetic methods; (2) the use of a fundamental discontinuous complex field, extended by adjunction to form the scalar field; (3) the use of a finite but large dimensionality, special subspaces yielding subgeometries; (4) extension to imaginary elements by use of a fundamental Hermetian rather than quadratic form; (5) the use of particles as weighted points, and of other weighted figures; (6) the use not only of Hermetian hyperspheres (shells), but also of analogous algebraic-geometric elements to which no points are incident; (7) The use of a unique particle A at infinity; (8) the use of a basic fixed simplex from which, and A, all constructions are made; (9) the incidental existence of a pure inversive Hermetian geometry of many dimensions; (10) the covariant and contravariant aspects of the geometric elements. Simple applications were made to pencils of shells, and to some elementary metric theorems.

3. *Distance sets*, by Professor J. C. Oxtoby, Bryn Mawr College, introduced by Dr. Whitman.

The speaker considered the following question: if two finite sub-sets A and B of an abelian group have the same set to differences, when can it be inferred that A is congruent to B, or to $-B$ by a group translation? A Theorem of S. Piccard and some recent results of A. L. Patterson bearing on this problem were discussed. Patterson has shown that the problem is intimately connected with the question of the unique determination of crystal structure by means of x-ray analysis.

4. *Photographic astrometry*, by Professor Peter van de Kamp, Swarthmore College, introduced by Professor Dresden.

Photographic astrometry is the study of the accurate positions (*i.e.*, directions) of star images on photographs. This paper was limited to long focus photography, and particular attention was given to studies made from photographs taken in the focal plane of the Sproul visual refractor at Swarthmore College. By a linear transformation, the measurements on all plates of a certain star field are reduced to the same origin, scale, and orientation. With suitable precautions, relative positions are obtained with an accuracy of 0.01 second of arc.

With the aid of many slides, it was pointed out that the study of proper motion in the mass leads to results for solar motion and mean parallax; the study of individual proper motions in multiple systems and clusters leads to dynamical information about these systems; the study of orbital motion of double stars leads to information about stellar masses; perturbations in the motions of double and single stars indicate the presence of companion stars inaccessible to direct observation.

5. *Transcendentality of certain continued fractions*, by Professor G. C. Webber, University of Delaware, introduced by Professor Rees.

Professor Webber discussed the question of transcendence of certain classes of continued fraction expansions. Using the Siegel theorem that $(b_0, b_1, b_2, \cdots)$ is transcendental if the b_i form an arithmetic progression of order one, he showed

that $(b_0, a_1, a_2, \cdots, a_k, b_1, a_1, \cdots, a_k, b_2, \cdots)$ is also transcendental. Likewise, $(a, b, a+d, b+e, a+2d, b+2e, \cdots)$ is transcendental if a, d, and c are positive integers, c or d being even, and $e=c^2d$, $b=c^2a+(c^2d/2)$. The Gelfond theorem, that e^x is transcendental if $x\neq 0$ is algebraic, leads to the transcendence of $(1, 3u^2, 5, 7u^2, \cdots)$, and $(1, 3u^2-1, 1, 4, 1, 7u^2-1, 1, 8, \cdots)$ if u^2 is a positive integer.

6. *On the many-valued logics*, by Professor J. B. Rosser, National Defense Research Committee.

The usual mode of reasoning decrees that any proposition must have one of the two truth values, truth or falsehood. It is possible to allow many truth values, but then one must use different methods of reasoning. One such method was outlined by the speaker.

P. M. WHITMAN, *Secretary*

CALENDAR OF FUTURE MEETINGS

Twenty-Seventh Summer Meeting, Wellesley, Mass., August 12–14, 1944.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN, Pittsburgh, Pa., April, 1944
ILLINOIS, Normal, Ill., May 12–13, 1944
INDIANA, Indianapolis, November 10, 1944
IOWA, Cedar Rapids, April 15, 1944
KANSAS, Topeka, April 15, 1944
KENTUCKY, Lexington, April 29, 1944
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK, New York, April 22, 1944
MICHIGAN, Ann Arbor, March 18, 1944
MINNESOTA
MISSOURI
NEBRASKA, Lincoln, May 6, 1944
NORTHERN CALIFORNIA, San Francisco, January 27, 1945
OHIO, Columbus, April 6, 1944
OKLAHOMA
PHILADELPHIA, Philadelphia, November, 1944
ROCKY MOUNTAIN, Greeley, Colo., April 14–15, 1944
SOUTHEASTERN
SOUTHERN CALIFORNIA, Los Angeles, March 11, 1944
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE
WISCONSIN, Milwaukee, May, 1944

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DEVOTED TO THE INTERESTS OF

COLLEGIATE MATHEMATICS

VOLUME 51 NUMBER 4

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FOCAL PROPERTIES OF OPTICAL AND ELECTROMAGNETIC SYSTEMS*

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1. Preface. This is a Preface, not an Introduction. Like the Prefaces to the plays of George Bernard Shaw, it has only the remotest connection with what follows, and may be read first, last, or not at all. Anyone who reads it must form his own judgment regarding its truth insofar as it deals with the past and the present. As for the future, who knows?

Most mathematicians work with ideas which, by common consent, belong definitely to mathematics. They form a closed guild. The initiate foreswears the things of the world, and generally keeps his oath. Only a few mathematicians roam abroad and seek mathematical sustenance in problems arising directly out of other fields of science. In 1744 or 1844 this second class included almost the whole body of mathematicians. In 1944 it is such a small fraction of the whole that it becomes necessary to remind the majority of the existence of the minority, and to explain its point of view.

The minority does not wish to be labelled "physicist" or "engineer," for it is following a mathematical tradition extending through more than twenty centuries and including the names Euclid, Archimedes, Newton, Lagrange, Hamilton, Gauss, Poincaré. The minority does not wish to belittle in any way the work of the majority, but it does fear that a mathematics which feeds solely on itself will in time exhaust its interest. This attitude has been summarized recently by Birkhoff [1]: "It will probably be the new mathematical discoveries which are suggested through physics that will always be most important, for, from the beginning, Nature has led the way and established the pattern which mathematics, the language of Nature, must follow."

Apart from its effect on the future of mathematics proper, the isolation of mathematicians has robbed the rest of science of a support on which it counted in all previous epochs. It is true that an immense amount of mathematical work now appears in physical and engineering journals. The technical scientist of today knows a great deal more mathematics than his grandfather knew. But does anyone seriously think that the role played in the past by the greatest mathematicians of their day can be taken over by technical scientists to whom mathematics is merely a tool? Of course they cannot take over that role, and they would be the first to admit it. Out of the study of nature there have originated (and in all probability will continue to originate) problems far more difficult than those constructed by mathematicians within the circle of their own ideas. Scientists have relied on the mathematician to throw his energies against these problems. They know that the mathematician is not merely a skilled user of certain tools already made—they can use those tools with no inconsiderable skill themselves. Rather they look to the qualities peculiar to the mathemati-

* Presented at the meeting of the Mathematical Association of America in Chicago, November, 1943.

cian—his logical penetration and capacity to see the general in the particular and the particular in the general.

Through three centuries a magnificant cooperation raised our civilization to an intellectual height never before achieved by man. We have been far sharper than the Greeks, far more practical than the Romans. Whatever the historian of the future may say about the follies and vices of these three centuries—their wars, robberies and greeds,—he will at least say: "This was a great age of science!"

In all this the mathematician was the directing and disciplining force. He gave science its methods of calculation—logarithms, calculus, differential equations, and so on—but he gave it much more than this. He gave it a blue-print. He insisted that thought be logical. As each new science came up, he gave it—or tried to give it—the firm logical structure that Euclid gave to Egyptian land surveying. A subject came to his hands a rough stone, trailing irrelevant weeds; it left his hands a polished gem.

At present science is humming as it never hummed before. There are no obvious signs of decay. Only the most observant have noticed that the watchman has gone off duty. He has not gone to sleep. He is working as hard as ever, but now he is working solely for himself. In his present attitude towards the rest of science there may be a shade of resentment. Theoretical physicists stole the public applause, and tried to use him as a mere calculator. However that may be, the fact is that the brain of science is no longer where it used to be. It is not dead or out of order; it has simply been removed by a painless operation and set up apart from the body it so long directed.

In brief, the party is over—it was exciting while it lasted. I do not share Birkhoff's optimism. Nature will throw out mighty problems, but they will never reach the mathematician. He may sit in his ivory tower waiting for the enemy with an arsenal of guns, but the enemy will never come to him. Nature does not offer her problems ready formulated. They must be dug up with pick and shovel, and he who will not soil his hands will never see them.

Change and death in the world of ideas are as inevitable as change and death in human affairs. It is certainly not the part of a truth-loving mathematician to pretend that they are not occurring when they are. It is impossible to stimulate artificially the deep sources of intellectual motivation. Something catches the imagination or it does not, and, if it does not, there is no fire. If mathematicians have really lost their old universal touch—if, in fact, they see the hand of God more truly in the refinement of precise logic than in the motion of the stars—then any attempt to lure them back to their old haunts would not only be useless—it would be denial of the right of the individual to intellectual freedom. But each young mathematician who formulates his own philosophy—and all do—should make his decision in full possession of the facts. He should realize that if he follows the pattern of modern mathematics he is heir to a great tradition, but only part heir. The rest of the legacy will have gone to other hands, and he will never get it back.

A man who spends his time deploring the loss of a vanished glory is a nuisance and a bore. If, indeed, the mathematician of the future is to use his finest efforts within the circle of purely mathematical ideas, let us make plans for the future with that fact admitted as an essential premise. I do not mean plans for mathematics itself, but plans for the future relationship of mathematics to the rest of science. Just now we are in an awkward and slightly embittered period. The awkwardness and bitterness arise from a failure on both sides to realize what has taken place. The non-mathematical scientist does not know that the mathematician has given up his general supervision of science, and is embittered by the mathematician's lack of interest. The mathematician does not realize that he has lost the leadership, and is embittered by the fact that his finest creative efforts carry so little weight in the body of science as a whole.

What we all need is more mutual understanding and respect. I do not mean just refraining from overt criticism, but a genuine and humble attempt to understand the motivation and thought-processes of other men of science. If this is achieved, there is a hope that, though the mathematician is no longer the great leader, he will remain the counsellor and friend of all science. It will not be the glory of the past, but at least a modus vivendi. Our science started with mathematics and will surely end not long after mathematics is withdrawn from it (if it is withdrawn). A century hence there will be bigger and better laboratories for the mass-production of facts. Whether these facts remain mere facts or become science will depend on the extent to which they are brought into contact with the spirit of mathematics.

2. The variational principle in optics. There are two basic laws in optics—the law of reflection and the law of refraction. They form the axioms or postulates of a mathematical theory—geometrical optics. Starting from them, the mathematician can study the behavior of light rays passing through an optical instrument (camera, telescope, or microscope) without any knowledge of physics. However, the wise mathematician will ask the physicist what questions would be most interesting to investigate, since otherwise his researches may wander through lack of a goal. One of the interesting problems—in fact, the main problem—is that of the formation of images.

The laws of reflection and refraction are extremely simple. A knowledge of high-school mathematics is sufficient to understand them. But mathematics at the high-school level is not always easy. The geometry of the point, line, and circle, for example, provides problems of great difficulty because there is no guiding principle for their solution. Workers in optics find the same need for a guiding principle.

It is a rule of mathematics that axioms should not be redundant. It should not be possible to prove one axiom from another. So, to get things straight, we wipe out the laws of reflection and refraction from our sheet of axioms, and write instead:

$$\delta \int n ds = 0. \tag{1}$$

From this single axiom, known as Fermat's Principle, the whole of geometrical optics follows logically. The laws of reflection and refraction are reinstated, not as axioms but as deductions from equation (1). Actually, we have widened the scope of our theory by adopting (1), rather than the laws of reflection and refraction. We can use (1) to discuss situations with which the elementary laws cannot cope. In spite of this, and although Fermat's dates were 1601–1665, most physicists seem to regard Fermat's Principle rather as a curiosity.

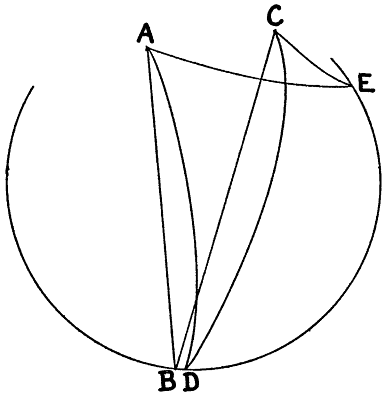

FIG. 1*a*. Reflection at a spherical mirror.

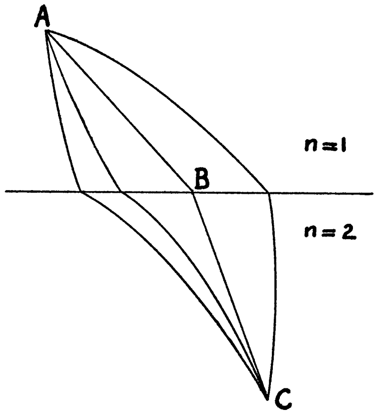

FIG. 1*b*. Refraction at a plane.

Let us see what (1) means. The integrand n is the *refractive index*. To the mathematician, it is a given function of position (and possibly of direction also); ds is an element of arc-length. The symbol δ tells us that a light ray is such that $\int nds$, taken along the ray, has a stationary value when compared with the same integral taken along a neighboring path with the same end points. (The integral for any path is called its *optical length*.) Very often in optical problems the stationary value is actually a minimum, but this is not always the case. Fig. 1*a* shows reflection at a spherical mirror; the optical length of the ray ABC has a

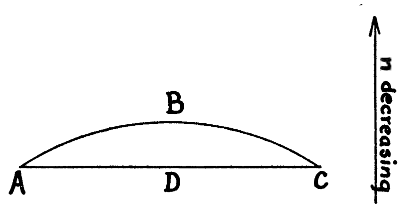

FIG. 2. Case where n varies continuously.

stationary optical length (in this case the same as geometrical length, if we take $n=1$) when compared with a neighboring curve ADC; but it is not a minimum (compare AEC). Fig. 1*b* shows refraction at a plane; here the ray ABC has a minimum optical length when compared with any other curve joining A and C.

In this latter case n is a discontinuous function, having one constant value below the horizontal line and another constant value above that line. Fig. 2 shows a ray ABC in the case where n varies continuously. Although the geometrical length of ABC is greater than that of the straight line ADC, its optical length is less. (Here we have the mathematics of the mirage.)

3. Electron optics. Optics is a very old subject—perhaps the oldest branch of mathematical physics. In recent times physicists have invented an instrument called the electron microscope. To the superficial mind there is only one feature in common between the electron microscope and the ordinary microscope—they both enable us to observe very small objects. But to the mathematical mind, intent only on the logical deduction of behavior on the basis of axioms, the two instruments are one and the same, or nearly so.

What are the axioms of the electron microscope? They are the axioms of electromagnetic theory, and we shall not trouble to write them down here. On the basis of these axioms, it is possible to prove that the path of an electron obeys a variational principle. This principle can be written precisely in the form (1). If we like (and we do like), we may wipe out the axioms of electromagnetic theory, and base electron optics on the axiom (1).

To the physicist, the integrand n in (1) has entirely different associations according as he is thinking of optics or electronic paths. To the mathematician it is just a given function. The only essential difference is that in electronic problems it is always a continuous (indeed, analytic) function, whereas in optics discontinuous functions most frequently occur. This distinction is insignificant in some arguments, important in others. For the detailed arguments developed later in this paper the distinction is important, as we find it necessary to assume analyticity. Consequently, the results are more significant for electron optics than for the classical optics.

4. The focal property. For three hundred years men have been grinding lenses and mirrors for a single purpose, and electron microscopes are constructed for that same purpose—the formation of images. It is therefore only fitting that a mathematician, sitting down to study the logical implications of the variational principle (1), should set before himself as an important problem the theory of the formation of images. Naturally, the first step is to get an idea (in mathematical terms) of what is meant by an image.

The easiest way to talk about these things is to use the language of physics without relaxing the clarity of mathematical thought. It is wise, however, not to insist on too great a precision of logic. Let us get the general picture, and leave the loose ends to another day.

In Fig. 3, O is a source of light and G a lump of glass. By means of a screen with a hole in it we cut off all the rays from O except a thin bundle. This bundle of rays strikes G and is refracted through it. Let us now take another screen and hold it in the path of the light, so that there is an illuminated patch on it. To make matters more precise, let us hold it always perpendicular to some assigned

ray of the bundle. What do we see as we move the screen to various positions between O and G and behind G?

Nothing exciting happens as we move the screen from O out towards G. The patch expands but remains of one shape. But on the other side of G the patch changes shape as the screen is moved. In each of two positions of the screen, the patch gets very thin.* In fact, if the initial angle of the bundle at O is regarded as infinitesimal, and small quantities of the second order are neglected, these thin patches are indistinguishable from straight lines. They are called *focal lines* (L_1, L_2).

There is so much confusion nowadays at to what is physics and what is mathematics, that is it well to pause for a moment and ask: Is this physics or mathematics? You can avoid the question by saying that it is mathematical physics, but let us rule that answer out of order. Then I would say without hesitation that it is mathematics, dressed up (if you like) in the language of physics.

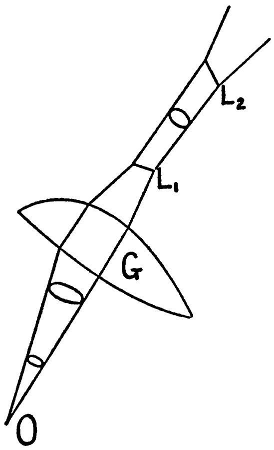

FIG. 3. Focal lines.

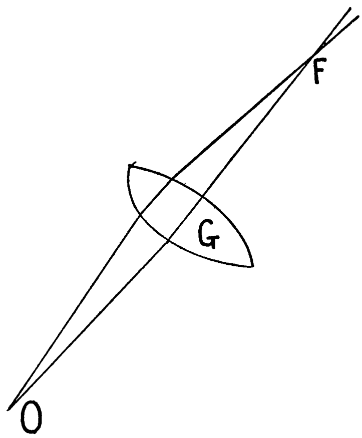

FIG. 4. The principal focus or image.

It is true that you can set up an experiment in a laboratory, and verify (with a certain amount of fuzziness) that the statements are physically true. But I have in mind a logical deduction of these statements from the equation (1) without the least trace of logical fuzziness. The deduction, of course, is not included here. It belongs, in the language of modern mathematics, to "differential-geometric aspects of the calculus of variations." It seems simpler to call it "geometrical optics."

But let us return to the question of the image. In Fig. 3 there is no image. But if we are lucky or clever in choosing the shape of the glass G, things may turn out as shown in Fig. 4. Here the two focal lines have disappeared. Instead there is a point F through which all the rays of the bundle very nearly pass. In

* An over-simplified statement. Sometimes the thin patches do not occur as stated, but exist only on an "imaginary" screen cutting the "imaginary" bundle of rays obtained by producing the final rays backward through G.

fact, they miss it only by a distance which is an infinitesimal of the second order. The point F is a *principal focus*, or an *image* of the *object* O.

Here we come to a climax, which should not be missed. The condition shown in Fig. 3 is *general*, that shown in Fig. 4 is *exceptional*. If, in the path of rays from O, you set up a lump of glass at random, the probability of getting an image is zero. An image occurs when the two roots of a quadratic equation coincide; that is an event of zero probability if the coefficients in the equation are chosen at random. (As a matter of fact, the coefficients are not completely random, since the roots are actually always real. But that does not affect the zero probability.)

Had Galileo stuffed chips of glass at random into his telescope tube, it is highly improbable that he would ever have seen the crescent of Venus. Clearly he had a design. What was its essential feature? How are we to change a probability zero into a probability unity?

The answer is simple: give the instrument symmetry of revolution, and point the axis of symmetry at (or near) the object. Practical physicists have known this so long that they forget that it is not quite obvious. When they came to construct electron microscopes, they used the same plan with success. Mathematical proofs have been given both for optical instruments and for the electron microscope. The most satisfying argument for the optical instrument is one based on Hamilton's angle-characteristic [2] (or eikonal [3]). A simpler argument has been given [4], and this may be used also for the electron microscope in the case where the paths of the electrons are straight lines before entering the instrument and after emerging from it.

When the electromagnetic field extends over the whole path of the electron (so that it is curved throughout), the arguments mentioned above cannot be used. The proofs given in this case [5] are rather involved, and I thought it would be interesting to discuss the problem using only the basic variational principle (1), without recourse to electromagnetic details until the last possible moment. When this plan is combined with the use of the complex variable, we get a very compact type of argument. Incidentally (as so often happens when a new method is tried) there is something in the way of a new result. It is found that symmetry less complete than symmetry of revolution is sufficient.

5. The differential equations and cases of symmetry. The variational equation (1) may be written

$$\delta \int m(x, y, z, x', y')dz = 0, \tag{2}$$

where x, y, z are rectangular Cartesian coordinates, the prime denotes d/dz, and

$$m(x, y, z, x', y') = n \cdot ds/dz. \tag{3}$$

The Euler equations of (2) read

$$\frac{d}{dz}\frac{\partial m}{\partial x'} - \frac{\partial m}{\partial x} = 0, \qquad \frac{d}{dz}\frac{\partial m}{\partial y'} - \frac{\partial m}{\partial y} = 0. \tag{4}$$

Assuming analyticity, we expand m:

$$
\begin{aligned}
m = m_0 &+ m_1 x + m_2 y + m_3 x' + m_4 y \\
&+ \tfrac{1}{2}(m_{11}x^2 + 2m_{12}xy + m_{22}y^2) \\
&+ \tfrac{1}{2}(m_{33}x'^2 + 2m_{34}x'y' + m_{44}y'^2) \\
&+ x(m_{13}x' + m_{14}y') + y(m_{23}x' + m_{24}y') + M,
\end{aligned} \tag{5}
$$

where M consists of terms of the third and higher orders. The coefficients are functions of z.

We come now to the kernel of the matter—the question of symmetry. This is best discussed in terms of "covering operations." A covering operation is a transformation of coordinates such that the specification or description of the instrument is the same whether we use the old or the new axes. We shall consider only covering operations which consist of rotations about the z-axis.

Let us rotate the axes Oxy into $O\bar{x}\bar{y}$ by turning them through an angle α. This gives the transformation

$$x = \bar{x} \cos \alpha - \bar{y} \sin \alpha, \qquad y = \bar{x} \sin \alpha + \bar{y} \cos \alpha, \tag{6}$$

with similar formulas for x', y'. As far as we are concerned mathematically, the "instrument" is essentially the function m. If m is an arbitrary function, it will be changed by the transformation (6). Rotation through the angle α is a covering operation if, and only if,

$$m(x, y, z, x', y') = m(\bar{x}, \bar{y}, z, \bar{x}', \bar{y}'). \tag{7}$$

The only possible types of rotational symmetry are

(i) symmetry of revolution (α arbitrary);

(ii) ν-gonal symmetry ($\alpha = 2\pi/\nu$, $\nu = 2, 3, \cdots$).

Fig. 5 illustrates some of these types of symmetry. As rotation through an arbitrary angle may be produced by an infinite succession of infinitesimal rotations, we may describe symmetry of revolution as ν-gonal symmetry with $\nu = \infty$.

On substituting (6) in (5) and comparing the form of the new expression with that of the old expression, we find that ν-gonal symmetry implies

$$m_1 = m_2 = m_3 = m_4 = 0, \tag{8}$$

with the following further relations, if $\nu > 2$:

$$m_{11} = m_{22}, \quad m_{33} = m_{44}, \quad m_{13} = m_{24}, \quad m_{14} = -m_{23}, \quad m_{12} = m_{34} = 0. \tag{9}$$

Changing the notation for the coefficients, we have, for ν-gonal symmetry with $\nu = 3, 4, \cdots$,

$$
\begin{aligned}
m = p_0 &+ \tfrac{1}{2}p_1(x^2 + y^2) + \tfrac{1}{2}p_2(x'^2 + y'^2) \\
&+ p_3(xx' + yy') + p_4(xy' - yx') + M;
\end{aligned} \tag{10}
$$

for diagonal symmetry ($\nu=2$), the expression (5) is simplified merely by the omission of the linear terms.

We note that the terms written explicitly in (10) are invariant under an arbitrary rotation. Hence, in any approximation which neglects M, *ν-gonal symmetry implies symmetry of revolution if* $\nu>2$.

Symmetry of revolution of course implies ν-gonal symmetry for all values of ν; for an instrument of revolution, we have (10) ,with further knowledge that M is a function of the invariants

$$x^2+y^2, \qquad x'^2+y'^2, \qquad xx'+yy', \qquad xy'-yx'.$$

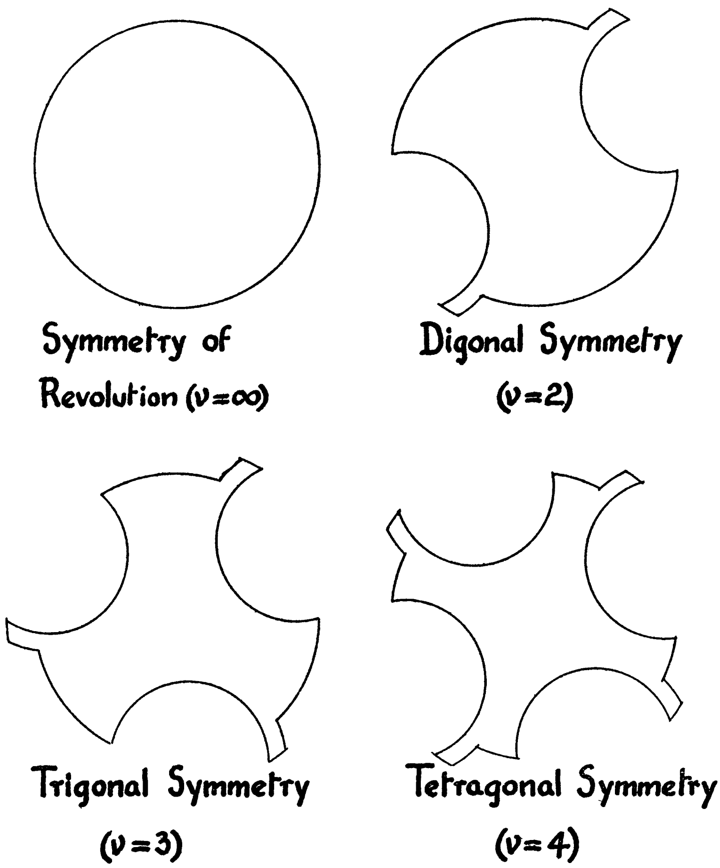

FIG. 5

6. Introduction of the complex vector and reduction of the equations to a single equation with real coefficients. Let us now confine our attention to the case of ν-gonal symmetry which $\nu>2$, so that m is in (10). Symmetry of revolution is included as a particular case. The Euler equations (4) read

$$
(11)\qquad
\begin{aligned}
&\frac{d}{dz}(p_2x' + p_3x - p_4y) - (p_1x + p_3x' + p_4y') = N_x,\\
&\frac{d}{dz}(p_2y' + p_3y + p_4x) - (p_1y + p_3y' - p_4x') = N_y,
\end{aligned}
$$

where

$$
(12)\qquad N_x = -\frac{d}{dz}\frac{\partial M}{\partial x'} + \frac{\partial M}{\partial x}, \qquad N_y = -\frac{d}{dz}\frac{\partial M}{\partial y'} + \frac{\partial M}{\partial y}.
$$

In terms of the complex vector $\zeta = x+iy$, the two equations (11) may be written as the single equation

$$
(13)\qquad \zeta'' + P\zeta' + Q\zeta = T,
$$

where

$$
(14)\qquad
\begin{aligned}
P &= (p_2' + 2ip_4)/p_2,\\
Q &= (p_3' + ip_4' - p_1)/p_2,\\
T &= (N_x + iN_y)/p_2.
\end{aligned}
$$

We shall not consider the singular case $p_2=0$; if $p_2=0$, the character of the differential equations (11) is altered drastically, since the second derivatives of x and y no longer appear on the left-hand side.

The fact that we can write (11) in the simple form (13) is itself remarkable. This could not be done but for the pattern of the coefficients in (11); this pattern arises from the symmetry of the instrument.

We now remove the second term from (13) by introducing the new complex vector

$$
(15)\qquad u = \zeta \exp\left(\frac{1}{2}\int P(z)dz\right).
$$

We obtain

$$
(16)\qquad u'' + Ru = T \exp\left(\frac{1}{2}\int P(z)dz\right),
$$

where

$$
(17)\qquad
\begin{aligned}
R &= Q - \tfrac{1}{2}P' - \tfrac{1}{4}P^2\\
&= [p_2(p_3' - p_1) - \tfrac{1}{2}p_2p_2'' + \tfrac{1}{4}p_2'^2 + p_4^2]/p_2^2.
\end{aligned}
$$

The reader should not miss the fact that something very remarkable has just occurred. From *complex* expressions for P and Q we have obtained a *real* R! It is Miracle No. 1 in the proof of the focal property of an electron microscope. It raises the probability of the existence of an image from 0 to $\frac{1}{2}$, as we shall see presently.

For paths adjacent to the axis of symmetry, ζ (or u) is a small quantity of the first order. The complex quantity T is a small quantity of the second order at least, for ν-gonal symmetry ($\nu > 2$), and of the third order for symmetry of revolution. Unless we wish to study aberrations, we neglect T and write for (16)

$$u'' + Ru = 0. \tag{18}$$

This business of neglecting small quantities, apparently of higher order, without a thorough investigation, is (in the eyes of the pure mathematician) the greatest vice of the applied mathematician. Many a shining mathematical monument (the theory of small oscillations, for example) was built by ruthless builders who threw into the junkpile all inconvenient residues. "Forward!" they cried, "The problem must be linearized. We shall come back another day, and look into these residues." They seldom do. In our problem, the residue T contains the theory of aberrations. They are very important to the physicist, and have been investigated. But we shall stick to the linearized equation (18); it contains the secret of the focal property.

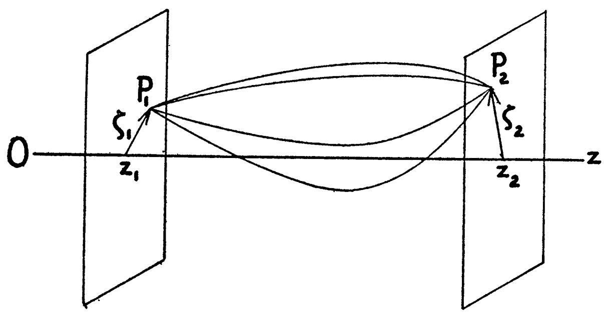

FIG. 6. Formation of an image.

If, however, we go back to (10) and delete M, *i.e.*, if we decide to deal with an instrument for which m is given by (10) with M deleted, then the equation (18) becomes exact, and it is no longer necessary to offer apologies. We have a choice between an inexact mathematical treatment of a general m and an exact treatment of an artificially simplified m. Whichever we choose, we arrive at (18).

Let us recall the general situation. Fig. 6 shows the instrument, Oz being the axis of symmetry. Extremals (or electron paths) start from the point P_1 with coordinates $z = z_1$, $\zeta = \zeta_1$. If the instrument has the focal property, then all extremals starting from P_1 in arbitrary directions pass through a common point P_2 (say, $z = z_2$, $\zeta = \zeta_2$). If these extremals are projected on the plane $z = 0$, they will appear as in Fig. 7. These curves are solutions of (13) with T omitted, *i.e*, of

$$\zeta'' + P\zeta' + Q\zeta = 0. \tag{19}$$

It seems too good to be true that the solutions of this equation should have the remarkable property of convergence shown in Fig. 7. And it is, in general, if P

and Q are arbitrary complex functions of the independent variable z. In our problem they are not arbitrary, but are so related that R is real.

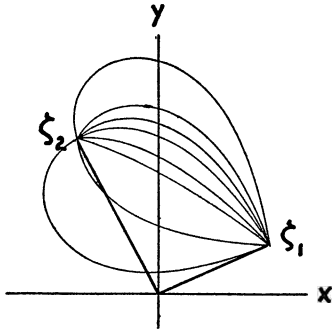

FIG. 7. Paths projected on normal plane.

Let us return to (18), remembering that though R is real, the solutions u in which we are interested are in general complex. Fig. 8 shows the complex u-plane, with the initial point P_1 ($u=u_1$). Let $f(z)$, $g(z)$ be independent *real* solutions of (18). The general complex solution is

$$u = \alpha f(z) + \beta g(z), \tag{20}$$

where α and β are arbitrary complex constants. The following particular solu-

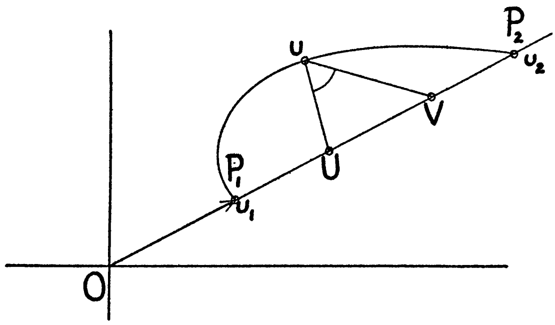

FIG. 8. The complex plane of u.

tions satisfy the initial condition $u=u_1$ for $z=z_1$

$$U = u_1 f(z)/f(z_1), \qquad V = u_1 g(z)/g(z_1). \tag{21}$$

As z varies, the points U, V move on the straight line OP_1.

Any other solution of (18) is of the form (20), and if this solution passes through $u=u_1$ for $z=z_1$, the constants satisfy

$$u_1 = \alpha f(z_1) + \beta g(z_1). \tag{22}$$

Then (20) may be written in either of the forms

(23a) $$u = u_1 f(z)/f(z_1) - \beta[f(z)g(z_1) - f(z_1)g(z)]/f(z_1)$$

(23b) $$u = u_1 g(z)/g(z_1) + \alpha[f(z)g(z_1) - f(z_1)g(z)]/g(z_1).$$

Thus

(24) $$\frac{u - U}{u - V} = -\frac{\beta g(z_1)}{\alpha f(z_1)}.$$

This is independent of z, and so the angle between the vectors $u-U$ and $u-V$ remains constant as z changes. The ratio of the magnitudes of these vectors is also constant. From the constancy of the angle it follows that if U and V have a second coincidence (in addition to the initial coincidence at P_1), then u will be sucked into that coincidence. Such a coincidence (an image, in fact) occurs if the equation

(25) $$f(z_2)/g(z_2) = f(z_1)/g(z_1)$$

has a real solution z_2, other than the trivial $z_2=z_1$. For it is clear from (23) that such a value of z will make $u=U=V$. The position of the image is

(26) $$z = z_2, \quad u = u_2 = u_1 f(z_2)/f(z_1),$$
$$\zeta_2 = \zeta_1 \frac{f(z_2)}{f(z_1)} \exp\left(-\frac{1}{2}\int_{z_1}^{z_2} P(z)dz\right).$$

Since the equation (25) does not involve the initial lateral displacement ζ_1 (or u_1), it is clear that the existence or non-existence of an image does not depend on this lateral displacement. The focal property may be studied, if we like, by considering only the case $\zeta_1=0$, *i.e.*, the case where the object point is placed on the axis of symmetry. If the object point $z=z_1$, $\zeta=0$ is imaged at $z=z_2$, $\zeta=0$, then all the object points on the plane $z=z_1$ are imaged on the plane $z=z_2$. The transformation from one plane into the other is a uniform magnification accompanied by a rotation about the axis. If we prefer, we may speak of the "complex magnification"

(27) $$\frac{\zeta_2}{\zeta_1} = \frac{f(z_2)}{f(z_1)} \exp\left(-\frac{1}{2}\int_{z_1}^{z_2} P(z)dz\right).$$

This may be split into a real magnification (positive or negative)

(28) $$\frac{f(z_2)}{f(z_1)} \exp\left(-\frac{1}{2}\int_{z_1}^{z_2} P_1(z)dz\right),$$

and an angle of rotation

(29) $$-\frac{1}{2}\int_{z_1}^{z_2} P_2(z)dz,$$

where $P=P_1+iP_2$.

Before introducing the electromagnetic field, let us see where we have got to on the basis of the variational principle (1) alone. We have reduced the question of image formation to the discussion of the simple differential equation (18). The fact that R is real stands out as of prime importance. The next question of importance is this: Is R positive or negative? The variational principle alone does not restrict the sign of R. Without discussing the sign of R, however, we have brought the matter to a climax in the equation (25). It seems legitimate to say that there is probability $\frac{1}{2}$ that an arbitrary real equation should have a real root. Hence, symmetry alone raises the probability of an image from 0 to $\frac{1}{2}$.

As for symmetry, we have found that digonal symmetry is no help towards image formation, but that ν-gonal symmetry with $\nu > 2$ is just as effective as symmetry of revolution. This means that in an electrostatic microscope circular holes are not essential. Trigonal or tetragonal holes might be used, and the same remark applies to the coils in a magnetostatic microscope. They might be triangular or square—but not rectangular. These suggestions are, however, more philosophical than practical, because it seems fairly certain that aberrations would be least in the case of symmetry of revolution.

7. The existence of real images in an electromagnetic instrument. If we attack the problem of the electron microscope directly through the equations of electromagnetism [6], we find the equation (19) with the following values for P and Q:

$$(30)\qquad P = \frac{1}{2}\frac{W'}{W} + i\Omega_0'\left(\frac{e}{2mW}\right)^{1/2}, \qquad Q = \frac{1}{4}\frac{W''}{W} + \frac{1}{2}i\Omega_0\left(\frac{e}{2mW}\right)^{1/2},$$

and hence by (17)

$$(31)\qquad R = \frac{3}{16}\frac{W'^2}{W^2} + \frac{1}{2}\Omega_0'^2\frac{e}{2mW}.$$

Here

e = electronic charge (negative),
m = electronic mass (positive),
$V_0(z)$ = axial electrostatic potential,
$\Omega_0(z)$ = axial magnetostatic potential,
$W(z) = C - V_0(z) < 0$
C = constant of energy.

The radicals in (30) are positive.

The above formulas are most easily derived for the case of symmetry of revolution, but they hold also for ν-gonal symmetry with $\nu > 2$.

There is an important difference between the general R of (17) and the R for the electromagnetic field as given by (31). The electromagnetic R is positive definite; it vanishes only if the field is absent. This is Miracle No. 2 in the theory of the electron microscope.

Let us take any point ζ_0, z_0, not on the axis of symmetry. Let us start a trajectory from this point in a direction making $u'=0$, u being given by (15). This does not, in general, imply $\zeta'=0$; thus the direction will not be parallel to the axis. Since R is positive, it is evident that the differential equation (18), under these initial conditions, defines a function $u(z)$ which vanishes at least once for $z<z_0$ and at least once for $z>z_0$. Let us call these points z_1 and z_2.

Now if we place an object on the axis at z_1, we get an image at z_2, since the vanishing of u implies the vanishing of ζ. The plane perpendicular to the axis at z_1 is imaged on the plane perpendicular to the axis at z_2, in accordance with (26). We have therefore this result: *Every statical electromagnetic system with an axis of ν-gonal symmetry (or symmetry of revolution) forms images (principal foci) of some objects, provided $\nu>2$.*

The reader familiar with the language of optics may ask whether the images of which we speak are "real" or "virtual." We say therefore that the virtual image plays no part in the theory as developed here. By "image" we mean "real image."

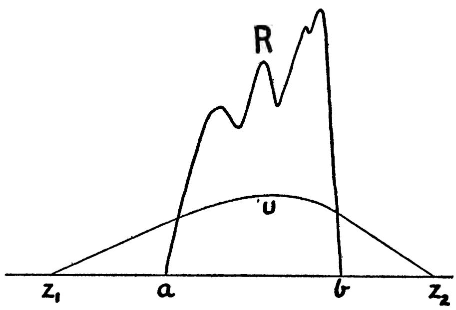

FIG. 9. Graphs of R and u.

We cannot assert that *every* object has an image, for in the above deduction we did not choose z_1 arbitrarily. The field bends in to a principal focus a set of trajectories which diverges weakly enough, or does not diverge at all, as when the object is at infinity. But the field may not be strong enough to bend in a strongly divergent set of trajectories. However, we can secure an image of any point if we are allowed to increase the strength of the field, *i.e.*, increase the positive coefficient R in (18). With the understanding that we are allowed to do this, we have reached our goal. The probability of securing an image has been raised from 0 through $\frac{1}{2}$ to 1.

I wish to thank my colleague, Professor T. Radó, for the interest he has taken in this work, and for his criticisms and suggestions. Among other results connected with the differential equation (18), he has pointed out one which concerns the case where $R(z)>0$ for $a<z<b$ and $R(z)=0$ for z outside this range. Then there is certainly an image if

$$z_1 < a - \left(\int_a^b R\,dz\right)^{-1}$$

This case is illustrated in Fig. 9, which shows graphs of R and u.

References

1. G. D. Birkhoff, American Scientist, 31, 1943, 310.
2. J. L. Synge, Geometrical Optics, Cambridge, 1937, 68.
3. C. Carathéodory, Geometrische Optik, Berlin, 1937, 84.
4. E. G. Ramberg and G. A. Morton, Journal of Applied Physics, 10, 1939, 465–478.
5. J. Picht, Einführung in die Theorie der Elektronenoptik, Leipzig, 1939, 15, 70.
6. J. L. Synge and B. A. Griffith, Principles of Mechanics, New York, 1942, 383, 385.

INTERESTING THE ENGINEERING STUDENT*

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1. Introduction. I hope no one will expect a complete solution to the problem mentioned in the title. While I have taught engineering students in two different institutions over a considerable period, this is only a partially solved problem as yet to me. From time to time I have experimented with different methods to attract their interest to mathematics. This paper might be termed a preliminary report, for my experiences are continuing and I expect to have additional ideas concerning this problem in years to come.

2. The problem. Although we speak of him in the singular, the engineering student varies in many characteristics and qualities. Nowhere is this variation more striking than in the mathematics classroom. At one end of the scale we have the students who were particularly excellent in mathematics in high school and perhaps entered the study of engineering because, so far as they knew, it was the only field in which mathematics could be used to earn a living. At the other end of the scale there are the students who chose the profession of engineering because they were particularly skilled with hand tools or wished an occupation with outdoor work. This may seem facetious but is actually the reason why many students enter our engineering colleges. This is most unfortunate as such students have little aptitude for the mathematics and other theory encountered in the engineering curriculum and belong really in trade schools.

There is no problem of interest with the student of the first type. In fact our problem, if there is one, is to see that he does not become too interested in mathematics. To a mathematician this is probably heresy, but from an engineering viewpoint there is certainly a possibility that the student may become too deeply interested in mathematical theory to the detriment of its engineering applications. In such a case, the student should be encouraged to specialize in engineering design where full use can be made of his mathematical interest or to change from engineering into the field of mathematics itself. Some of these students are not in this field only because the proper vocational counseling was not available to them during their high school days.

* Delivered June 20, 1943, before the Society for the Promotion of Engineering Education at Chicago.

Probably a student at the other end of the scale cannot be interested in mathematics at all, and needs some advice which will lead him from the engineering profession into a trade school or other mechanical work where his ability with tools is a real advantage.

In between these two extremes lie many different layers of ability and types of personalities, and the problem of interest in mathematics is concerned primarily with this intermediate and largest group. Too many of these students find mathematics dull and exacting, and breathe a sigh of relief when their required mathematics courses are finished. Then, in the language of the student, he can close his books and forget mathematics. This is precisely the point of view we must fight if our mathematics teaching is to be successful. A student who has this feeling will never develop into an engineer of the highest quality, and it is to the interest of both the engineering profession and ourselves as mathematics teachers to see that this attitude does not develop. Unfortunately these two groups have not always cooperated in putting mathematics in the proper perspective for students of engineering.

3. Faults of the mathematics teacher. Too often the mathematician has been at fault in this respect. He has too frequently looked on the liberal arts students who elect to major in mathematics as the group deserving his best efforts, and on his engineering class as a part of his job, necessary to earning his salary, but not requiring his full energy in teaching the power and beauty of mathematics.

I once heard a young mathematics instructor refer to his engineering class as "my dummies." Needless to say, this attitude did not result in proper teaching. Very shortly his performance became so poor the engineering class was taken from him and he soon left the field of teaching altogether. While this is an extreme case, it has been my observation that many professional mathematicians attempt to avoid teaching engineering classes. From a purely professional point of view this is most unfortunate as these classes contain some of the finest products of our high schools. As mentioned earlier, some of these students do not belong in engineering at all, but in the field of mathematics. A considerable number of fine graduate students in mathematics were originally in engineering classes, and one could easily name several professional mathematicians who began as engineering students and in some cases hold engineering degrees. Thus the mathematician looking for students to be led along the path of mathematics should realize that in his engineering classes he has a fertile source of students, perhaps the most fertile source of all.

Moreover, in the engineering classes there is a certain reality of interest in mathematics not found by the liberal arts student who takes a course or two as a cultural elective. Here are students who are to use their mathematics in the profession of engineering. This should be more of a stimulus to the teacher than the girl who elects a course in the calculus with no intention whatever of using the calculus at any time. For this reason alone it seems that the teacher of mathematics truly interested in his subject would find the engineering classes exciting.

4. Faults of the professional engineer. On the other hand, teachers of technical courses as well as the practising engineer have sinned frequently by giving students the impression that their mathematics is not important.

Very often the engineer becomes an executive or enters the sales division, and forgets that he is not really practising his profession at all. In these abortive lines naturally he has little need for his mathematics, and often he is quite vocal in his scorn for mathematics. He continues to think of himself as an engineer and overlooks the fact that he himself, rather than the engineering profession, has ceased to use mathematics. Similarly a doctor who became a drug manufacturer might advocate that courses in anatomy in the medical curriculum be replaced by cost accounting.

Sometimes we find engineering professors who fear and mistrust mathematical methods. Usually this is caused by their lack of knowledge of the technique and powers of mathematical methods. This leads to curious incidents. I know one engineering professor who told his class that all the engineer needs of mathematics is arithmetic and perhaps a little trigonometry. You can imagine the profound impression this produced in the minds of these young and immature young men, and the difficulties we had in the next few weeks combating this impression. The sequel to this story is that this same professor came to the mathematics office a few weeks later for some assistance in reading an article in his own field. A knowledge of Bessel functions was required for its reading and this was unknown territory to him. I am delighted to say that he was reminded of his earlier remarks and it was suggested that he try arithmetic and a little trigonometry on the article.

Mathematics teachers are sometimes embarrassed by questions from junior and senior students who inquire why Professor X does not use mathematics in teaching the theory of his subject. One hates to face such questions for usually the facts are that Professor X is not familiar with modern mathematical methods and is applying an earlier rule or some method he has learned which does not involve calculus and beyond. Frequently mechanics is taught so as to avoid calculus wherever possible. This is deplorable on two counts. In the first place the mathematical methods are more elegant and powerful. In addition it is imperative that the students continue to apply their mathematical ideas so that these ideas and methods become firmly impressed on their minds. If the teacher of technical subjects is not to use these mathematical tools, perhaps the mathematical requirements in engineering education should be abolished and engineering taught as a trade profession.

Having relieved ourselves of this venom against both our own colleagues and the professional engineer, we may turn to one or two constructive ideas on the question of interesting engineering students in mathematics.

5. Engineering textbooks. This places me in a most embarrassing position for I wish to advocate the writing of some new textbooks for the mathematics taught in the engineering college. All my life I have taken the stand that there

were plenty of textbooks already available, in fact, that the supply was more than ample. With few exceptions, however, all our textbooks have been written from a commercial viewpoint. That is, each attempts to appeal to as diverse a group of college classes as possible in order to produce the largest possible sale. This results in a textbook which fits all needs to an epsilon approximation but sometimes the epsilon seems disturbingly large. The number of engineering students in this country is sufficient to justify a textbook being written specifically for them and I believe such a book or series of books well written would contribute much in this problem of interest.

In the problems which were collected and edited by a committee of the Society for the Promotion of Engineering Education under the chairmanship of J. W. Cell, and which have just been published in book form by the McGraw-Hill Book Company, a real start in this direction has been made. However, this is only a beginning and real satisfaction will not be achieved until problems of this type and similar material from the various fields of engineering find their way into the mathematics classroom, not as supplementary exercises, but as a regular part of the courses. Until we integrate them into the courses and into the textbooks used, the average student about whom this paper worries will not look on them as important.

In addition to our familiar x, y, z letters, these new textbooks should use the symbols in common use in the various divisions of engineering literature. In this way the student becomes accustomed to seeing his mathematics in various letters. While this seems trivial, it is a real hurdle to the average student. My friends in the chemistry faculty tell me the student who can solve a quadratic equation with ease when the variable is x and the location is the mathematics building, finds the greatest difficulty when quadratic equations confront him in a chemistry course using chemical letters. I see no reason why the student should be given twenty problems of the type $4x^2-7x+3=0$ when we can introduce into this set of problems examples of quadratic equations which occur in engineering subjects and he can see the same method applied to these problems.

The textbook should give the student some training in problems where coefficients have one or two decimal places in place of the more elegant whole numbers of mathematics. Most of his equations in technical work will have such approximate numbers and unless he has also seen them in the mathematics classroom he may not make the desired transfer of his training. In exponents, too, the students should meet a few numbers with decimal figures.

These proposed textbooks should show the student a few applications of complex numbers so as to remove the ghostly character of imaginary numbers. It should explain that the electrical engineer uses the letter j instead of i and why he does so. In this way the student continually feels the reality of his mathematics and it is tied closer to the profession which lies close to his heart.

It is unbelieveable that one can teach the three types of conics without showing the class where such curves occur in the world about him. Yet there are

several textbooks where these applications are not even mentioned. Here, as elsewhere, the subject of mathematics has been divorced from reality and the engineering student loses thereby.

In the calculus the situation is not so serious, for all textbooks contain many applications. The student applies his techniques to maxima and minima problems and to find moments, centers of gravity, volumes bounded by surfaces, *etc.* Here the supply of applications, while perhaps not perfect, is somewhat adequate.

It is in trigonometry, algebra, and analytic geometry that the student is not shown enough of the applications. Many students study trigonometry and see no purpose to it at all except in surveying and other measuring of plane areas. With the richness of applications of this subject, there is no reason why the student should not be acquainted with many other fields where this material finds use. In a trigonometry class recently, one of our instructors was asked about the usefulness of angles greater than 180° and he put the question back to the class. Just one member of the class knew of an application where it was necessary to use angles of more than 180°. Many of our textbooks teach these angles with no mention of their use. So the student, thinking entirely of his trigonometry in terms of triangles, sees no reason for studying them except for the need of a passing grade.

Throughout his entire training in mathematics, the application of the subject to engineering should be emphasized over and over again. This does not mean we are to teach engineering in our mathematics classrooms, but the problems and formulas of engineering should be introduced in the textbooks whereever it can be done without discussing technical matters. This is not done at present, partly because suitable applications are difficult to find and present to students with no technical background, and partly because the mathematics teacher is not sufficiently acquainted with the applications of his subject.

6. Algebra and its technique. If there is one subject above all in mathematics which the student finds dull and divorced entirely from his reality, that subject is algebra. Every mathematician knows that algebraic technique is the foundation of all mathematics and yet the student sees only an endless series of arbitrary rules and does not see the value of the subject. A student is forced to learn to handle exponents, to solve quadratic equations, to solve higher degree equations, and all the other techniques of elementary algebra. He sees no connection with bridges, dynamos or internal combustion engines. As a result I find this subject the most difficult in the matter of student interest.

Some of the proposals of the previous section will assist in removing this difficulty, but I believe it would improve matters also to combine algebra with analytic geometry. For example, determinants and systems of linear equations may be studied along with the subject of straight lines in analytic geometry. And the methods of handling quadratic equations would accompany the conics of analytic geometry. In this way we may motivate some of the study of algebraic technique.

In general this matter of technique is one of the most difficult problems the mathematics teacher faces. In mathematics we attempt to do two things simultaneously: to impart new ideas, and to teach the students the technique of handling these new methods. It is useless to teach the idea without drill on technique, and yet the student finds the technique laborious and often dull. A too heavy emphasis on technique is sure to produce boredom on the part of the student. In the long run more can be accomplished by attempting to hold the interest of the student even if this means slightly less drill and perfection in technique. The student who finds his mathematics exciting will perfect his technique satisfactorily, whereas the student who has technique drilled into him against his desire will hate the entire subject and never learn it at all.

7. Conclusion. Finally this matter of interesting the engineering student reduces to a problem in salesmanship. We must sell our students the idea that mathematics is interesting, that it is important, and that it will be useful to them later. It is necessary that the mathematics teacher and the teacher of technical subjects cooperate in this undertaking. It is to their interest as much as ours to see that the student learns his mathematics and approaches his technical subjects with clear and definite mathematical ideas. Only when both groups join in facing this problem will we make real progress toward our goal.

POLYGONS OF GREATEST AREA INSCRIBED IN AN ELLIPSE

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1. Introduction. In many books on the calculus the problem is given to determine the isosceles triangle of greatest area or the rectangle of greatest area which can be inscribed in a given ellipse. The purpose of this paper is to discuss polygons of greatest area inscribed in an ellipse and polygons of least area circumscribed about an ellipse. The term polygon as used throughout this discussion is to mean a simple convex polygon. A polygon of greatest area will be referred to as a maximum polygon and a polygon of least area will be referred to as a minimum polygon. Two lemmas will be used as a basis for the discussion.

2. Maximum polygon inscribed in a circle. The existence of a maximum polygon is assumed. However, this may be proved by elementary analytical methods. The coordinates of the vertices of a polygon of n sides inscribed in a circle with unit radius and center at origin are designated by $(\cos\theta_i, \sin\theta_i)$, $\theta_i < \theta_{i+1}$ and $\theta_{n+1} = 2\pi + \theta_1$. Let $u_i = \theta_{i+1} - \theta_i$ and the area of the polygon is

$$A = \tfrac{1}{2}\sum_{i=1}^{n} \sin u_i = \tfrac{1}{2}\sum_{i=1}^{n-1} \sin u_i - \tfrac{1}{2}\sin\sum_{i=1}^{n-1} u_i.$$

Hence, $\partial A/\partial u_j = \frac{1}{2}\cos u_j - \frac{1}{2}\cos\sum_{i=1}^{n-1} u_i = \frac{1}{2}\cos u_j - \frac{1}{2}\cos u_n$, $(j = 1, 2, \cdots, n-1)$, and a necessary condition that A be maximum is that $u_j = u_n = 2\pi/n$. This is also

a sufficient condition since $\Delta_p = |a_{jk}| = (-1)^p(\frac{1}{2})^p(p+1)\sin^p 2\pi/n = (-1)^p N_p$ $(N_p > 0)$ where $a_{jk} = \partial^2 A/\partial u_j \partial u_k$ and $u_j = u_n = 2\pi/n$, $(j, k = 1, 2, \cdots, p)$.

LEMMA 1. *The maximum polygon of n sides which can be inscribed in a circle is a regular polygon.*

Let $P_1, P_2, \cdots, P_n$ denote the consecutive vertices of a polygon inscribed in a circle. If any two adjacent sides, say P_1P_2 and P_2P_3, of this polygon are of unequal lengths then a larger inscribed polygon of n sides may be formed. For if all vertices except P_2 are kept fixed and P_2 is moved to the point of its arc which is the greatest distance from the chord P_1P_3 the area of the triangle $P_1P_2P_3$ will be increased and since the remainder of the polygon remains unchanged the area of the polygon is increased. Hence if any inscribed polygon of n sides is not regular, it is always possible to get an inscribed polygon of n sides which has a greater area. It is generally not possible to begin with a polygon which has two or more sides of unequal lengths and arrive at a regular polygon by repeating this process a finite number of times. However this process gives a bounded increasing sequence of areas whose limit must be the area of a regular polygon.

3. **Minimum polygon circumscribed about a circle.** The existence of a minimum polygon is also assumed. However, this may be proved analytically as in Lemma 1. Denote the points of tangency of the sides by $(\cos\theta_i, \sin\theta_i)$. The area is

$$A = \sum_{i=1}^{n} \tan\tfrac{1}{2}u_i = \sum_{i=1}^{n-1} \tan\tfrac{1}{2}u_i - \tan\tfrac{1}{2}\sum_{i=1}^{n-1} u_i.$$

A necessary and sufficient condition that A be minimum is that $u_i = 2\pi/n$ $i = 1, 2, \cdots, n$.

LEMMA 2. *The minimum polygon of n sides which can be circumscribed about a circle is a regular polygon.*

If in a circumscribed polygon of n sides the interior angles at two adjacent vertices are unequal, it is possible, by moving the side joining these vertices and leaving the other $n-1$ sides fixed in position, to get a circumscribed polygon of n sides which has a smaller area. By argument similar to that used above it follows that the minimum circumscribed polygon must have all interior angles equal and consequently is a regular polygon.

4. **Polygon inscribed in an ellipse.** There is a regular inscribed polygon of n sides with a vertex at any given point of the circle and these all have the same area. Hence by projection* there is a maximum inscribed polygon of n sides for an ellipse with a vertex at any point of the ellipse and these all have the same area. If the semi-axes of the ellipse are a and b we have the following result.

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* See Fine, H. B. and Thompson, H. D., Coordinate Geometry, 1909, p. 92, Art. 118. The transformation $x' = x$, $y' = by/a$ carries the circle $x^2 + y^2 = a^2$ into the ellipse $x'^2/a^2 + y'^2/b^2 = 1$. An area A in the circle is projected into an area b/aA in the ellipse.

THEOREM 1. *The area of the maximum polygon of n sides which can be inscribed in an ellipse is* $(n/2)ab \sin 2\pi/n$ *and the difference between the eccentric angles for every pair of adjacent vertices of such a polygon is* $2\pi/n$.

This area is the geometric mean between the area of a regular polygon of n sides in the major circle and the area of a regular polygon of n sides in the minor circle. Also by projection we have the following result.

THEOREM 2. *The area of the minimum polygon of n sides which can be circumscribed about an ellipse is* $nab \tan \pi/n$ *and the difference between the eccentric angles of the points of tangency of every pair of adjacent sides of such a polygon is* $2\pi/n$.

Since every regular polygon of n sides circumscribed about a circle has its vertices on another circle it follows that the minimum polygons of n sides circumscribed about an ellipse are also maximum polygons inscribed in another ellipse. It follows that the minimum polygons of n sides circumscribed about the ellipse

$$x^2/a^2 + y^2/b^2 = 1 \tag{1}$$

are the maximum polygons of n sides inscribed in the ellipse

$$x^2/a^2 + y^2/b^2 = \sec^2 \frac{\pi}{n} \cdot$$

Also the maximum polygons of n sides inscribed in the ellipse (1) are the minimum polygons of n sides circumscribed about

$$x^2/a^2 + y^2/b^2 = \cos^2 \frac{\pi}{n} \cdot$$

The transformation

$$\left\{\begin{aligned} x' &= x \cos \alpha + y \frac{a}{b} \sin \alpha \\ y' &= -x \frac{b}{a} \sin \alpha + y \cos \alpha \end{aligned}\right. \tag{2}$$

will carry any polygon inscribed in the ellipse (1) into another inscribed polygon of the same area. However if these polygons are maximum inscribed polygons for the ellipse there are other invariants besides the area which are of interest.

5. Invariant properties of maximum and minimum polygons. One of the results is expressed in the following way.

THEOREM 3. *The sum of the squares of the lengths of the sides of every maximum polygon of n sides inscribed in the ellipse* (1) *is* $2n(a^2+b^2) \sin^2 \pi/n$, *and the sum of the squares of the lengths of the sides of every minimum polygon of n sides circumscribed about the ellipse* (1) *is* $2n(a^2+b^2) \tan^2 \pi/n$.

The second of these relations follows immediately from the first in view of the above discussion, so it is sufficient to prove the first relation.

Let $(a \cos \theta_i, b \sin \theta_i)$, $i=1, 2, \cdots, n$, be the vertices of a maximum polygon of n sides inscribed in the ellipse. Since $\theta_{i+1}-\theta_i=2\pi/n$, $\theta_i=\theta_1+(i-1)2\pi/n$. Define $\theta_{n+1}=\theta_1+2\pi$. Let S denote the sum of the squares of the lengths of the sides of the polygon. Then

$$
\begin{aligned}
S &= \sum_{i=1}^{n} \left\{a^2(\cos \theta_{i+1} - \cos \theta_i)^2 + b^2(\sin \theta_{i+1} - \sin \theta_i)^2\right\} \\
&= 4 \sin^2 \frac{\pi}{n}\left\{a^2\sum_{i=1}^{n} \sin^2\left(\theta_i + \frac{\pi}{n}\right) + b^2\sum_{i=1}^{n} \cos^2\left(\theta_i + \frac{\pi}{n}\right)\right\}.
\end{aligned}
$$

Since

$$
\sum_{i=1}^{n} \cos^2 \{\alpha + (i-1)\beta\} = \frac{n}{2} + \frac{1}{2} \cos \{2\alpha + (n-1)\beta\} \sin n\beta \csc \beta,
$$

it follows that

$$
\sum_{i=1}^{n} \cos^2\left(\theta_i + \frac{\pi}{n}\right) = \sum_{i=1}^{n} \cos^2 \left\{\left(\theta_1 + \frac{\pi}{n}\right) + (i-1)\frac{2\pi}{n}\right\} = \frac{n}{2},
$$

and

$$
\begin{aligned}
\sum_{i=1}^{n} \sin^2\left(\theta_i + \frac{\pi}{n}\right) &= \sum_{i=1}^{n}\left\{1 - \cos^2\left(\theta_i + \frac{\pi}{n}\right)\right\} \\
&= n - \sum_{i=1}^{n} \cos^2\left(\theta_i + \frac{\pi}{n}\right) = \frac{n}{2}\cdot
\end{aligned}
$$

Therefore

$$
S = 2n(a^2 + b^2) \sin^2 \frac{\pi}{n}\cdot
$$

It is noted here that the sum of the squares of the lengths of the sides of the maximum polygon of n sides inscribed in an ellipse is the arithmetic mean between the sum of the squares of the lengths of the sides of a regular polygon of n sides inscribed in the major circle and the sum of the squares of the lengths of the sides of a regular polygon of n sides inscribed in the minor circle.

If r_i $(i=1, 2, \cdots, n)$ are the distances from the center of the ellipse (1) to the vertices of a maximum inscribed polygon then

$$
\begin{aligned}
R &= \sum_{i=1}^{n} r_i^2 = \sum_{i=1}^{n} \left\{a^2 \cos^2 \theta_i + b^2 \sin^2 \theta_i\right\} \\
&= a^2\sum_{i=1}^{n} \cos^2 \left\{\theta_1 + (i-1)\frac{2\pi}{n}\right\} + b^2\sum_{i=1}^{n} \sin^2 \left\{\theta_1 + (i-1)\frac{2\pi}{n}\right\} \\
&= \frac{n}{2}(a^2 + b^2).
\end{aligned}
$$

It follows that if t_i $(i=1, 2, \cdots, n)$ are the distances from the center of ellipse (1) to the points where the sides of a maximum polygon of n sides inscribed in (1) are tangent to the ellipse

$$x^2/a^2 + y^2/b^2 = \cos^2 \frac{\pi}{n}$$

then

$$T = \sum_{i=1}^{n} t_i^2 = \frac{n}{2} (a^2 + b^2) \cos^2 \frac{\pi}{n} \cdot$$

If p_i $(i=1, 2, \cdots, n)$ are the perpendicular distances from the center of the ellipse (1) to the sides of a maximum inscribed polygon of n sides then

$$\begin{aligned}
\frac{1}{P} = \sum_{i=1}^{n} \frac{1}{p_i^2} &= \sum_{i=1}^{n} \frac{a^2 \sin^2 \dfrac{\theta_{i+1} + \theta_i}{2} + b^2 \cos^2 \dfrac{\theta_{i+1} + \theta_i}{2}}{a^2 b^2 \cos^2 \dfrac{\pi}{n}} \\
&= \frac{1}{a^2 b^2 \cos^2 \dfrac{\pi}{n}} \sum_{i=1}^{n} \left\{ a^2 \sin^2 \left(\theta_i + \frac{\pi}{n} \right) + b^2 \cos^2 \left(\theta_i + \frac{\pi}{n} \right) \right\} \\
&= \frac{n(a^2 + b^2)}{2a^2 b^2 \cos^2 \dfrac{\pi}{n}} \cdot
\end{aligned}$$

Similar invariants for the minimum polygons of n sides circumscribed about ellipse (1) may be obtained from these.

It might be pointed out that if a polygon inscribed in ellipse (1) is not maximum then the sum of the squares of the lengths of the sides does not remain invariant under transformation (2). Transformation (2) leaves every ellipse of the family $x^2/a^2+y^2/b^2=k$ unchanged. A maximum polygon of n sides inscribed in an ellipse of this family is transformed into itself by (2) if α is an integral multiple of $2\pi/n$. A necessary and sufficient condition that a polygon of n sides inscribed in an ellipse of this family be a maximum inscribed polygon of n sides is that there exist a transformation (2) different from the identity which transforms the polygon into itself.

CLUBS AND ALLIED ACTIVITIES

EDITED BY J. S. FRAME

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to J. S. Frame, Michigan State College, East Lansing, Michigan.

ANNUAL REPORTS AND TOPICS FOR CLUB TALKS

Officers in charge of Mathematics Clubs are requested to send in their annual reports for 1943–44 promptly at the close of the academic year. The form adopted for printing reports this last year is different from that used in the past, and should be followed as closely as possible. Titles and speakers are displayed on separate lines. Officers are listed at the end.

The editor would be glad to receive articles for publication in this department which either have been or might be used as Mathematics Club talks on an undergraduate level.

HISTORICAL NOTE

The approximation for the smallest angle in a right triangle, $A \cong 172°a/(b+2c)$, which appeared in a recent issue of this MONTHLY (Vol. 50, no. 10, p. 622, Dec. 1943), was recognized by W. A. Hurwitz as having previously appeared in this MONTHLY in 1920 (Vol. 27, no. 8, p. 368) in an article by R. A. Johnson. Professor Hurwitz, then editor of Questions and Notes, gave a bibliography (p. 366) in which the formula is traced to Nicolaus Cusanus in 1514, and the proof to Ozanam in 1699. Despite its antiquity, however, this formula seems to be not well-known by mathematicians today.

CLUB REPORTS 1942-43

Pi Mu Epsilon, Montana State University

Talks included on the program for 1942–43 were:

Partially ordered systems, by Roy Dubisch.

Cryptography, by Walter Hook.

A geometrical presentation of complex numbers, by Francis Pott.

Lebesgue integration, by Eldon Whitesitt.

The annual banquet was held in March, and was in honor of Professor E. F. A. Carey who retires this year. In the fall the annual *Pi Mu Epsilon* Mathematical Prize Competition for freshmen was held, and prizes of twenty-five, fifteen, and ten dollars were awarded respectively to Dick Burgess, Jack Groene, and Mary Farrington. At the June Commencement the *Pi Mu Epsilon* prize for excellence in mathematics was awarded to Francis Pott, and the *Pi Mu Epsilon* prize for excellence in physics was awarded to Mark Jacobson. The award this year consisted of a copy of *Willard Gibbs* by Rukeyser. Officers for the year were: Director, Francis Pott: Treasurer, Mel-Iden Pollock.

Alfred University Mathematics Club

How to count, was the topic presented by Professor Lloyd Lowenstein at the first meeting of the club, covering material on the correspondence of numbers and sets, as well as a discussion of the transfinite. Other programs were:

The miniature world, a finite geometry, by W. B. Cottrell, President.

Equations and their roots, by Mrs. T. H. Dunkelberger.

Topology, by Lewis Butler.

Paradox lost and paradox regained, a program presented jointly by Jeanette Torrey and Robert Burgess, two seniors in the Mathematics Department.

The yearly project of the club was the binding of several years' collections of mathematical publications which were accumulating in the library. The annual Blue Key Mathematics Award was presented to President W. B. Cottrell on Moving-up Day. The activities of the club were conducted by its officers for the past year, who were: President, W. B. Cottrell; Secretary, Jeanette Torrey; Treasurer, Richard Rulon. New officers elected at the final meeting were: President, Lewis Butler; Secretary, Frank Olson; Treasurer, Richard Rulon. Professor Lowenstein is the Club's faculty adviser.

M. I. T. Mathematical Society

The M. I. T. Mathematical Society heard seven speakers during the past academic year, on the following subjects:

Relaxation oscillations, by Professor Norman Levinson.

Functions of several complex variables, by Professor W. T. Martin.

Vectors and tensors, by Professor D. J. Struik.

Continued fractions, by W. S. Loud.

Numbers, by R. Baumberger.

Motors, by D. L. Thomsen.

Permissible functions in physics, by E. J. Zadina.

The Society sponsored a contest for freshmen and sophomores in May, 1942. The winners were G. Raisbeck and L. C. Biedenharn. Officers for 1942–43 were: President, W. S. Loud; Program Manager, R. Baumberger.

Pi Mu Epsilon, Oklahoma A. and M. College

The mathematical activities at the Oklahoma A. and M. College have been under the sponsorship of the chapter of *Pi Mu Epsilon*. The fraternity held two open meetings, besides several smaller meetings to discuss pledging, initiation, *etc.* Topics discussed at the open meetings were:

The essence of mathematics, by Professor J. H. Zant, Acting Head of the Department of Mathematics.

The ratios of similitude, by Professor B. F. Behrendt, of the Department of Civil Engineering, discussed from the standpoint of the engineer.

Eighteen new members were initiated during the year, though most of them have already been called into the armed services. The officers are: President, Lawrence Hanna; Secretary, Joe Oursler; Faculty Sponsor, Professor J. H. Zant.

The Archimedeans, Winthrop College

Among the interesting programs presented during 1942–43 were:

A demonstration of the slide rule, by Jessie Cockfield.

Pantomimes portraying great mathematicians and their contributions.

Aspects of Chinese mathematics.

On one occasion *The Archimedeans* and the National Council of Teachers of Mathematics entertained a group of air cadets, then stationed at Winthrop College, at an informal gathering. Officers for 1943–44 are: President, Jessie Cockfield; Vice-President, Emily Jean Adams; Secretary, Ann Major; Treasurer, Sarah Ellen Leslie; Social Chairman, Eloise Dempsey; Faculty Adviser, Mrs. Blanche Badger.

Junior Mathematical Club, University of Chicago

During the year the following papers were presented by students:

Repeating phenomena in the decimal representation of numbers, by Daniel Zelinsky.

Infinite series, by C. F. Brumfiel.

Invariants of quadratic differential forms, by Robert Kates.

Conjugate nets in asymptotic parameters, presented by Janet MacDonald to the *Senior Mathematical Club.* As a prize for the best student paper in content and presentation, Miss MacDonald was awarded a copy of Bell's *Development of Mathematics.*

In addition to the student papers listed above, the following talks were presented to the Club:

Mathematics and meteorology, by Professor W. T. Reid.

Alignment charts and their applications to meteorology, by Albert Cahn, Institute of Meteorology, University of Chicago.

Mathematics for electronics, by Professor M. R. Hestenes.

Singular points on higher plane curves, by Professor M. I. Logsdon.

Concerning the validity of the multiplication law of radicals, by Dr. Luise Lange, Wilson Junior College.

Alignment charts, by Professor L. R. Ford, Illinois Institute of Technology.

Permutation groups in cryptanalysis, by Professor A. A. Albert.

The generalized harmonic oscillator, by Stuart Lloyd, Department of Physics, University of Chicago.

Applications of mathematical logic to neuron networks, by Walter Pitts, Department of Biophysics, University of Chicago.

Almost-periodic functions, by Dr. Y. K. Wong.

In addition to the teas given before the talks, the club sponsored a party in the Autumn Quarter and a picnic in the Spring Quarter. The officers for the year 1942–43 were: President, Anne Lewis; Treasurer, Daniel Zelinsky; Social Chairman, Janet MacDonald; Committee, Robb Jacoby and Hyman Zimmerberg. The officers elected for 1943–44 are: President, Daniel Zelinsky; Treasurer, Hyman Zimmerberg; Social Chairman, Mary Toft; Committee, Robb Jacoby, Robert Kates, Charles Nichols.

Kappa Mu Epsilon, Chicago Teachers College

Illinois Gamma Chapter of *Kappa Mu Epsilon* has now completed its first year of fraternal activity. Heading the list of program meetings were the following discussions:

The struggle for existence, by Dr. Ralph Mansfield of the College faculty—a talk on developing a theoretical and mathematical formula for the struggle for existence of the mosquito.

Celestial navigation, by Naval Lt. F. C. Chunn.

Old and new ways of solving equations, by Dr. L. R. Ford, chairman of the Mathematics Department of the Illinois Institute of Technology.

How mathematics can be used to obtain vital information for an educational testing program, by Dr. W. C. Krathwohl, head of the Department of Tests and Measurements, Illinois Institute of Technology.

In addition to their charter members, the group initiated eighteen into their organization during the past year. Officers for the year were: President Archimedes, Robert Healy; Vice-President Euler, John Wiegand; Secretary Galileo, Arletta Mae Loomis; Treasurer Kepler, Rita Cooney; Secretary Descartes and Faculty Sponsor, Joseph Urbancek.

The Cooper Union Mathematics Club

Activities of the Mathematics Club of the Cooper Union School of Engineering for the academic year 1942–43, ending February 1943, were rather restricted because of the continuing accelerated teaching program. However the following talks were well received:

The slide rule, by Mr. M. J. Minneman.

Vector analysis (two lectures), by Mr. E. Herzkorn.

Planetary motions, by Mr. J. Diamond.

Operator theory, by Mr. H. Grad.

The officers of the club for 1942–43 were: President, H. Grad; Vice-President, E. Herzkorn; Secretary, D. Frankl; Treasurer, J. Diamond.

Mathematics Club, Mount Mary College

During the past school year, the Mathematics Club of Mount Mary College has met regularly on the second Monday of each month. At these meetings, reports were given on great mathematicians, including

Descartes, by Pat Paisley.

Archimedes, by Therese Rathgeber.

Sir Isaac Newton, by Marge Farrell.

Euclid, by Anne Snyder.

An especially interesting biography of the adviser was presented by Virginia Manly. Meetings also included games, with War Stamps for prizes. To conclude activities for the year, an outing was planned and enjoyed by the club. Officers were: President, Jeanne Garvey; Secretary, Anne Snyder; Treasurer, Mary Jane Crowley; Adviser, Sister M. Felice.

Pi Mu Epsilon, Michigan State College

Although fewer meetings than usual were held during 1942–43, the fall and winter meetings enjoyed a large attendance. At three of the meetings addresses were heard as follows:

Archimedean solids, by Dr. B. M. Stewart.

Synthetic definitions of hyperquadrics, by Dr. L. Toralballa.

Four dimensional geometry, by Professor G. Y. Rainich of the University of Michigan, the address delivered after the annual dinner.

Two of the other meetings featured six or seven brief problems presented by student members, and one meeting was a business meeting. Officers of the chapter were: President, Alice Benedict; Vice-President, Anne Mandenberg; Secretary, John Harrington; Treasurer, Peter Trezize; Faculty Advisers, Dr. J. F. Heyda and Dr. W. L. Mitchell.

Mathematics Club, University of Buffalo

The 1942–43 season opened with a meeting held on October 31 in the Blue Room of the Norton Union, at which new members were welcomed, plans for the year were made, and mathematical games and Hallowe'en refreshments were enjoyed. Topics for the ensuing meetings were:

The probability in dice-throwing games, complemented by actual charts of the data which he had accumulated, by John Castle.

Special mathematical problems, by William Hoctor and Lois Obenauer.

The theory of the game of Nim, by Marjorie Easterbrook, followed by mathematical games.

Measurement of an arc, by John Castle.

The theory of sundials, with directions on how to make one, by Gordon Guernsey.

The last two talks were presented at the annual open-house meeting for high school teachers and students in April, the program for which was planned and conducted by Ruth Brendel. The March meeting was a bowling party to which the Army Air Corps cadets stationed on campus were invited. Officers for the year were: President, Annabel Miller; Vice-President, Jeanne Jerge; Secretary-Treasurer, Lois Obenauer.

Mathematics Club, University of Dayton

Semi-monthly meetings were held at which members of the Club presented papers:

Finite and infinite classes, by Louis Synck.

Do concentric circles intersect?, by John Stang.

The four color map problem, by William McHugh.

Calculating prodigies, by Joseph Overwein.

Why the slide rule works, by Paul Herking.

The game of Nim, by John Westerheide.

At a joint meeting with the engineering seminar group, sponsored by the Club in October, the topic was:

Desirable educational qualifications of an engineer, by Mr. A. E. Schnaitman of the Warner and Swasey Co. of Cleveland. In November the Club sponsored a *Symposium on applied mathematics*, featuring two speakers:

Applications of the calculus of finite differences, by Professor F. M. Tiller of Vanderbilt University.

Fundamental concepts of the calculus of finite differences, by Professor K. C. Schraut.

Other topics presented during the year at the regular meetings were:

The life and work of Leibnitz, by Professor K. C. Schraut.

The method of least squares, by Mr. G. C. Peckham.

At the formal initiation in February, thirty new members were issued certificates. Dr. Francis J. Molz, Associate Dean of the Division of Science, was elected to honorary membership. William Fitzgibbons delivered the vice-president's charge to the new members, addressing them on the subject:

Aims and purposes of the Mathematics Club.

This address was subsequently published in the *Exponent*, which is the University's literary magazine.

Near the end of the year a banquet attended by fifty-five members was held. At the final meeting of the year the Dean of Science First and Second Awards were conferred upon William Fitzgibbons and Joseph Overwein respectively, for having delivered before the Club the most interesting papers of the year. At commencement the *Mathematics Club Alumni Awards for Excellence in Advanced Mathematics* were conferred upon Jack Homan in the Senior Class and Louis Synck in the Junior Class. The officers of the Club were: President, Joseph Overwein; Vice-President, William Fitzgibbons; Secretary-Treasurer, Theodore Schuler. Professor K. C. Schraut was Faculty Adviser.

Mathematical Society, Hofstra College

Speakers and topics for the year were as follows:

Non-Euclidean geometry, by Professor E. R. Stabler.

Denumerability of sets, by Ellen Fletcher.

Probability, by Harry Durham.

Vibration and flutter in aeronautics, by Dr. R. H. Tripp of the Grumman Aircraft Corporation.

At one meeting a quiz contest was held. In May the Club had a joint picnic with the Hofstra chapters of *Kappa Mu Epsilon* and *Beta Beta Beta* (honorary fraternities in mathematics and biology respectively). Officers for the year 1942–43 were: President, Mario Juncosa; Vice-President, Ellen Fletcher; Secretary, Phyllis Reynolds; Treasurer, Harry Durham; Historian, Edward Ryder. The Club Adviser was Professor E. R. Stabler.

Brooklyn College Mathematics Society

The Brooklyn College Mathematics Society adopted a program which attempted to cover the varied interests of the members, and which included the following talks:

War courses in mathematics at Brooklyn College, by Professor MacNeish.

Numeration, by Professor Boyer.

Finite configurations, by Dr. Singer.

Constructions by inversion, by Gerard Washnitzer, a member of the Society.

Linear transformations of polygons, a lecture by Professor Douglas based on his original work.

The Society sponsored the *Math Mirror,* which has been issued annually for the last eleven years, and which featured articles by members of the Society. The Society also sponsored an Integration Contest for the students of calculus at Brooklyn College. The officers for the fall term 1942 were: President, Frances Brand; Vice-President, Julian Kielson; Secretary, Audrey Dirnfeld; Treasurer, Joseph Chiarulli; Publicity Director, Norman Zabb. For the spring term 1943, they were: President, Irving Reiner; Vice-President, Abraham Mark; Secretary, Audrey Dirnfeld; Treasurer, Margot Henn; Publicity Director, Seymour Chays. The Faculty Advisers were Dr. Richardson for the fall term and Mrs. Kormes during the spring term.

DISCUSSIONS AND NOTES

EDITED BY MARIE J. WEISS, Sophie Newcomb College, New Orleans, La.

The Department of Discussions and Notes is open to all forms of activity in college mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

A SPEED TEST QUESTION. A PROBLEM IN GEOGRAPHY

E. J. MOULTON, Columbia University

How long does it take you to get a correct answer to the following mathematical problem? You may be surprised to know that most mathematicians require at least ten minutes. You are urged to time yourself for a little amusement. Read the problem carefully, solve it mentally, write your answer down, and note the time required. (We add the remark that you are assured that this is a legitimate mathematical problem, in which the earth is assumed to be a sphere.)

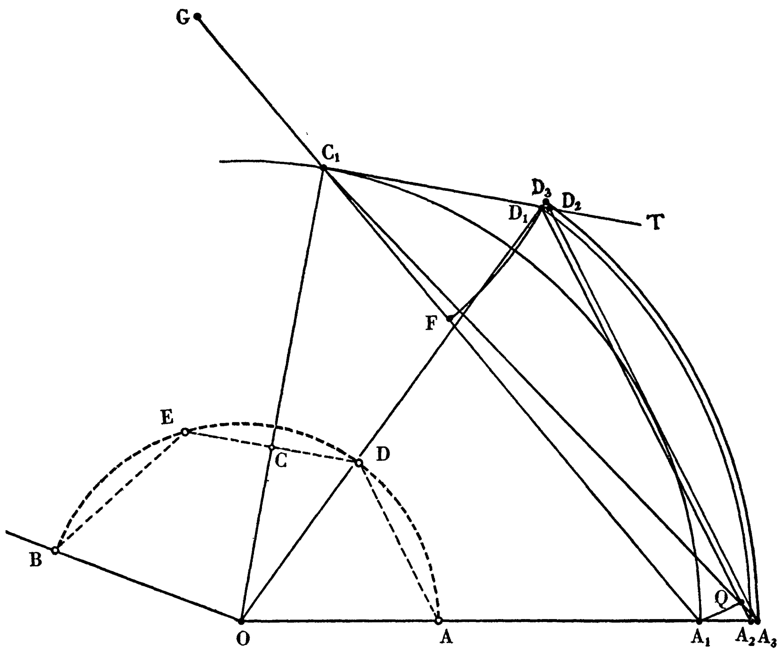
THE PROBLEM. *Starting at a point on the earth's surface, a man walked due south ten miles, then due east ten miles, then due north ten miles, and found that he had returned to the starting point. Where must he have started?*

For comments on the problem see page 220.

A SIMPLE CONSTRUCTION FOR THE APPROXIMATE TRISECTION OF AN ANGLE

R. L. DURHAM, Buena Vista, Virginia

1. Introduction. Since it is impossible to locate the exact trisectors of an arbitrary angle under the terms of Euclid's Postulates alone, the customary methods make use of other curves than circles. (See, *e.g.*, W. W. R. Ball, *Mathematical Recreations and Essays*, 10th edition, page 344, or 11th edition, page 333.) There is also the problem of *approximating* to the trisectors with ruler and compasses. Most of the efforts in this direction have been either too crude or too complicated. It seems reasonable to claim, however, that the present construction attains remarkable proximity without undue complication.



For an explanation of the general ideas involved, consider three equal arcs AD, DE, EB of a circle with center O, and let C be the midpoint of the middle arc DE. Then $AD=2DC$. Thus a *given* angle AOB will be trisected by OD if we can locate a point D on the arc AB, twice as far from the point A as from the line OC (which *bisects* the given angle). A hyperbola of eccentricity 2, with focus A and directrix OC, would intersect the circle in such a point D, but that is beyond the prescription "ruler and compasses." Instead, we proceed to construct a first approximation OD_1, and a second approximation OD_2.

In the figure, AOB is drawn as an obtuse angle for the sake of perspicuity. But the method gives a far better approximation when applied to an acute angle.

2. The first approximation. For a given angle AOB, draw a circle with center O (and conveniently large radius), to meet OA and the bisector OC in A_1 and C_1. Draw the tangent C_1T (perpendicular to OC). Then D_1 is defined to be a point on C_1T, twice as far from A_1 as from C_1. It is most easily located by means of a circle of Apollonius, which is the *locus* of points twice as far from A_1 as from C_1. For this purpose, trisect the chord A_1C_1 at F, so that $A_1F=2FC_1$, and take a point G in A_1C_1 (produced) so that $FC_1=C_1G$. Then D_1 is the point where C_1T meets an arc through F with center G.

The line OD_1 fails to trisect the angle. For, although D_1 is twice as far from A_1 as from OC, yet it lies outside the circle through A_1. We now seek a second approximation OD_2, which fails similarly but with a much smaller discrepancy.

3. The second approximation. With center O, draw an arc through D_1 to meet OA_1 (produced) in A_2. With center D_1, draw an arc through A_1 to meet D_1A_2 in Q. Let A_3 be the point of intersection of C_1Q and OA_2 (both produced). With center O, draw an arc through A_3 to meet OD_1 (produced) in D_3. Then the desired point D_2 is where the chord A_3D_3 meets C_1T.

4. Discussion. To see, in a general way, why OD_2 is close to the trisector, we observe that $D_1Q=D_1A_1=2D_1C_1$, whence, by similar triangles, $D_2A_3=2D_2C_1$. Thus D_2 is twice as far from A_3 as from OC. This point D_2, on the chord A_3D_3, lies inside the circle through A_3; but the distance D_2D_3 is small, and the difference OD_3-OD_2 is smaller still, with the result that OD_2 is almost indistinguishable from the true trisector on any ordinary drawing, especially when the angle AOB is small.

To estimate the amount of the discrepancy, let the given angle be 2θ, so that $\theta=\angle A_1OC_1$. We write

$$\alpha = \angle D_1OC_1, \qquad \gamma = \angle D_2OC_1,$$

so that α and γ are the first and second approximations to $\theta/3$. Then we find that

$$3 \tan \alpha = (1 - \cos \theta)^{1/2}(7 + \cos \theta)^{1/2} - \sin \theta,$$

$$\cot \gamma = 2 \cos \frac{\theta - \alpha}{2} \csc \theta + \cot \theta.$$

For instance, when $\theta=90°$, γ is about $30°\ 10'$. Again, when $\theta=45°$, γ is about $15°\ 0'\ 12''$. From the nature of this method it appears that the discrepancy $\gamma-\theta/3$ is positive, and diminishes with θ. Thus the construction trisects any acute angle AOB with a maximum error of twelve seconds.

For a more precise investigation we could use the expansions in powers of θ, which begin with

$$\alpha = \frac{\theta}{3} - \frac{7}{6}\left(\frac{\theta}{6}\right)^3 - 125\left(\frac{\theta}{12}\right)^5 + \cdots,$$

$$\gamma = \frac{\theta}{3} + \frac{7}{144}\left(\frac{\theta}{3}\right)^5 - \cdots,$$

A TABLE FOR COMPUTING PERIMETERS OF ELLIPSES

ALAN WAYNE, Rhodes School

The length of a quadrant Q_a of an ellipse of semi-major axis a, semi-minor axis b, and eccentricity k is

$$Q_a = aE = a\int_0^{\pi/2}(1 - k^2 \sin^2\theta)^{1/2}d\theta, \qquad (0 < k < 1), \tag{1}$$

where E denotes a complete elliptic integral of the second kind. Also,

$$k^2 = 1 - (k')^2, \quad \text{where} \quad k' = b/a, \qquad (a > b). \tag{2}$$

For given semi-axes, Q_a is usually computed by finding k^2 and using the corresponding value of E from a table [1, pp. 204–205]. A second method is to find arc sin k [1, pp. 2–19] and then E [2, p. 263]. Again, E may be found from tabulated values corresponding to those of $(k')^2$ [3, p. 80].

For more rapid computation, the writer devised the table of values of E corresponding to those of k', as given here.

SEMI-MINOR AXIS	LENGTH OF QUADRANT OF UNIT ELLIPSE $Q_1 = E = Q_a/a$									
$k' = b/a$	0	1	2	3	4	5	6	7	8	9
.0	1.000	000	001	002	003	005	007	009	011	013
.1	016	019	022	025	028	032	035	039	043	046
.2	050	055	059	063	068	072	077	082	087	091
.3	096	102	107	112	117	123	128	134	139	145
.4	151	156	162	168	174	180	186	192	199	205
.5	211	217	224	230	237	243	250	256	263	270
.6	276	283	290	297	304	311	317	324	331	339
.7	346	353	360	367	374	381	389	396	403	411
.8	418	426	433	440	448	455	463	470	478	486
.9	493	501	509	516	524	532	540	547	555	563
1.0	571									

From this table, the perimeter of any ellipse may be found readily, since to each given ellipse with semi-axes a and b there corresponds a "unit ellipse" which has semi-axes 1 and k', and length of quadrant $Q_1 = E$, so that $Q_a = aQ_1$.

For example, to find the perimeter of an ellipse with $a = 3.74''$ and $b = 2.36''$, find $k' = 2.36/3.74 = .631$, and from the table, $Q_1 = 1.298$, so that $P = (4)(3.74)(1.298) = 19.4''$.

With a slide rule the process takes less than a minute. No more than three slide settings are required to find the perimeter to three significant figures, which is adequate for most practical work.

References

1. H. B. Dwight, Mathematical Tables, McGraw-Hill, New York, 1941.
2. R. S. Burington, Handbook of Mathematical Tables and Formulas, Handbook Publishers, Sandusky, Ohio, 1940.
3. E. Jahnke and F. Emde, Tables of Functions with Formulae and Curves, Dover Publications, New York, 1943.

COMMENTS ON THE PROBLEM IN GEOGRAPHY (p. 216)

E. J. MOULTON, Columbia University

When this problem is proposed orally to mathematicians, the usual response after a short time, varying from one second to a few minutes, is that *the man must have started at the North Pole.* When a mathematician is then told that his answer is wrong—and this answer *is* wrong—he is likely either to look for some non-mathematical explanation of his error or to become slightly belligerent in defense of his answer. However, he is assured that the problem is a bona fide question in the geometry of a sphere, and that his answer is mathematically incorrect. In spite of years of training in logical thinking, he has slipped on the word "must" in the question, "Where must he have started?" He has found one point where the traveller *may* have started, but there is another point where he may have started.

On second thought, which may last ten minutes or longer, mathematicians discover infinitely many points from which the man may have started; the usual answer then given is, *the man must have started either at the North Pole or at a point on a parallel of latitude which is ten miles north of a parallel of latitude whose circumference is ten miles.* But *this answer* also *is wrong*—the man *may* have started at any one of these points, but you should not assert that he *must* have started at one of them.

A correct (but not very satisfying) answer is that

(1) *he must have started at some point on the earth's surface.*

A more satisfactory answer, which is also correct, is that

(2) *he must have started at some point of a locus S which consists of the North Pole and the circles which are ten miles north of the parallels of latitude whose circumferences are $10/n$ miles, where n ranges over the positive integers.*

This locus S is interesting. It consists of an isolated point and an infinite set of circles which have as a limit circle the parallel of latitude which is ten miles from the South Pole; this limit circle is not a part of the locus.

It is also interesting to see what modifications are required in the preceding discussion if the number ten is replaced by a number x, and to consider the locus S as a function of x on the range $x=0$ to $x=12{,}500$ (calling the circumference of the earth 25,000 miles). I consider this function of x to be one of the most interesting which I have encountered, possessing as it does some startling discontinuities.

RECENT PUBLICATIONS

EDITED BY VIRGIL SNYDER, Cornell University, and H. P. EVANS, University of Wisconsin

All books for review should be sent directly to the editors of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y., and not to any of the other editors or officers of the Association.

Differential and Integral Calculus. Fourth Edition. By C. E. Love. New York, The Macmillan Company, 1943. 15+483 pages. $3.25.

The fourth edition of Love's Calculus has appeared in a new format employing a larger page size and larger type and is illustrated with a set of greatly improved figures.

The textual material has in general been simplified by the inclusion of fuller explanations. The order of presentation of the topics of the Calculus has been changed to include within the first six chapters most of the applications of the differentiation of algebraic functions and the applications of the integration of simple algebraic functions to areas and plane motion. Vectors and parametric equations are also treated within these chapters.

Continuity and discontinuity are discussed in the seventh chapter. There is an advantage in the position of this topic within the text for the teacher who feels that an introduction to the Calculus increases the student's interest in and understanding of such a discussion.

No great changes have been made in the presentation of later chapters. The Integral Test for series has been omitted, and a development of the equation of the catenary has been included. Talks of natural logarithms, hyperbolic functions, and trigonometric functions have been inserted.

FRED KIOKEMEISTER

Geometry with Military and Naval Applications. By W. F. Kern and J. R. Bland. New York, John Wiley and Sons; London, Chapman and Hall, 1943. 7+152 pages. $1.75.

In 1934 the authors of this text in solid geometry produced "solid Mensuration," which was reviewed in this journal (Vol. XLIII, page 232) by R. A. Johnson. A second edition with proofs appeared in 1938. This newest geometry is essentially the 1938 edition dressed up for current use by replacing many of the standard problems with others having a military or naval setting. A few sections of the text have been omitted and 17 pages on trigonometric functions and the solution of right triangles have been added to the appendix.

All three texts are excellent for a course in solid geometry in which the emphasis is on mensuration rather than geometric theorems and their proof. This particular volume will make mensuration more interesting to those students who are in the armed forces or are anticipating early induction.

L. A. DYE

Basic Physics for Pilots and Flight Crews. By E. J. Knapp. New York, Prentice-Hall, Inc., 1943. 118 pages. $1.65.

The author states in his Preface, "The purpose of this book is to teach the fundamentals of physics as a preparation for Meteorology, Theory of Flight, and Engine Operation. It is designed to meet the requirements of primary flight training or air cadet training."

This pocket size handbook presents, in condensed form, certain fundamental principles of Mechanics of Solids and Fluids and of Heat. The following laws are stated and illustrated by problems: Newton's Laws of Motion; Laws of Sliding Friction; Law of Conservation of Energy; Boyle's Law; Charles' Law; Dalton's Law; General Laws of Gases. As a rule these laws are merely stated without reference to experimental data which lead to their formulation.

Problems on addition of vectors are solved by the graphical method only. Subtraction of vectors is not considered. Problems on balanced forces acting at a point in a plane are also solved graphically, without analytical treatment.

Gravitational units are used for the measurement of force. The fundamental equation of motion becomes, consistently, $F=(W/g)\cdot a$.

Discussion of curvilinear motion is limited to the statement that centripetal force equals $(W/g)(V^2/R)$.

Discussion of kinetic energy is limited to kinetic energy of translation. Kinetic energy of rotation is not mentioned.

The subject of elasticity and Young's Modulus is not mentioned.

The student is not warned that the general gas laws hold strictly for an ideal gas only, and that large corrections must be made for easily condensible gases.

These omissions have been made purposely, without doubt, for the sake of condensation. In general, it may be said that definitions and laws are stated accurately; that the text is simply and clearly worded, and that the problems and exercises should be within the capacity of a basic student to solve. On completion of the course, the student should have become familiar with a number of physical formulas and should have learned how to apply these formulas to the solution of simple problems. The handbook may well be useful as a short cut to these objectives. It is not a text on elementary physics.

F. M. GREENLAW

Plane and Spherical Trigonometry. By A. L. Nelson and K. W. Folley. With Mathematical Tables. Revised edition. New York, Harper and Brothers, 1943. 14+247+135 pages. $2.40.

In the reviewer's estimation both the present edition and the original version of Nelson and Folley's Trigonometry are excellent examples of what might be called the classical treatment of the subject. Trigonometry done in the traditional manner is, of course, a somewhat crystallized topic; and naturally the various expositions differ, for the most part, but slightly in broad outline. The war, however, has provided an opportunity to freshen this rather stale subject;

and instructors will welcome the fact that the present authors among others have taken advantage of the situation in an effective manner. If one feels the compunction to "follow the book," the teaching of stereotyped freshman courses is apt to be a dull business; and for this reason at least novelty in textbooks should certainly be more welcome than say mere sterile perfection of detail.

The two editions differ in the aspects emphasized, the applications, the tables, and necessarily but unfortunately in format. The first edition was designed primarily to prepare the student for more work in mathematics and the stress is on the functional aspects as opposed to the computational. Thus the analytic portion is completed as soon as feasible; the general definition of the trigonometric functions, to mention one item, is presented at the outset. On the other hand, the second edition, as an impact of the war, places somewhat more emphasis on the computational part than did the earlier exposition. The trigonometric functions are presented first in connection with right triangles, and the solution of oblique triangles is accorded extra weight by early introduction and by the allotment of two chapters to the subject. The first sets of problems on oblique triangles (chapter III) are limited to numbers of two or three digits and hence are well suited to both elementary arithmetic and the slide rule. A later chapter (V) presents the logarithmic treatment. Chapter III is brightened considerably by the introduction of some interesting and timely material on aeronautics which undoubtedly will serve both to relieve the monotony for the instructor and to arouse the student.

In addition to the more common topics there are articles on: significant digits, abridged multiplication, projection, the mil, vectors (addition and resolution into components), aeronautical applications, logarithms (a thorough treatment), graphing by addition of ordinates, complex numbers (elementary operations, polar form, De Moivre's theorem, powers and roots, *etc.*), spherical triangles (thorough and inclusive), applications of spherical triangles (great circle sailing, plane sailing, traverse sailing, middle latitude sailing, dead reckoning, Mercator projection, Mercator sailing, the celestial sphere, the astronomical triangle, determination of latitude and longitude, determination of the time of day and of the time of sunrise and sunset).

The book contains about 155 pages of tables including haversines, logarithms of numbers, logarithms of trigonometric functions, trigonometric functions, natural logarithms, conversions of degrees to radians, squares, cubes, square roots, cube roots, reciprocals, circumferences and areas of circles, and some important constants.

Conclusion. This book is certainly one of those that should be examined by anyone who is considering the selection of a text on the subject. By proper choice of topics and their order it can be made to serve either the war needs for which it was designed or to contribute to the foundation for more mathematics courses. It contains an abundance of problems (2,624), most of which are not in the old edition; and the exposition leaves little to be desired.

H. V. CRAIG

Analytic Geometry. By F. H. Steen and D. H. Ballou. New York, Ginn and Co. 1943. 7+206+9 pages. $2.40.

This book is designed for beginners. A feature of the text is the inclusion of "orals" preceding many of the sets of exercises, to serve as orientation of the purpose of the exercises following. The normal form of the equation of the line and the formula for the distance between a point and a line are derived without use of trigonometry; this method is then followed by a short outline of the traditional one. The area of a triangle in terms of the coordinates of its vertices is given as a determinant, with a foot-note explaining its use. Otherwise, determinants of the third order are not used. The idea of a sign associated with a non-linear equation is not given. The explanation of what the book calls squaring an equation is not helpful. The foot-note concerning imaginaries needs more explanation to be intelligible. The treatment of the circle and of conic sections is clear and concise. It includes that of pencils of lines and of circles. The conics are presented by means of focal distances, but followed by an outline of the method of eccentricity. The use of the complementary equation, page 66, is hazy.

Rotation and elimination of the product term in the equation of a conic are discussed without use of trigonometry, again followed by an alternate method, based on the traditional procedure. A chapter on curve tracing includes symmetry as to an axis, tangents at the origin, asymptotes parallel to the axes and vacant strips parallel to the axes. Trigonometric curves, logarithmic and exponential curves, sums of ordinates and compound curves are presented, the last too sketchily. Parametric representation is discussed and applied to the cycloids; polar coordinates are discussed rather fully, including the tangent at the pole, if the curve goes through the pole.

Coordinates in space are treated in four chapters in the same manner as in the plane. Determinants of the second order are employed in the discussion of angles, but nowhere else. Quadric surfaces are described in a brief outline which includes curves of intersection and projecting cylinders. The book is provided with an index and the answers to the exercises. It has an attractive appearance with an open page, clear type, and over 200 well drawn figures.

VIRGIL SNYDER

Selected Topics in Higher Mathematics for Teachers. Published by the Committee on Higher Mathematics of the Association of Teachers of Mathematics of New York City, 1943. 107 pages. $0.50. (For copies, address the chairman, Max Peters, at Utrecht High School, Brooklyn.)

This book, printed by the photo-offset process, consists of five monographs covering some topics treated in in-service courses sponsored by the above association for the following purposes: first, "to make available to teachers valuable enrichment material drawn from appropriate fields of higher mathematics," second, "to make more profound the teacher's understanding of the concepts he teaches."

The first monograph, on Topology, by Samuel Greitzer, is marred by several obscure statements. For instance, in the derivation of Euler's formula appears the following sentence whose obscurity seems partly due to a misprint and partly to a desire to avoid buckling down to the consideration of all possibilities: "Let us therefore take any polyhedron and reduce it by removing edges and with a simplified figure having one vertex, one face and one or more edges." There is also some tendency to assume unduly knowledge which could not be obtained by reading the monograph. There is, however, a wealth of fascinating illustrations and the presentation is such as to stimulate the reader to look more carefully into the subject.

The second monograph, on Vectors in Plane Geometry, by Harry Sitomer, is a *very* convincing and interesting attempt to show how the use of vectors can simplify certain proofs of plane geometry. The text is interspersed with exercises which are well chosen to teach the reader the concepts involved and the possibilities of the method. The only essential improvement the reviewer could suggest would be to make the figures more accurate—notably the one at the top of page 30.

The third monograph, on the Theory of Numbers in Secondary Mathematics, by Mannis Charosh, presents a good case for the training of teachers in some of the elements of the theory of numbers, though in two instances the author stretches his argument too far: the number tricks on page 42 are explained by algebra and the application to the roots of $\sin x=\frac{1}{2}$ seems somewhat far-fetched. The principle types of application given are: finding integer values of F that will make $C=(5/9)(F-32)$ an integer, determining quadratic equations with rational roots, finding integer values of a and b so that the two expressions $x^2+ax\pm b$ have factors with rational coefficients, and determining oblique triangles whose sides and area are integers. The reviewer trusts that even this monograph will not induce teachers to give only problems with integral answers.

The fourth monograph, on the Postulational Approach to Mathematics, by Mannis Charosh, is a good treatment of a difficult subject. A good discussion is given of axiom (self-evident truth) versus postulate. The author's illustration of a mathematical system complete with consistency, independence, and categoricalness serves the purpose admirably.

The fifth monograph, An Introduction to Non-Euclidean Geometry, by Max Peters, describes some notable attempts to avoid the fifth axiom of Euclid and follows this by an introduction to Lobatchewsky's geometry and Riemannian geometry. The monograph is very well written and should be within the understanding of those for whom it is intended.

Though there are a number of misprints, pages 35 and 36 are interchanged, and though in some cases there is a little tendency to be careless in statement and superficial in treatment, it is the reviewer's opinion that this book is an admirable beginning. The fact that the teachers themselves participated in and pushed the venture is very significant and it is to be hoped that this is only the first of a series of such books.

B. W. JONES

Basic Mathematics for Pilots and Flight Crews. By C. V. Newsom and H. D. Larsen. New York, Prentice-Hall, 1943. 6+153 pages. $2.00.

This text provides for the prospective pilot, navigator or bombardier a review of mathematics at the high school level, and offers a wealth of problems designed to stimulate his interest in and understanding of the applications of elementary mathematics associated with the flight of aircraft. The authors state that little of the mathematics offered will be new to the student who has had a year of algebra and a year of geometry.

The reviews of arithmetic, algebra and geometry contained in this text emphasize the application of the subject to the problems of flight. In this respect the discussion of the basic mathematics involved and the illustrations offered differ widely from the usual high school text.

A chapter on elementary vectors and vector diagrams with applications and problems is provided. A chapter is devoted to the circular slide rule and its use in navigation. Answers are provided for about half of the problems.

While this text is essentially at the high school level the point of view taken by the authors in their discussions together with the nature of the applications makes this text a valuable one for the prospective student of aeronautics.

H. N. HUBBS

Solid Geometry and Spherical Trigonometry. By H. C. L. Leighton. New York, D. Van Nostrand Co., 1943. 19+192 pages. $2.20.

The contents of this book, as indicated by the title, are preceded by a summary of the contents of plane geometry, including the axioms, postulates and an outline of those theorems which are not mentioned again in the body of the text. Among the added postulates of the new part are featured that:

a plane is determined by a line and a point not on the line;

if two planes intersect, their intersection is a straight line.

The scheme of instruction is to present a list of pertinent questions for orientation, without providing proofs for all of them. This requires a large number of figures; each part of a complicated demonstration is featured in this way. Special attention is given to Napier's Rules for the right spherical triangle. It is pointed out that any spherical triangle can be broken up into right spherical triangles, each of which could then be discussed in this way. A later chapter derives the law of sines and the law of cosines in the usual manner. A short chapter on applications includes a description of true bearing, great circle sailing, and the Mercator chart. A final chapter provides the derivation of the half-angle formulas; they are not used in the earlier part. The printing and press-work are excellent. A clear, open page is presented, and the figures, nearly three hundred of them, are well done.

VIRGIL SNYDER

Business Mathematics. By C. L. Richtmeyer and J. W. Foust. New York, McGraw-Hill Book Company, second edition, 1943. 15+401 pages. $2.75.

The first edition of this useful text for students of business mathematics appeared in 1936 (see review this MONTHLY, Vol. XLIII, No. 9, p. 570). The book has now been considerably enlarged, the present edition containing 318 pages, exclusive of tables, answers, etc., as compared with 182 pages originally.

The organization and treatment in the first edition are retained in the second, but certain changes have been made which increase the usefulness of the book. Among these changes are:

1. The material on elementary arithmetic has been expanded. The work on integers and fractions which was treated in one chapter in the first edition occupies three in the second.
2. Self-tests which appeared only in the first chapter of the earlier edition now occur at the end of each chapter. These tests should prove to be a valuable feature of the book.
3. Lists of supplementary references are given at the end of each chapter.

Other changes include expansion of the chapter on graphs, computations with approximate numbers, and installment buying. The work on bar graphs is more complete and includes several additional illustrative problems. Approximate numbers are introduced early in the text and are used to good advantage throughout the book. Also, a short account of correlation is included in the chapter on statistics. Problem and exercise lists have been revised and new problems added.

R. G. PUTNAM

Mathematics of Flight, By James Naidich. New York, McGraw-Hill Book Co., 1943. 6+409 pages. $2.75.

The purpose of this text is to give the necessary foundations to a student who wishes to fly, build, or analyze the mechanics of airplanes. The goal is all-embracing and the results are correspondingly lacking in depth.

The mathematical content is of high-school level and includes the barest essentials of arithmetic, algebra, plane geometry, and trigonometry. There is a tremendous effort to drill the student on these essentials and their applications to simple problems in the theory of airplanes. Such an effort could be justified only with a very young student, with one of low intelligence, or with one of very weak background.

The aerodynamical content consists in a few of the basic concepts such as center of gravity, stability, lift, drag, angle of attack, and banking, with purely intuitive explanations and great emphasis on practical applications. Much space is given to pictures of new models of aircraft. The presentation of these topics is excellent and should be quite successful in giving a young student a feeling for the subject.

W. KAPLAN

Original Tables to 137 Decimal Places of Natural Logarithms for Factors of the Form $1 \pm n \cdot 10^{-p}$, Enhanced by Auxiliary Tables of Logarithms of Small Integers. By H. S. Uhler. Published by the author, New Haven, Conn., 1942. 21+97 pages.

This is a set of radix tables similar to, but larger in extent than, those of Wace, Flower, and others. Their preparation was stimulated by the author's desire to extend his tables of $1/n!$ (Trans. of Conn. Acad. Arts and Sci., vol. 32, 1937) to larger n. Table I gives the logarithms of the integers $2, 3, \cdots, 10$ and 10^{10k}, $k=2, 3, 4, 5, 6, 7, 8, 9, 11$ to 137 places. Table II gives the logarithms of numbers of the form $1-n \cdot 10^{-p}$ for $n=1, 2, \cdots, 9$; $p=1, 2, \cdots, 69$ to 137 places, while Table III gives the logarithms of numbers of the form $1+n \cdot 10^{-p}$ for $n=1, 2, \cdots, 9$; $p=1, 2, \cdots, 21$. Tables V and VI are skeleton forms for Tables II and III for $p>20$. Finally, Table IV gives the values of $\log_e 10$ and $\log_{10} e$ to 325 places as well as logarithms of the primes from 11 through 113 to 148 places.

The method outlined by the author in his introduction to find the logarithm of a number N is essentially the one usually attributed to Weddle. N is multiplied by a preliminary reduction factor N' and then by a sequence of factors of the form $1 \pm n \cdot 10^{-p}$ to produce a number $1 \pm x$ where x is so small that $\log(1 \pm x)$ can easily be evaluated to the desired accuracy by the series expansion for $\log(1 \pm x)$. Since logarithms of the factors $1 \pm n \cdot 10^{-p}$ are known and $\log N'$ can be found with the aid of Tables I and IV, $\log N$ can be calculated. To find an antilogarithm, the process is reversed by first finding the factored number and then expanding the factors, *i.e.*, using the method known by the name of Hearn.

The introduction contains several examples of both procedures, the method used to calculate the tables, and checks used to insure complete accuracy.

D. E. RICHMOND

Analytic Geometry. By E. E. Smith, Meyer Salkover, and H. K. Justice. New York, John Wiley and Sons, Inc., 1943. 12+298 pages. $2.50.

This text is not for a brief course in analytic geometry, although the authors point out that certain sections may be omitted in order to enable it to serve in that capacity. It contains sufficient material for a full year's course of three hours a week or a semester's course of five or six hours a week. There are fifteen chapters (202 pages) of plane analytic geometry and five chapters (66 pages) of solid analytic geometry. These are followed by an appendix containing useful formulas from geometry, algebra, and trigonometry; the Greek alphabet; the proof of the invariance of Δ in connection with the general conic; and tables of trigonometric functions, logarithms, and values of e^x and e^{-x}. Answers to odd-numbered problems are listed.

One is impressed by the excellent figures in the book. They are so clearly marked that all the required features are unmistakably indicated.

Throughout the book discussions of general equations precede those of special cases. This is particularly noticeable in the treatment of the general

equation of the second degree in both plane and solid analytic geometry. It makes for a considerable amount of unity since each particular special case is referred back to the general equation for comparison.

Although the topics considered are those generally treated in any book on the same subject, there are many novel features in the presentation. Among these are the less usual constructions given for conics; the use of the terms normal axis, normal angle and normal intercept in connection with the straight line; the discussion of factorable equations; the definition of an asymptote; the derivation of the equation of a conic through five given points; the use of slope as well as other devices in curve tracing; the determination of parametric equations from a given Cartesian equation; and the use of the term direction parameters rather than direction numbers or direction components.

Topics sometimes slighted in other texts are treated at greater length here. For example, curve tracing in polar coordinates is discussed from the standpoint of intercepts, extent and symmetry in a manner analogous to that used for Cartesian coordinates. Some books give the impression that the treatment for polar coordinates is entirely different from the other. Empirical equations are given a generous amount of attention, using the method of average equations rather than the method of least squares. The treatment of tangents, normals and diameters is quite complete. The thoroughness of these discussions is accomplished, however, without emphasis on unimportant details. In fact, many steps in the derivations of formulas are omitted and there is frequent use of the expression "it may be shown that." There is wisdom in these omissions in that the overall pattern of the discussion is more evident, but they might prove difficult for the poorer student.

Determinants are used throughout the book with only a brief reference to them in the list of useful formulas. Students should really have a knowledge of determinants in order to use the book to the best advantage.

There are a few typographical errors, particularly in the small type material, but no glaring errors were detected. Difficulties of war time printing may be blamed for the majority of defects. On the whole the authors have developed a sound text which should take its place among the better books on the subject.

HARRIET F. MONTAGUE

Air Navigation for Beginners. By S. G. Lamb. New York, Norman W. Henley Publishing Co., 1942. 103 pages. $1.50.

This small volume was designed for the young student who knows nothing about navigation and little about mathematics. Neither trigonometric functions nor logarithms are used. The problems considered require only arithmetic or simple geometrical constructions. Teachers who prefer to have their students use a method first and find out why it works afterward will be disappointed that the author mentions the Dalton Dead Reckoning Computer, the Nautical Almanac, and Hydrographic Office Publications No. 211, 208, and 214 without giving the reader any idea of the way in which these navigational aids are used.

If one is willing to accept the author's approach, one will find little to criticize in the material included. The first two chapters cover fundamentals such as latitude, longitude, direction and distance on the earth. Chapter III describes the simpler instruments of navigation. At no point in the book will the reader find the sextant, the gyro-compass, and other complex instruments described. Chapters IV to VII cover briefly but fairly adequately Maps and Charts, Piloting and Radio Bearings, Dead Reckoning, The Wind Triangle and Wind Triangle Applications. Chapter VIII describes the radio range system of the Civil Aeronautics Administration as it existed in the United States before the war.

Chapters IX, X, and XI concern Flight Instruments and Corrections, Aerial Astronomy, and Time. There follows twelve pages of questions and answers and one page of abbreviations.

This book may be recommended as an introduction to air navigation for high school students.

C. H. SMILEY

NEW BOOKS RECEIVED

An Introduction to Pure Solid Geometry. By G. S. Mahajani. Second Edition. Poona, (4), Aryabushan Press, 1943. 10+104 pages. Rs. 3-0-0.

Lessons in Elementary Analysis. By G. S. Mahajani. Third Edition. Poona (4), Aryabushan Press, 1942. 13+298 pages. Rs. 6-4-0.

An Intermediate Course in Differential Equations. By E. D. Rainville. New York, John Wiley and Sons, Inc.; London, Chapman and Hall, Ltd. 6+213 pages. $2.75.

Geometry with Military and Naval Applications. By W. F. Kern and J. R. Bland. New York, John Wiley and Sons, Inc.; London, Chapman and Hall, Inc., 1943. 7+152 pages. $1.75.

Statistical Adjustment of Data. By E. W. Deming. New York, John Wiley and Sons, Inc.; London, Chapman and Hall, Ltd., 1943. 10+261 pages. $3.50.

Elements of Trigonometry. Plane and Spherical. By L. M. Kells, W. F. Kern, J. R. Bland, and J. B. Orleans. New York and London, McGraw-Hill Book Co., Inc., 1943. 10+363 pages. $1.80.

Plane and Spherical Trigonometry. By D. H. Ballou, and F. H. Steen. Boston, Ginn and Company, 1943. 6+179+9 pages. $2.20.

Spherical Trigonometry with Tables. By D. H. Ballou and F. H. Steen. Boston, Ginn and Company, 1943. 4+68+84 pages. $1.25. (This is latter half of Plane and Spherical Trigonometry, pages 109–176 published with Tables.)

Principles of Air Navigation. By B. A. Shields. New York and London, McGraw-Hill Book Co., Inc., 1943. $2.20.

Grösse Masszahl und Einheit. By M. Landoldt. Zürich, Rascher and Co., 1943. Fr. 5.50.

Mathematics for Mariners. By C. E. Dimick and C. C. Hurd. New York, D. Van Nostrand Company, Inc., 1943. 7+254 pages. $2.75.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND H. S. M. COXETER

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to H. S. M. Coxeter, 24 Strathearn Boulevard, Toronto 10, Canada.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 616. *Proposed by J. S. Cromelin, Chicago*

A $60'$ ladder and a $77'$ ladder rest with their lower ends against a building on the east side of a street, their upper ends against the opposite building on the west side. A third ladder rests with its lower end against the building on the west side, its upper end against the building on the east side. It crosses the first ladder at a height of $17'$, the second at a height of $19'$. What is the length of the third ladder, and how wide is the street (to the nearest inch)?

E 617. *Proposed by V. Thébault, San Sebastián, Spain*

From a given tetrahedron we derive another by taking as vertices the points of contact of the insphere with the faces. Show that the dihedral angles at pairs of opposite edges of the first tetrahedron are supplementary if and only if the second tetrahedron is trirectangular.

E 618. *Proposed by Thorold Gosset, Cambridge, England*

Prove that, for any value of θ between 0 and π,

$$\sin\theta + \frac{\sin 3\theta}{3} + \frac{\sin 5\theta}{5} + \cdots = \frac{\pi}{4}.$$

E 619. *Proposed by W. B. Clarke, San José*

Prove that the four triangles of the complete quadrangle formed by the circumcenters of the four triangles of any complete quadrilateral are similar to those triangles.

E 620. *Proposed by Alan Wayne, Rhodes School, New York*

Find integral sides for a triangle in which one angle is six times another.

SOLUTIONS

Two Remarkable Squares

E 582 [1943, 454]. *Proposed by V. Thébault, San Sebastián, Spain*

Find a square of ten digits such that the two numbers formed by the first five and last five digits are consecutive.

Solution by D. H. Browne, Buffalo, N. Y. Instead of the equation

$$x^2 = abcde\cdot 100001 \pm 1,$$

consider the congruence

$$x^2 \equiv \pm 1 \pmod{11\cdot 9091}.$$

Since -1 is a quadratic non-residue (mod 11), the upper sign must be taken. Thus $x-1$ and $x+1$ must be multiples of 11 and 9091 (or vice versa) which differ by 2, say

$$11u - 9091v = \pm 2.$$

Since $9091 \equiv 5 \pmod{11}$, we have $5v \equiv \mp 2$, whence $v \equiv \pm 4 \pmod{11}$. Among such values of v, 4 and 7 are the only ones for which $9091v$ has five digits. Hence the two solutions

$$36365^2 = 1322413225, \qquad 63636^2 = 4049540496.$$

Also solved by R. L. Caswell, Daniel Finkel, N. G. Gunderson, Irving Kaplansky, Walter Penney, E. D. Schell, E. P. Starke, and the proposer.

Radical Axes of Spheres

E 583 [1943, 454]. *Proposed by N. A. Court, University of Oklahoma*

Given four spheres (A), (B), (C), (D), with centers A, B, C, D, let a plane parallel to ABC cut DA, DB, DC in points U, V, W. Show that the radical axis of the three spheres having U, V, W for centers and coaxal with the respective pairs of spheres (D) and (A), (D) and (B), (D) and (C), coincides with the radical axis of the spheres (A), (B), (C).

Solution by P. D. Thomas, Wright Junior College, Chicago. A sphere orthogonal to two spheres is orthogonal to any sphere of the pencil determined by them. Hence the radical sphere of (A), (B), (C), (D), being orthogonal to these, is also orthogonal to (U), (V), (W), and coincides with the radical sphere of (U), (V), (W), (D). Its center lies on the radical axis of (A), (B), (C), and on the radical axis of (U), (V), (W). Hence these two radical axes, perpendicular to the parallel planes ABC and UVW, must coincide.

Also solved by Howard Eves and the proposer. The corresponding proposition in the plane is due to C. Servais (*Mathesis*, 1891, p. 238, q. 717).

Curve of Flight by Directional Radio

E 586 [1943, 512]. *Proposed by Frank Hawthorne, Allegheny College*

What is the path of an airplane "homing" from A to B by directional radio from B only, if acted on by a constant wind?

Solution by W. N. Brown, Glendora, California. Use polar coordinates with origin at B and initial line in the direction of the wind. Let (a, α) be the given position A, and (r, θ) a subsequent position. If v is the constant velocity of the airplane relative to the air, and w is the constant velocity of the wind, the resultant velocity, along the tangent to the path, is

$$\frac{dr}{rd\theta} = \frac{v + w \cos \theta}{w \sin \theta}.$$

Integrating, and evaluating the constant, we obtain the polar equation

$$r = a \frac{\tan^{v/w} \frac{1}{2}\theta \sin \theta}{\tan^{v/w} \frac{1}{2}\alpha \sin \alpha}.$$

Also solved by Howard Eves, referring to Agnew's *Differential Equations* (New York, 1942), p. 88.

Altitudes of an Orthocentric Tetrahedron

E 587 [1943, 512]. *Proposed by V. Thébault, San Sebastián, Spain*

Let AA', BB', CC', DD' be the altitudes of an orthocentric tetrahedron $ABCD$, with orthocenter H. Show that

$$\frac{BC\cdot DA}{B'C'\cdot D'A'} = \frac{CA\cdot DB}{C'A'\cdot D'B'} = \frac{AB\cdot DC}{A'B'\cdot D'C'} = \frac{HA\cdot HB}{HC'\cdot HD'} = \frac{HC\cdot HD}{HA'\cdot HB'} = \cdots.$$

Solution by Howard Eves, Syracuse University. Since the four points A, A', B, B' are concyclic,

$$AB/A'B' = HB/HA' = HA/HB'.$$

Similarly $DC/D'C' = HC/HD' = HD/HC'$. Therefore

$$\frac{AB\cdot DC}{A'B'\cdot D'C'} = \frac{HB\cdot HC}{HA'\cdot HD'} = \frac{HB\cdot HD}{HA'\cdot HC'} = \frac{HA\cdot HC}{HB'\cdot HD'} = \frac{HA\cdot HD}{HB'\cdot HC'}.$$

Similar results for the other pairs of opposite edges establish the theorem.

Also solved by L. M. Kelly and the proposer.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known text-books or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4115. *Proposed by H. F. Sandham, Trinity College, Dublin, Ireland*

From a point P on the circumcircle of a triangle lines are drawn inclined at angles θ to the sides of the triangle and meeting them in three collinear points. Prove that as P varies, the line on the three points envelops a three-cusped

hypocycloid. Prove that this hypocycloid is the locus of a point on a circle, radius $R/2 \sin \theta$, which rolls inside another circle, radius three times that of the rolling circle, whose center X is equidistant from the circumcenter C and the orthocenter O, and is such that angle $OXC = 2\theta$, R being the circumradius.

4116. *Proposed by N. A. Court, University of Oklahoma*

Given the tetrahedron $(T_1) = SA_1B_1C_1$, the tangent plane to its circumsphere at the diametric opposite of S meets the edges SA_1, SB_1, SC_1 in the points A_2, B_2, C_2. The tangent plane to the circumsphere of the tetrahedron $(T_2) = SA_2B_2C_2$ at the diametric opposite of S meets the edges of (T_2) through S in the points A_3, B_3, C_3 thus forming the tetrahedron $(T_3) = SA_3B_3C_3$, *etc.* Find the locus of the incenters of these tetrahedrons.

4117. *Proposed by J. Rosenbaum, Bloomfield, Conn.*

A polygon $A_1A_2 \cdots A_n$ may be transformed into a polygon $B_1B_2 \cdots B_n$ by locating the points B_i on the sides A_iA_{i+1} so that the ratio of A_iB_i to B_iA_{i+1} is equal to a constant r. Prove that if T_1 and T_2 are two such transformations for the ratios r_1 and r_2, then $T_1 \circ T_2 = T_2 \circ T_1$, and generalize.

4118. *Proposed by Otto Dunkel, Washington University*

Show that

$$\sum_{t=0}^{n} (-1)^{n+t} \frac{t^{n+4}}{t!(n-t)!} = \frac{(n+4)(n+3)\cdots n}{6!8} [15n^3 + 30n^2 + 15n - 2], \quad n \geqq 0,$$

and that each member of this equality is a non-negative integer. If n is a negative integer, the right member is an integer; what meaning may be given to the result in this case?

4119. *Proposed by V. Thébault, San Sebastián, Spain*

The straight lines joining the vertices of a triangle to the points of contact of the inscribed circle with the respective opposite sides meet in a point P. Show that the six points of contact of circles tangent to two sides and orthogonal to a given circle with center P are on a circle concentric with the inscribed circle.

SOLUTIONS

An Inequality for Triangles

4070 [1943, 124]. *Proposed by P. Erdös, Princeton, N. J.*

Let ρ denote the length of the radius of the inscribed circle of the triangle ABC, let r denote the circumradius and let m denote the length of the longest altitude. Show that $\rho + r \leqq m$.

Correction. The proposer intended to exclude obtuse angled triangles.

I. *Solution by L. A. Santaló, Rosario, Argentina.* Let $A_1A_2A_3$ be a triangle with no obtuse angle and with angles $A_1 \geqq A_2 \geqq A_3$; let O, I, H_i, B_i be the circumcenter, incenter, foot of the altitude from A_i, point of contact of incircle (I)

with side $a_i = A_jA_k$. If $I \equiv O$ the triangle is equilateral and $\rho + r = m$. If $A_1 = \pi/2$ and $A_1A_2 = A_1A_3$, then $\rho + r = m$.

Since $A_2B_1 \leqq B_1A_3$ and angle $OA_3B_1 \leqq$ angle IA_3B_1, the center O does not lie outside the triangle B_1A_3I. Also A_3O and A_3H_3 are symmetric with respect to A_3I. Let O_3 on A_3H_3 be the symmetric of O with respect to A_3I so that $A_3O_3 = r$, and let I_3 be the orthogonal projection of I on A_3H_3 so that $I_3H_3 = \rho$. Since angle $O_3IB_3 > \pi/2$, the point O_3 lies on segment A_3I_3, and it follows that $\rho + r \leqq A_3H_3 = m$.

If $A_1A_2A_3$ has an obtuse angle it is not easy to determine the relation between m and $\rho + r$. If $A_1A_2 = A_1A_3$ and angle $A_1 \to \pi$, then $m \to 0$; hence there are triangles for which $\rho + r > m$. On the contrary, if $A_1 > \pi/2$, $A_1A_2 < A_1A_3$ with A_1A_3 fixed, and we let $A_1 \to \pi/2$ and $A_1A_2 \to 0$; then we see that there are obtuse angled triangles for which $\rho + r < m$.

II. *Solution by Alfred Brauer and I. S. Cohen, University of North Carolina.* It is easy to see that this theorem is not always true. For example, let the triangle be isosceles and let its vertex angle approach 180°. Then $\rho \to 0$, $r \to \infty$, $m \to 0$, so that $\rho + r \leqq m$ cannot hold. The proposer subsequently indicated that the triangle should have been acute.

In the following, we prove, in fact, considerably more: Let the angles A, B, C of the triangle be such that

$$A \leqq B \leqq C. \tag{1}$$

Then the above theorem is true if $B \geqq 45°$; on the other hand it is definitely false if $B < 2 \text{ arc sin } \frac{1}{2}(\sqrt{3} - 1) = 42° \ 56' +$. If $42° \ 56' + < B < 45°$, then the theorem may be true or false, and it is definitely false if also $C \geqq 135°$. For an acute triangle, the theorem then follows from the fact that then $B \geqq 45°$.

Since A is the smallest angle, the longest altitude will be the one drawn from the vertex A. It can be shown that

$$\frac{m - \rho - r}{r} = \cos (C - B) - \cos C - \cos B = f(B, C).$$

It follows from (1) that

$$B \leqq C, \qquad B + C < 180°, \qquad 2B + C \geqq 180°.$$

These inequalities define in the (C, B)-plane a certain triangular domain, and we are interested in the values of $f(B, C)$ in this domain.

For a fixed B $(0 < B < 90°)$, let $f(B, C)$ be considered as a function of C in $B \leqq C \leqq 180°$. Then

$$\frac{\partial f}{\partial C} = - \sin (C - B) + \sin C$$

vanishes if and only if

$$C = 90° + \tfrac{1}{2}B,$$

and it is easily verified that this gives a maximum of f; moreover this is the only maximum in $B \leqq C \leqq 180°$ (for fixed B).

We now distinguish the cases $B \geqq 60°$ and $B < 60°$. If $B \geqq 60°$, then for the significant values of C we have

$$B \leqq C \leqq 180° - B \leqq 90° + \tfrac{1}{2}B.$$

Therefore, for fixed B, $f(B, C)$ is increasing in the significant interval, and so $f(B, C) \geqq f(B, B) = 1 - 2 \cos B \geqq 0$. Thus the theorem is proved when $B \geqq 60°$.

If, now, $45° \leqq B < 60°$, then the significant values of C are defined by $180° - 2B \leqq C < 180° - B$. Then the maximum $C = 90° + \tfrac{1}{2}B$ lies in this interior of the interval, and we must consider the function at both endpoints. Now at the left endpoint, $f(B, 180° - 2B) = \cos 2B(1 - 2 \cos B) \geqq 0$. At the right endpoint, $f(B, 180° - B) = -\cos 2B \geqq 0$. The theorem is now proved for all triangles for which $B \geqq 45°$.

To see when the theorem will not be true, we note that it will certainly be false for those values of B for which $f(B, C)$ is negative at the maximum $C = 90° + \tfrac{1}{2}B$. At this maximum we have $f(B, 90° + \tfrac{1}{2}B) = 2 \sin^2 \tfrac{1}{2}B + 2 \sin \tfrac{1}{2}B - 1$. Since the roots of the quadratic $2x^2 + 2x - 1$ are $-\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{3}$, we have $f(B, 90° + \tfrac{1}{2}B) < 0$, if $-\tfrac{1}{2} - \tfrac{1}{2}\sqrt{3} < \sin \tfrac{1}{2}B < -\tfrac{1}{2} + \tfrac{1}{2}\sqrt{3}$, that is, if

$$B < 2 \text{ arc sin } \tfrac{1}{2}(\sqrt{3} - 1) = 42°56' + .$$

Thus the theorem is certainly false if B is less than this angle.

If $42°\ 56' + \leqq B < 45°$, the theorem may be true or false, depending on the value of C. We show that if $C \geqq 135°$, then it is false. Namely,

$$f(B, C) = \cos (C - B) - 2 \cos \tfrac{1}{2}(C + B) \cos \tfrac{1}{2}(C - B).$$

Since $C \geqq 135°$, B is $< 45°$, it follows that $C - B > 90°$, and $\cos (C - B) < 0$. Since $\tfrac{1}{2}(C + B)$ and $\tfrac{1}{2}(C - B)$ are between $0°$ and $90°$, the second term is also negative.

Solved also by H. Eves and I. Kaplansky.

Editorial Note. The solution by Kaplansky used the function $f(B, C)$ above with somewhat similar results. Eves showed that the theorem is not true for all obtuse triangles; and, for right and acute angled triangles, he gave a synthetic proof based on the equality $x_1 + x_2 + x_3 = \rho + r$ given in Johnson's *Modern Geometry*, art. 298, f. where the x_i's are absolute normal coordinates of the circumcenter of any triangle $A_1A_2A_3$. Using this equality and the relation $h_i a_i = a_1 x_1 + a_2 x_2 + a_3 x_3$, where a_i and h_i are lengths of sides and corresponding altitudes, we easily obtain a proof different from that of Eves. If $a_1 \leqq a_2 \leqq a_3$, we have at once $h_3 \leqq \rho + r \leqq h_1$. Returning to the function $f(B, C)$ there are two angles C_1, C_2 for which $f(44°, C) = 0$ which are approximately $95°\ 45.8'$, $128°\ 14.2'$, and $f(44°, C)$ is positive or negative according as C lies within or outside the interval from C_1 to C_2.

Evaluation of a Determinant

4071 [1943, 124]. *Proposed by Harry Langman, Brooklyn, N. Y.*

Setting

$$\frac{1}{1^{t+1}}+\frac{1}{2^{t+1}}+\frac{1}{3^{t+1}}+\cdots+\frac{1}{n^{t+1}}=b_t,$$

show that

$$\begin{vmatrix} b_0, & -b_1, & b_2, \cdots, & (-1)^{n-1}b_{n-1} \\ -(n-1), & b_0, & -b_1, \cdots, & (-1)^{n-2}b_{n-2} \\ 0, & -(n-2), & b_0, \cdots, & (-1)^{n-3}b_{n-3} \\ \cdot & \cdot & \cdot\ \cdots & \cdot \\ 0 & 0 & 0 & b_0 \end{vmatrix} = 1.$$

Solution by W. A. Bowers, Cornell University. Consider the n numbers $\beta_1, \beta_2, \cdots, \beta_n$; let s_k denote the sum of their kth powers; and p_k the elementary symmetric function involving k factors. Then by Newton's formula, see Dickson, *Elementary Theory of Equations*, p. 70, we have

$$s_k - p_1s_{k-1} + p_2s_{k-2} - \cdots + (-1)^{k-1}p_{k-1}s_1 + (-1)^k kp_k = 0,\ k = 1, 2, \cdots, n.$$

Solving these equations for p_n by Cramer's rule and making certain simplifications and rearrangements in the determinants, we have

$$\begin{vmatrix} s_1 & -s_2 & s_3 \cdots & (-1)^{n+1}s_n \\ -(n-1) & s_1 & -s_2 \cdots & (-1)^n s_{n-1} \\ 0 & -(n-2) & s_1 \cdots & (-1)^{n-1}s_{n-2} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot\ \cdots & -s_2 \\ \cdot & \cdot & \cdot\ \cdots & s_1 \end{vmatrix} = n!p_n.$$

We obtain the desired result by setting $\beta_k = 1/k$ and $s_k = b_{k-1}$. The result is so easily obtained that I should think it could be found in the literature.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to B. W. Jones, White Hall, Cornell University, Ithaca, New York.

A volume on mathematics in honor of Professor J. Rey Pastor is being published by Rosario University, Argentina. Invited contributors from the United States are G. D. Birkhoff, Jaime Lifshitz, M. S. Vallarta, George Polya, Arthur Rosenthal, Edward Kasner, Arthur Korn; Jacques Hadamard, John DeCicco, J. V. Uspensky.

A portrait of Professor Emeritus R. C. Archibald, given by a number of his friends, was presented to Brown University.

Professor V. W. Adkisson of the University of Arkansas has been appointed chairman of the department of mathematics.

Assistant Professor A. T. Brauer of the University of North Carolina has been promoted to an associate professorship.

Dr. W. S. Erickson of St. Olaf College, Northfield, Minnesota, has been appointed to a professorship at Minot State Teachers College, Minot, North Dakota.

Assistant Professor R. N. Haskell of the University of Texas has been promoted to an associate professorship.

Assistant Professor G. E. Hay of the University of Michigan has been granted leave of absence to work on an NDRC project at Brown University.

Dean H. M. Hosford of the College of Arts and Sciences of the University of Arkansas has been appointed vice-president.

Professor R. L. Jeffery of Acadia University has been appointed to a professorship at Queen's University, Kingston, Ontario.

Professor H. W. March and Assistant Professors Churchill Eisenhart and Stanislaw Ulam of the University of Wisconsin are on leave for war research.

Dr. J. C. C. McKinsey of New York University has been appointed to an assistant professorship at Montana State College.

Professor F. D. Murnaghan of The Johns Hopkins University has been appointed to a visiting professorship at Brown University for the current year.

Professor C. V. Newsom of the University of New Mexico has been appointed head of the department of mathematics at Oberlin College beginning July 1, 1944.

Assistant Professor J. F. Randolph of Cornell University has been appointed to a professorship at Oberlin College.

Dr. Henry Scheffé of Princeton University has been appointed to an assistant professorship at Syracuse University.

Colonel R. H. Somers (U. S. Army retired) has been appointed visiting lecturer in mathematics at Dartmouth College.

Dr. G. L. Walker of the University of Delaware has been promoted to an assistant professorship and has been granted leave of absence to teach at Cornell University.

Associate Professor G. A. Williams of Oregon State College has been promoted to a professorship.

Dr. Y. K. Wong of the University of Chicago has been appointed lecturer at the University of North Carolina.

The following appointment to an instructorship is announced:
Dartmouth College: D. B. Kirk.

Professor Emeritus T. S. Fiske of Columbia University died January 10, 1944.

Professor John Matheson of Queen's University, Kingston, Ontario, died January 24, 1944. He was a charter member of the Mathematical Association.

Professor T. R. Rosebrugh of the University of Toronto died January 24, 1943. He was a charter member of the Mathematical Association.

WAR INFORMATION

EDITED BY C. V. NEWSOM

Send news reports upon the utilization of mathematicians or mathematics in war activities to C. V. Newsom, University of New Mexico, Albuquerque, New Mexico.

DEFERMENT OF INSTRUCTORS IN THE NAVY COLLEGE PROGRAM

The following memorandum was issued February 8, 1944, by the Office of L. E. Denfeld, Acting Chief of Naval Personnel. Schedule "A" referred to in the bulletin has been sent to the presidents of all universities and colleges participating in the Navy College Program.

"College officials charged with the administration of the Navy College Program should take all steps necessary to prevent the loss of needed members of their instructional staffs through induction into the Armed Forces.

(a) They should request all instructors to file their qualifications with the National Roster of Scientific and Specialized Personnel, Washington, D. C., since the Roster is utilized for informational purposes by the Selective Service system.

(b) They should file Selective Service Form 42A with the Local Board, in all appropriate cases, and request the deferment of all instructors considered essential.

(c) Where replacement of personnel is contemplated but cannot be effected immediately, they should prepare a Replacement Schedule in accordance with Local Board Memorandum No. 158-A. Intra-term replacements should not be scheduled. The Navy Liaison Officer at State Headquarters is authorized to assist in preparation.

(d) Where the request for the deferment is refused by the Local Board, they should appeal within the time allowed—ten days from the date the Local Board

mails the notice of reclassification to the registrant. Transfer of this appeal to the Board where the instructor is employed should be requested.

(e) Where the appeal has been dismissed by the State Director, they should communicate with the Bureau of Naval Personnel, Navy Department, and request the Bureau to make representations at the Washington level. In order that the Bureau may have the necessary facts, the attached schedule 'A' should be completed and forwarded with such requests via the Naval Officer in Charge for his endorsement and transmission to the Bureau of Naval Personnel.

"The following policies govern the disposition of requests by institutions to obtain the services of former instructors now members of the Navy:

(a) Instructors, now members of the Navy, on active duty, will not be placed on an inactive status so that their services may be utilized by educational institutions.

(b) Instructors, now members of the Navy, will not be detailed to an active duty status to perform teaching or administrative assignments under their former employer.

(c) In the case of the irreplaceable teacher whose induction has taken place, causing serious jeopardy to the Navy V-12 Program, institutions may submit a full statement of the facts to the Bureau of Naval Personnel, via the Officer-in-Charge of the Navy Unit at the institution, for consideration of methods of remedial action. The statement of facts should include the following information in addition to that covered by the attached schedule 'A':

Date of Induction________________________

Service to which Assigned____________________

Service Number_________________________

The statement of fact should include a certification by the President of the College that the teacher has been employed and would in the future, if available, be employed full time in connection with the training of naval personnel (or members of the Armed Forces, giving detailed explanation) on active duty, and that no other member of the faculty or college staff is available and qualified to replace the teacher.

"The principles outlined herein for Navy V-12 Units apply equally to Naval Flight Preparatory Schools."

DESCRIPTION OF THE PROFESSION OF MATHEMATICS

The following "description of the profession of mathematics" has been distributed by the National Roster of Scientific and Specialized Personnel to offices of the United States Employment Service.

Occupational Summary

The greatest percentage of the personnel in this profession is engaged in teaching mathematics at the college and university level. A relatively large staff is required since the training is basic to most sciences, pure and applied. The mathematical subjects taught include algebra, calculus, trigonometry, geometry,

differential equations, and applied mathematics. There is a present trend toward the wider utilization of mathematicians in industry, although not many mathematicians are now employed as such. Teachers of mathematics at high school level are not always sufficiently trained to be considered mathematicians, and the scope of their teaching is generally limited to elementary algebra and geometry.

Major Branches

Since most mathematicians are engaged in teaching the subjects enumerated above, only the small non-teaching group specialize in the subdivisions listed below.

1. *Analysis.* Industrial mathematicians are usually employed in a consultative capacity to a research engineering staff. As such, they are called upon to advise concerning mathematical analytical methods for solving engineering problems. These methods include the use of calculus, differential equations, and vector and tensor analysis.
2. *Applied Mathematics* includes its applications in the field of ballistics, cryptanalysis, harmonic analysis and in probability and mathematical statistics. The latter specialty involves the gathering of data by sampling methods, evaluating such information, and preparing it for the guidance and use of technicians in related fields.
3. *Other Specialized Areas* include the theory of numbers, special functions, and the advanced areas of geometry—differential, projective and topologic.

Functional Specializations

As indicated in the occupational summary, mathematicians specialize in any one or more of the following areas.

1. Research
2. Teaching (often combined with research)
3. Industrial applications

Educational Qualifications

To be considered a mathematician, one should have at least a Bachelor of Arts or a Bachelor of Science degree, with major in mathematics. Normally an additional year of training would be required for professional standing but the great need during the national crisis necessitates the use, at a professional level, of all persons holding a bachelor's degree with major in mathematics.

Professional Affiliations

Mathematicians are organized into three learned societies: The American Mathematical Society, the Mathematical Association of America and the Institute of Mathematical Statistics. Membership is a strong indication of professional standing.

Alternate Designation

Since mathematicians are very likely to have specialized in some field of application, it is often true that the designation of the individual may be in terms of that field. Thus, in the field of mathematical physics may be found physicists or mathematicians. Similarly advanced electrical engineering is dependent upon applied mathematics. Applied mathematicians may be employed for advanced work and designated in terms of the fields listed below:

Electrical engineering	Metallurgy
Mechanical engineering	Communications
Chemistry	Astronomy
Biology	Economics
Physics	Ballistics

Industry

Mathematicians are employed in educational institutions, by government both Federal and state, in research institutions, and to some extent in industry.

FROM THE NATIONAL ROSTER ON NEW SELECTIVE SERVICE REGULATIONS

In the MONTHLY, February, 1944, appeared a report pertaining to new regulations on student deferment issued by Selective Service Headquarters. Several communications have been sent by the National Roster to presidents of all institutions of higher education in the country giving additional information upon the new rulings. Some important excerpts from these letters follow.

Letter of January 20. "Graduate students are not covered by Activity and Occupation Bulletin 33-6." (The status of graduate students as teachers is now governed by LBM 115, amended January 6, 1944. This Memorandum states, "A person may be considered as engaged in full-time teaching if he devotes not less than 15 hours per week in contact with students in actual classroom or laboratory instruction.")

Letter of January 27. "Because of the pressing national need for young men in the armed services, the Selective Service System has decided that it can allow the deferment of a total of only 10,000 undergraduate students in the fields of engineering, physics, chemistry, geology and geophysics. The national quota of 10,000 has been distributed by fields as follows: engineering, 6,775; physics, 850; chemistry, 2,250; geology and geophysics, 125. The total number of students presently deferred in these fields is considerably in excess of the 10,000 allowed. The National Roster has no authority in connection with the establishment of overall national quota. Its function is purely administrative. Therefore, it now becomes the obligation of the individual college to determine, within the quota assigned to it, the students for whom deferment may be requested. This should be done in accordance with provisions of Activity and Occupation Bulletin No. 33-6, as amended January 6, 1944.

"The initial quota assigned to your institution has been computed by multiplying the present number of students in Class II and 'Class II pending' (which you recently reported to us) by the ratio of the national quota to the total number of such students reported by all institutions."

THE CURTAILMENT OF THE ARMY COLLEGE PROGRAM

The press notice issued February 18, 1944, by the War Department in regard to the curtailment of the Army Specialized Training Program is reproduced below.

"The shortage of personnel from which the Army is now suffering has led the War Department to drastic decisions during the past week. Because of the inability of the Selective Service to deliver personnel according to schedule, the Army is now short 200,000 men who should have been in uniform before the end of 1943. The increased tempo of offensive operations together with the mounting casualties demanding immediate replacements in the field have created a situation which has necessitated drastic economies in the employment of personnel throughout the United States, and a decision to reduce the soldiers in colleges taking the Army Specialized Training from 145,000 to 35,000. This last measure has been rendered necessary by the imperative requirement at this time for these men who have already had their basic training and a certain amount of specialized training for which their services are now urgently needed.

"After exhausting all other sources, it was determined that the type of trained military personnel needed could be obtained only by decreasing the number of combat units or by drawing from the reservoir of men in ASTP training. It was decided that military necessity required that existing combat units be maintained.

"The 35,000 remaining in the program will be primarily those trainees taking advanced courses in medicine and dentistry, or engineering and include 5,000 pre-induction students. The students withdrawn will be those already basically trained and on active duty. Seventeen-year-olds in the Army Specialized Training Program Reserve will not be affected, nor will this Reserve phase of the program be curtailed.

"The student soldiers now in the Army Specialized Training Program were selected for their high intelligence, adaptability, and potential leadership. They are the type who can be expected to assume the responsibilities of non-commissioned officers and of skilled technicians. Experience to date in this war has demonstrated to the Army that the combat arms, particularly the infantry, need a substantial proportion of men with these qualities to insure continued success in operations. All experience also has shown conclusively that losses are considerably lower in units which have intelligent and aggressive leadership among non-commissioned officers.

"Reassignment from ASTP to other duty before April 1 will be made, so far as military necessity permits, at the completion of a particular training

course or a term in that course. Colleges will be reimbursed for the unexpired portion of contracts covering students withdrawn from ASTP.

"The War Department believes, on the basis of experience, the infusion of thousands of highly intelligent student soldiers into the ground forces, which will see more action as the tempo of our offensive increases, will help to increase our striking power. Consequently, around 80,000 of the men to be transferred from ASTP will be assigned to the Army Ground Forces where the skills and capacity for leadership are now most needed. Most of the remainder will be assigned to other units destined for overseas service. The policy will be to make certain that the skills and the qualities of leadership which these thousands of student soldiers possess are used on assignments where they can function most effectively."

THE ARMY AND NAVY COLLEGE RESERVE PROGRAMS

On March 3, 1944, the Army and the Navy issued a joint press release explaining the expanded Reserve Programs to be instituted in many colleges and universities which have previously participated in the Army and the Navy College Training Programs. To be in the first group of trainees selected for the new Reserve Programs, a student must have made a satisfactory grade upon the "Army-Navy College Qualifying Test" given March 15.

The test of March 15 was the third opportunity given male civilians, who meet the age and academic requirements, to enter the College Training Programs. Those who took the test were required to indicate on the day of the test their preference for the Army Program or that of the Navy. Young men graduated from high school or in their final term and who will have reached their 17th but not their 22nd birthday by July 1, 1944, were permitted to express a preference for the Army; those who will have reached their 17th but not their 19th birthday by July 1 could express a preference for the Navy. Taking the test did not constitute enlistment in any branch of the armed services; that is, having taken the test, a student is not obligated to enter the Program if he is accepted.

Seventeen-year-olds who qualified on the test, and who expressed Navy preference and are accepted, will receive the same training as other students in the Navy College Program, which is designed to provide officers for the Navy. These seventeen-year-olds will be enlisted in Class V-12, U. S. Naval Reserve, and will not be inducted as in the case of 18- and 19-year-olds. Like other students in the Navy College Program, they will be placed in uniform, with pay, under military discipline.

Successful contestants in the March 15 test, who will be less than 17 years and nine months old on July 1, 1944, and who have stated an Army preference, will form the group who enter the AST Reserve Program. This age restriction will, except in the event of an unforeseen military emergency, assure a minimum of six months of intensive academic work at the college level before the individual may be called to active Army duty for his basic military training. Those who

are less than 17 years and six months old on July 1, 1944, will receive, subject to the same conditions, at least nine months of academic work under the AST Reserve Program.

It is hoped that a sufficient number of seventeen-year-olds will be found qualified so that a substantial expansion of the AST Reserve Program can be made. This would tend to replace some of the loss in men trained at the college level occasioned by the recent return to active troop duty of many ASTP trainees.

The majority of eligible students who qualify for the military scholarships offered under the AST Reserve Program will be assigned to one of the following curricula: (1) applied sciences; (2) chemical and biological sciences preparatory to advanced medical and dental studies; (3) mathematics and physics. Assignment will be based on the qualifications and aptitudes of the individual. A limited number of specially qualified men may be assigned to a new foreign language course.

A limited number of men between 17 years, nine months old and 22 years old on July 1, 1944, who took the March 15 Army-Navy College Qualifying Test, will be selected for the regular Army Specialized Training Program after their induction into the Army and following their basic military training. In general, those selected will be men who have had prior academic training which qualifies them for advanced engineering and language courses of the Army Specialized Training Program. These men will be selected at the reception centers and will be further screened by a Specialized Training and Reassignment (STAR) board upon completion of their military training.

Students accepted for the regular Navy College Program will attend colleges and universities under contract to the Navy for varying numbers of 16-week terms depending on the type of course they pursue. Prospective deck officers for the Navy will receive four terms of college training in sixteen months after which they will pursue a four months' course at a Naval Reserve Midshipman's School before being commissioned as ensigns. Others will take courses of greater length leading to commissions as physicians, dentists, chaplains, engineers, supply officers and other specialists.

THE NEED FOR WOMEN IN THE ARMED SERVICES

The armed forces are making a new call for recruits among American college women who have had specialized training. Women with training in mathematics are needed in every branch of service.

The Army reports that on January 1, 1944, there were 62,859 officers and enlisted women in the WAC. Its authorized strength is 200,000. A college woman with technical training stands a good chance of being selected as an officer candidate. Moreover, it is now possible to apply for a specific type of work or for a specific station assignment.

At the present time there are approximately 50,000 WAVES, and recruiting drives are expected to double this number by the end of 1944. The Navy is now selecting college graduates more on the basis of their ability to do certain jobs

than on their personal qualifications. At the present time a search is being conducted especially for women trained in aerology, mathematics, physics, navigation, supply, or medicine.

The SPARS are seeking college women for officer candidates who have had three or more years of work experience. Women capable of supervising enlisted personnel are urgently needed. Opportunities are especially good for pay and supply officers, and for those having the background to do work in communications and ordnance.

The MARINES need women in the 25 to 35 age group for officer candidates. Those who can qualify as specialists are urged to apply.

In addition to the important service which a woman can render her country by enrolling in a branch of the armed forces, she will become eligible for the same postwar benefits as those available to men who are veterans of this war. In particular, valuable educational opportunities will probably be made available.

CALENDAR OF FUTURE MEETINGS

Twenty-Seventh Summer Meeting, Wellesley, Mass., August 12–14, 1944.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN, Pittsburgh, Pa., April, 1944
ILLINOIS
INDIANA, Indianapolis, November 10, 1944
IOWA, Cedar Rapids, April 15, 1944
KANSAS, Topeka, April 15, 1944
KENTUCKY, Lexington, April 29, 1944
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK, New York, April 22, 1944
MICHIGAN
MINNESOTA, St. Paul, May 6, 1944
MISSOURI
NEBRASKA, Lincoln, May 6, 1944
NORTHERN CALIFORNIA, San Francisco, January 27, 1945
OHIO, Columbus, April 6, 1944
OKLAHOMA
PHILADELPHIA, Philadelphia, November, 1944
ROCKY MOUNTAIN, Greeley, Colo., April 14–15, 1944
SOUTHEASTERN
SOUTHERN CALIFORNIA, Los Angeles, March 10, 1945
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE
WISCONSIN, Milwaukee, May, 1944

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DEFORMATIONS OF PLANE RECTILINEAR COMPLEXES*

S. S. CAIRNS, Queens College

1. The deformation theorem. On a euclidean plane, consider two fixed triangles, $A_1A_2A_3$ and $B_1B_2B_3$. Let P_1, P_2, and P_3 be three variable points of the plane. Suppose that $P_i(i=1, 2, 3)$ moves in continuous fashion from A_i to B_i. It is then easily seen that the variable triangle $P_1P_2P_3$ can be required to remain nondegenerate throughout the motion unless one of the cyclic orderings $A_1A_2A_3$ and $B_1B_2B_3$ is counterclockwise and the other is clockwise.

We will consider a more general problem of the same sort. It will be understood throughout that all the geometric configurations are on a euclidean plane. Let K_0 be a collection of triangles, any two of which are distinct or else have in common either a single vertex or two vertices and the edge joining them. Let $(A_1, \cdots, A_n)$ be the set of all points each of which is a vertex of at least one of the triangles K_0. Such a collection of triangles as K_0 is called a *(plane rectilinear simplicial 2-dimensional) complex*. Since all the complexes with which we deal will be plane, rectilinear, simplicial, and 2-dimensional, these modifiers will be omitted. A similar convention will apply to all words enclosed in parentheses in our definitions. The vertices, edges, and triangles of K_0 are referred to as the *cells* of the complex, of dimensions 0, 1, and 2 respectively.

Let K_1 be a second complex with vertices $(B_1, \cdots, B_n)$ where $B_iB_jB_k$ is a cell of K_1 when and only when $A_iA_jA_k$ is a cell of K_0. We refer to K_0 and K_1 as being *isomorphic (with vertices similarly numbered)*.

The Deformation Problem. Is it possible for a set $(P_1, \cdots, P_n)$ *of points to move continuously so that* (1) P_i *goes from* A_i *to* B_i $(i=1, \cdots, n)$ *and* (2) *throughout the motion,* $(P_1, \cdots, P_n)$ *are the vertices of a complex,* K, *isomorphic to* K_0? *Whenever the answer to this question is affirmative, we will say that* K_0 *is deformable into* K_1.

One can visualize the problem as follows. Think of $(A_1, \cdots, A_n)$ as pegs movable in the plane. Suppose that a piece of rubber band joins A_i to A_j when and only when A_iA_j is a cell of K_0; and suppose further that the rubber remains taut no matter how close together A_i and A_j are brought. Is it then possible to move the pegs $(A_1, \cdots, A_n)$ into the respective positions $(B_1, \cdots, B_n)$ without any rubber bands running foul of one another or of the pegs?

It is not difficult to concoct examples of various sorts in which the answer to the question just raised is negative. The object of the present paper is to prove a deformation theorem for an interesting class of complexes sufficiently restricted to avoid the complications of the most general case.†

* This paper contains the substance of an address entitled "Introduction of a Riemannian Geometry on a Manifold," delivered by the writer at the Summer Meeting of the Association, September 11, 1943, in New Brunswick, New Jersey.

† For a general discussion, directed toward an application to the problem of introducing an analytic Riemannian geometry on a manifold, see S. S. Cairns, Isotopic deformations of geodesic complexes on the 2-sphere and on the plane, to appear in the Annals of Mathematics.

2. A special deformation theorem. Euler's relationship.

THEOREM. *Let K_0 and K_1 be isomorphic complexes with vertices $(A_1, \cdots, A_n)$ and $(B_1, \cdots, B_n)$ respectively. Suppose that* (1) $A_i \equiv B_i$ $(i=1, 2, 3)$ (2) K_0, *and therefore* K_1, *exactly covers the triangle* $A_1A_2A_3$, *and* (3) *all vertices save* A_1, A_2, *and* A_3 *are interior to the triangle* $A_1A_2A_3$. *Then* K_0 *is deformable into* K_1. *It can be required that* A_1, A_2, *and* A_3 *remain fixed during the deformation.*

The most general rectilinear deformation theorem in the plane can be shown to follow from this special theorem, applied to corresponding parts of such complexes as K_0 and K_1.

Our discussion will involve the well-known Euler relationship

$$\alpha_0 - \alpha_1 + \alpha_2 = 1 \tag{2.1}$$

where α_0, α_1, and α_2 are respectively, the numbers of vertices, edges, and triangles belonging to K_0.

By the *star*, $S(A_i)$, of the vertex A_i, we will mean A_i, together with all the 1-dimensional and 2-dimensional cells of K_0 having A_i as a vertex. Let ν_i $(i=1, \cdots, n)$ be the number of 1-dimensional cells belonging to $S(A_i)$. Then

$$\begin{aligned} \sum_{i=1}^{n} \nu_i &= 2\alpha_1 \\ \sum_{i=1}^{n} \nu_i &= 3\alpha_2 + 3. \end{aligned} \tag{2.2}$$

For, (1) in the first summation, each edge can be thought of as counted twice (once for each end-point) and (2) in the second summation, each of the α_2 triangles of K_0, and the exterior of $A_1A_2A_3$, can be thought of as counted three times. Multiplying (2.1) by six, eliminating α_1 and α_2 with the aid of (2.2), and replacing α_0 by n, we deduce

$$6n - 3\sum_{i=1}^{n} \nu_i + 2\sum_{i=1}^{n} \nu_i - 6 = 6 \tag{2.3}$$

or the familiar result

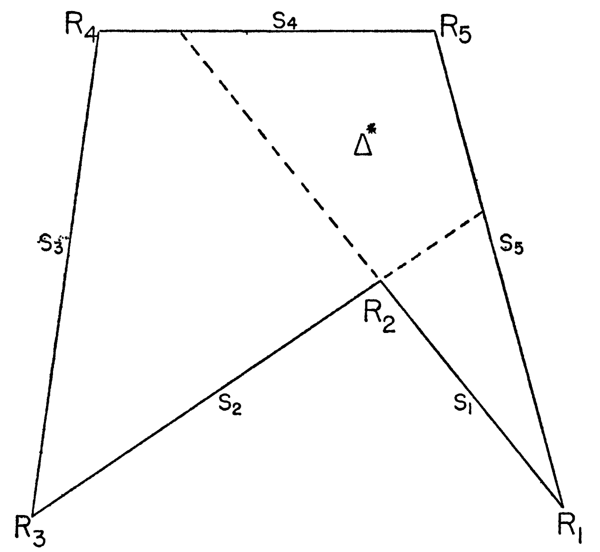
$$\sum_{i=1}^{n} (6 - \nu_i) = 12. \tag{2.4}$$

Except in the trivial case $n=3$, which implies that there are no interior vertices, each of the numbers ν_i exceeds two. Hence equation (2.4) implies that at least four of these numbers are less than six. Therefore

$$\nu_j < 6 \quad \text{for some } j \text{ of the set} \quad (4, \cdots, n). \tag{2.5}$$

Our proof of the above theorem involves an argument recurrent in n, the number of vertices. As a preliminary, we discuss a certain property of polygonal regions having fewer than six vertices.

3. Polygonal regions with $\nu<6$ vertices. Let $(R_1, \cdots, R_\nu)$ be the vertices of a polygonal region, Δ, where $\nu=3$, 4, or 5. Let $s_i=R_iR_{i+1}$ $(i=1, \cdots, \nu-1)$, and $s_\nu=R_\nu R_1$. If we regard Δ as the floor plan of a room with ν corners, then the result we need can be partly expressed by saying that there is always a non-vacuous locus, Δ^*, of vantage points from which one can observe all parts of the



room. Simple examples reveal that this statement is false for rooms with more than five corners. The locus Δ^*, in case $\nu<6$, will be referred to as the *core* of Δ.

LEMMA 3.1. *The core, Δ^*, of Δ is a non-vacuous, convex, polygonal region, and has a vertex in common with Δ.*

Proof. Let H_i $(i=1, \cdots, \nu)$ be that open half-plane with s_i on its boundary whose intersection with Δ also has s_i on its boundary. It is easy to show that Δ^* is the intersection

$$\Delta^* = H_1 \cdot H_2 \cdots H_\nu. \tag{3.1}$$

The reader can then complete the proof, breaking it into various cases classified according to (1) the value of ν and (2) the number and arrangement of re-entrant vertices of Δ.

(*A*) *We note that Δ^* is the locus of a point, P, such that the triangles PR_iR_{i+1} $(i=1, \cdots, \nu-1)$ and $PR_\nu R_1$ are all distinct and non-degenerate.*

In our application of this work, we will regard $(R_1, \cdots, R_\nu)$ as variable, subject to the requirement that they always determine a non-singular polygon, $R_1 \cdots R_\nu R_1$, bounding a region Δ. Both Δ and Δ^* vary continuously with the R's.

4. Proof of the theorem in the simplest case. Corresponding parts of isomorphic complexes will sometimes be denoted by the same symbol. Thus, for example, we will use Δ_j to denote the region covered by the star $S(A_j)$ in K_0, the region covered by $S(P_j)$ in K, or that covered by $S(B_j)$ in K_1. The context will make the meaning clear. The number j will be a fixed index such that condition (2.5) is fulfilled.

By a slight displacement of vertices, if necessary, let it be arranged that no three vertices of Δ_j be collinear. Let A_k be a vertex common to Δ_j and its core, Δ_j^* (Lemma 3.1). We will let the variable complex, K, have the initial position in which $P_i \equiv A_i$ $(i=1, \cdots, n)$. We first permit only P_j to vary, all other vertices of K being fixed. Then Δ_j^* is the locus of possible positions of P_j such that K remains isomorphic to K_0.

(*A*) *Let P_j move along the line-segment from A_j to A_k, all vertices other than P_j remaining coincident with the corresponding A's. Until P_j coincides with P_k (at position A_k), K remains isomorphic to K_0. When coincidence is reached, there results a complex, K_0', with $(n-1)$ vertices. We will say that K_0 is reduced to K_0' by the coincidence of P_j and P_k. The structure of K_0' is completely determined by that of K and by the choice of the vertices P_j and P_k.*

Hypothesis. For some value $n>3$, the deformation theorem of §2 *holds with $(n-1)$ in place of n.*

Verification of the hypothesis for $n=4$. If $n=4$, then the only complex with $(n-1)$ vertices is the fixed 2-cell $A_1A_2A_3$, plus boundary, and the deformation problem has a trivial solution.

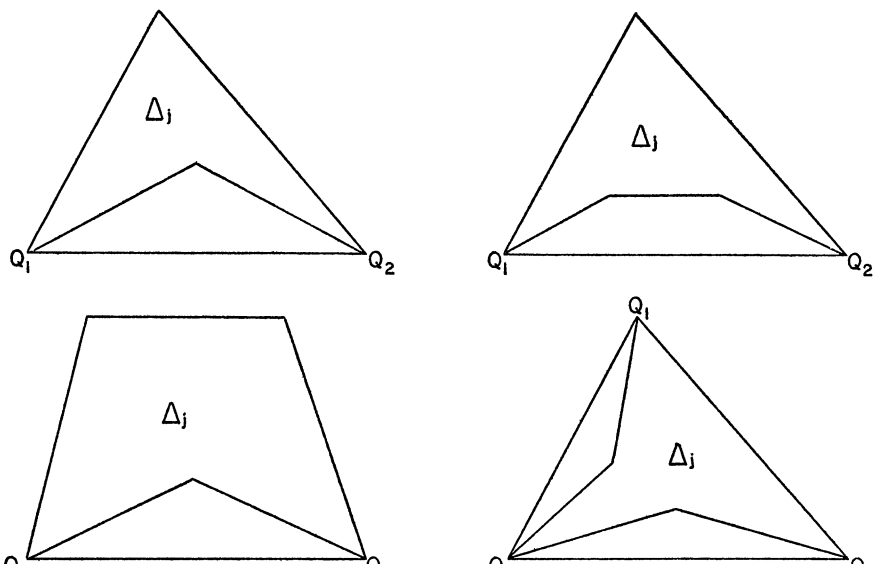
We will take the general step of the recurrency by deducing the existence of a deformation of K_0 into K_1 from the assumed existence of a deformation carrying K_0' [see (*A*) above] into an arbitrary isomorphic position subject to the restrictions of §2. The argument would be fairly straightforward if K_1 could necessarily be reduced to an isomorph of K_0' by moving P_j along the segment B_jB_k. However, K_1 may be such that B_k is not on the boundary of Δ_j^*. This difficulty will be obviated by the use of a complex, K_2, isomorphic to K_0 and K_1, in which Δ_j and Δ_j^* have all possible vertices in common.

CASE I. *No two non-adjacent vertices on the boundary, β_j, of Δ_j are joined by an edge of K_0.*

LEMMA 4.1. *In the present case, there exists a complex, K_2, isomorphic to K_0, with vertices $(C_1, \cdots, C_n)$, in which Δ_j is convex, with no three of its vertices collinear.*

Proof. We establish this lemma by commencing with the three vertices $C_i \equiv A_i$ $(i=1, 2, 3)$, and filling in the inner vertices in the following way. If all the vertices (A_1, A_2, A_3) are on β_j, which implies that $n=4$, then the conditions of the lemma are automatically fulfilled. If not, suppose $A_i \equiv C_i$ is not on β_j. Then the star $S(C_i)$, with vertices suitably denoted, can be introduced inside the triangle $A_1A_2A_3$, so that the part of the triangle not covered by $S(C_i)$ is a convex region, D, with no three vertices collinear. If all vertices of D correspond to

vertices of β_j, then C_j can be inserted anywhere in D, and the conditions of the lemma will be fulfilled. If not, then we fill in the missing cells of the star of some boundary vertex of D not on β_j; and we do it in such fashion that the portion of the triangle $A_1A_2A_3$ not covered by the sum of these cells and those previously introduced shall consist of a number of convex regions, with no three vertices collinear. We then deal with one of these regions exactly as we have just dealt with D. The recurrent process thus indicated leads ultimately to a situation where the entire triangle $A_1A_2A_3$ is covered, excepting a single convex region, Δ_j, whose vertices correspond to the vertices of Δ_j. We then introduce C_j into this region, and the conditions of the lemma are fulfilled.



(B) *Under the conditions of the foregoing lemma, Δ_j coincides with its core Δ_j^*; and K_2 can be reduced to a complex with $(n-1)$ vertices by a motion of C_j into any vertex on the boundary of Δ_j.*

Now suppose K_0 reducible to K_0' after the fashion described in (A) above. Then, by (B), K_2 is reducible to an isomorph, K_2', of K_0'. By the hypothesis of the recurrency, K_0' is deformable into K_2'. We now deduce that K_0 is deformable into K_2. Throughout any deformation of K_0' into K_2', the region Δ_j originally covered by $S(A_j)$ remains a polygon of $\nu_j<6$ sides. Instead of initially reducing K_0 to K_0', let P_j be moved to the centroid of the core, Δ_j^*, of Δ_j. Then let the vertices other than P_j move precisely as in some deformation of K_0' into K_2', while P_j remains at the centroid of the varying core Δ_j^*. Finally, let P_j move into C_j along a line-segment, while all other vertices remain fixed. This completes a deformation of K_0 into K_2. In similar fashion, K_1 can be deformed into K_2. It follows at once that K_0 is deformable into K_1, and the deformation theorem is proved for the present case.

5. Completion of the proof. CASE II. *Some non-adjacent vertices of the boundary, β_j, of Δ_j are joined by edges of K_0.* The various possibilities are shown in the preceding figures. Clearly, K_2 cannot be constructed, as in Case I, so that Δ_j will be convex. To say that Δ_j is convex is equivalent to saying that every vertex of Δ_j is also a vertex of Δ_j^*. In each of the present possibilities, those non-adjacent vertices [Q_i in the figures] of β_j which are joined by edges of K_0 cannot possibly be on the boundary of Δ_j^*. However, the following result holds and can be proved by a method similar to that used for Lemma 4.1.

LEMMA 5.1. *A complex, K_2, exists, isomorphic to K_0, in which Δ_j and Δ_j^* have in common all vertices save the non-adjacent vertices of β_j which are joined by edges of K_2.*

After Lemma 5.1 is proved, the discussion of Case II proceeds along the same lines as in Case I.

ASSOCIATED FREQUENCY DISTRIBUTIONS IN BIOMETRY*

R. H. COLE, University of Western Ontario

1. Introduction. The biometrician, studying minute structures, is frequently forced to limit his measurements to those which can be made on random plane sections of these structures. For example, sections of cells, alveoli, or glomeruli may be examined under the microscope and measured with respect to their diameters and areas. Such measurements yield some insight into the size of the original structures and are valuable for comparative purposes, but they fail to yield directly the statistical constants associated with a frequency distribution of the sizes of the structures themselves. There are instances† in the literature where an estimation of the mean volume of a set of structures has been made from the mean area of their random sections, but the present writer knows of no case in which an attempt has been made to determine additional statistical constants.

In view of its extensive applicability, the nature of the relationship existing between a distribution of the sizes of three dimensional structures and the distribution of the sizes of their plane sections merits more attention than it has received. The present discussion treats the problem as it applies to spheres and, further, suggests a method for adapting the results to nonspherical structures.

The mathematical framework required for the resolution of this problem is applicable to other similar problems. For example, the relationship between a distribution of velocity magnitudes and the distribution of their components in

* This problem was suggested by Dr. W. S. Hartroft, University of Western Ontario. A preliminary report was presented to the Royal Society of Canada, May 26, 1943, in conjunction with the report of Dr. W. S. Hartroft and Dr. C. C. Macklin on Pulmonic Alveolar Size.

† *Cf.*, for example, Roy Cohn, The postnatal growth of the lung, The Journal of Thoracic Surgery, vol. 9, 1939–40, pp. 274–277.

a given direction may be deduced by the same method that is developed here. That is to say, both problems lead to a consideration of the relationship existing between associated frequency distributions. It is desirable, therefore, to develop the relationship with respect to abstract variates. This will be done in sections 2 and 3, specific reference to the biometric problem being resumed in section 4. In that section we shall interpret the variate x, of sections 2 and 3, as the radius of a sphere and the variate y as the radius of a plane section of a sphere.

2. The general relationship. Let x be a variate which has a probability function, $\Phi(x)$, on the range (a, b). Suppose, further, that with each member of the universe of x there is associated a variate y which has a probability function, $\eta(x, y)$, on the range $(\alpha(x), \beta(x))$. If α is the greatest lower bound of $\alpha(x)$ and β the least upper bound of $\beta(x)$, we shall extend the range of definition of $\eta(x, y)$ to the rectangular region, $a \leqq x \leqq b$, $\alpha \leqq y \leqq \beta$, by stipulating that it be zero at every point not contained in the original region of definition.

Under simple sampling, then, the probability of a specified y associated with a specified x is given by $\eta(x, y)\Phi(x)$. The discussion may be made more general if we foresee the possibility of employing a method of sampling which is not simple but which weights the probability of x by a factor $s(x)$. When such sampling is performed, the probability function of the sample is $s(x)\Phi(x)$. Hence, the probability of a specified y associated with a specified x of the sample is $\eta(x, y)s(x)\Phi(x)$. If the probability of a specified y associated with any x of the sample is denoted by $\phi(y)$, we have

$$\phi(y) = \int_a^b \eta(x, y)s(x)\Phi(x)dx, \qquad \alpha \leqq y \leqq \beta. \tag{2.1}$$

If the functions $\phi(y)$, $\eta(x, y)$, and $s(x)$ are known, relation (2.1) is a Fredholm integral equation of the first kind for which a solution $\Phi(x)$ is required. Instead of employing the theory of integral equations, however, we shall approach the problem from the point of view of the practical statistician. That is, we shall suppose that a frequency distribution of y is known and shall attempt to determine therefrom the associated frequency distribution of x. Consequently, let us subdivide the interval (a, b) into n subintervals by the points $x_0 = a, x_1, x_2, \cdots, x_n = b$, and the interval (α, β) into m subintervals by the points $y_0 = \alpha, y_1, y_2, \cdots, y_m = \beta$.

The expected relative frequency of y on the subinterval $y_{i-1} < y \leqq y_i$ may be denoted by f_i and is given by

$$f_i = \int_{y_{i-1}}^{y_i} \phi(y)dy.$$

Whence, in view of (2.1),

$$f_i = \int_a^b \int_{y_{i-1}}^{y_i} \eta(x, y)s(x)\Phi(x)dydx, \qquad i = 1, 2, \cdots, m. \tag{2.2}$$

It is necessary, at this point, to make some assumption concerning the function $\Phi(x)$. Assuming, therefore, that $\Phi(x)$ is constant on (x_{j-1}, x_j), and denoting the relative frequency of x on that subinterval by F_j, we may write

$$\Phi(x) = \frac{F_j}{x_j - x_{j-1}}, \quad x_{j-1} < x \leqq x_j, \qquad j = 1, 2, \cdots, n.$$

Accordingly (2.2) may be written

$$\sum_{j=1}^{n} a_{i,j}F_j = f_i, \qquad i = 1, 2, \cdots, m, \tag{2.3}$$

where

$$a_{i,j} = \frac{1}{x_j - x_{j-1}} \int_{x_{j-1}}^{x_j} \int_{y_{i-1}}^{y_i} \eta(x, y)s(x)dydx.$$

The multiplication of the members of system (2.3) by the positive integer N will convert them from relations between expected *relative* frequencies to relations between expected frequencies from a sample of N. We may therefore interpret F_j and f_i as representing either frequencies or relative frequencies.

3. The inverse relationship. The relations (2.3) form a system of m equations in the n unknowns $F_1, F_2, \cdots, F_n$. We shall be interested in the case $m=n$, for which the existence of a unique solution depends on the value of the determinant $|a_{i,j}|$. It is frequently possible in specific cases to choose the subdivision points of the intervals (a, b) and (α, β) so that $|a_{i,j}|$ is triangular in form with the components of the main diagonal not zero. We shall assume, therefore, that

$$a_{i,j} = 0, \qquad i > j,$$
$$a_{j,j} \neq 0, \qquad j = 1, 2, \cdots, n,$$

and proceed to an explicit solution of system (2.3) which may now be written

$$\sum_{j=i}^{n} a_{i,j}F_j = f_i, \qquad i = 1, 2, \cdots, n. \tag{3.1}$$

If $i=n$ in (3.1) we get

$$F_n = \frac{1}{a_{n,n}} f_n.$$

Substituting this value for F_n in the other members of (3.1) and transposing, we obtain

$$\sum_{j=i}^{n-1} a_{i,j}F_j = f_{i,1}, \qquad i = 1, 2, \cdots, n-1, \tag{3.2}$$

where

$$f_{i,1} = f_i - \frac{a_{i,n}}{a_{n,n}} f_n.$$

Since (3.2) is of the same form as (3.1) this process evidently admits of repetition. Assume, then, that at the kth stage we have reduced the given set of equations to

$$\sum_{j=i}^{n-k} a_{i,j}F_j = f_{i,k}, \qquad i = 1, 2, \cdots, n-k, \tag{3.3}$$

where

$$f_{i,k} = f_{i,k-1} - \frac{a_{i,n-k+1}}{a_{n-k+1,n-k+1}} f_{n-k+1,k-1}. \tag{3.4}$$

If $i=n-k$, (3.3) yields

$$F_{n-k} = \frac{1}{a_{n-k,n-k}} f_{n-k,k}. \tag{3.5}$$

Substituting this value of F_{n-k} in (3.3) and transposing, we obtain

$$\sum_{j=i}^{n-k-1} a_{i,j}F_j = f_{i,k+1}, \qquad i = 1, 2, \cdots, n-k-1, \tag{3.6}$$

where

$$f_{i,k+1} = f_{i,k} - \frac{a_{i,n-k}}{a_{n-k,n-k}} f_{n-k,k}. \tag{3.7}$$

If $i=n-k-1$, (3.6) yields

$$F_{n-k-1} = \frac{1}{a_{n-k-1,n-k-1}} f_{n-k-1,k+1}. \tag{3.8}$$

Since (3.6), (3.7), and (3.8) may be obtained from (3.3), (3.4), and (3.5) respectively, by replacing k by $k+1$, the validity of the last mentioned set is established by induction when $k=0, 1, 2, \cdots, n-1$; when $k=0$ (3.4) must be replaced by $f_{i,0}=f_i$. The solution of the system (3.1) is given, therefore, by formulae (3.5) and (3.4) which, for convenience, may be rewritten in the equivalent form

$$F_{n-k} = \frac{1}{a_{n-k,n-k}} f_{n-k,k}, \qquad k = 0, 1, 2, \cdots, n-1, \tag{3.9}$$

$$f_{i,k} = f_{i,k-1} - R_{i,k-1}f_{n-k+1,k-1}, \tag{3.10}$$

$$k = 1, 2, \cdots, n-1;\ i = 1, 2, \cdots, n-k,$$

where

$$R_{i,k-1} = \frac{a_{i,n-k+1}}{a_{n-k+1,n-k+1}}. \tag{3.11}$$

4. Application to plane sections of spheres. Consider a universe of spheres imbedded in a three dimensional space. Let their radii be represented by the variate x having a probability function $\Phi(x)$ on (a, b). Let a sample be selected by making random plane cuts through the imbedding space and considering each cut sphere a member of the sample. Such sampling is not simple since a sphere of radius x is x times as likely to be cut as a sphere of radius 1. We must therefore introduce a sampling function, $s(x)$, defined by

$$s(x) = cx. \tag{4.1}$$

The value of the constant c is evidently determined by the condition

$$\int_a^b cx\Phi(x)dx = 1.$$

However, since the relations (2.3) are linear, the effect of failing to choose c in this manner is to multiply each of the frequencies $F_1, F_2, \cdots, F_n$ by the same constant. This does not alter the essential character of the distribution, hence c may be chosen with a view to simplifying the computation.

The manner of selecting the sample of spheres has automatically yielded a random plane section from each sphere of the sample. Let the radii of these plane sections be represented by the variate y. The probability function of y from a sphere of radius x, denoted by $\eta(x, y)$, may be found as follows.

Let λ be the distance from the center of the sphere of a section of radius y. Then λ and y satisfy the relation

$$y^2 + \lambda^2 = x^2. \tag{4.2}$$

The probability function of λ is $1/x$, all distances from the center of the sphere being equally likely. Accordingly the probability differential, dp, is given by

$$dp = 1/x \, d\lambda.$$

Differentiating (4.2) and substituting for λ from the same relation yields the fact that

$$d\lambda = -\frac{ydy}{(x^2 - y^2)^{1/2}}\cdot \tag{4.3}$$

Under this change of variate, therefore,

$$dp = -\frac{ydy}{x(x^2 - y^2)^{1/2}}\cdot$$

The probability of finding y between y and $y+dy$ is the same as the probability of finding λ between λ and $\lambda+d\lambda$, where dy and $d\lambda$ are related by (4.3). Absorbing the negative sign in the differential dy, therefore, we have

$$\eta(x, y) = \begin{cases} \dfrac{y}{x(x^2 - y^2)^{1/2}} & \text{if} \quad y < x \\ 0 & \text{if} \quad y > x \end{cases} \tag{4.4}$$

Let $y_j = x_j$, $j = 1, 2, \cdots, n$, and denote by F_j and f_j the frequency of x and y, respectively, on the subinterval (x_{j-1}, x_j). The expected relationship between these frequencies is given by (2.3), where, in view of (4.1) and (4.4)

$$a_{i,j} = \begin{cases} \dfrac{c}{x_j - x_{j-1}} \displaystyle\int_{x_{j-1}}^{x_j} \int_{x_{i-1}}^{x_i} (x^2 - y^2)^{-1/2} y\,dy\,dx, & \text{if} \quad i < j \\ \dfrac{c}{x_j - x_{j-1}} \displaystyle\int_{x_{j-1}}^{x_j} \int_{x_{j-1}}^{x} (x^2 - y^2)^{-1/2} y\,dy\,dx, & \text{if} \quad i = j \\ 0, & \text{if} \quad i > j. \end{cases}$$

Integrating with respect to y, and defining $I_{i,j}$ by the relation

$$I_{i,j} = \begin{cases} \dfrac{1}{x_j - x_{j-1}} \displaystyle\int_{x_{j-1}}^{x_j} (x^2 - x_i^2)^{1/2} dx, & \text{if} \quad i < j \\ 0, & \text{if} \quad i \geqq j, \end{cases}$$

it may be verified that

$$a_{i,j} = c(I_{i-1,j} - I_{i,j}), \qquad i, j = 1, 2, \cdots, n.$$

It follows that $|a_{i,j}|$ is triangular in form with the elements of the main diagonal not zero. The hypotheses of section 3 are therefore satisfied and the inverse relationship is given by (3.9), (3.10), and (3.11).

Although the formula for the evaluation of $I_{i,j}$ is familiar, the task of evaluating $(n-1)^2/2$ such quantities is sufficiently arduous to discourage the application of these results to a given collection of data. For this reason it is desirable to mold the results so that they admit of readier application, provided this can be done without extensive sacrifice of accuracy. To this end, we shall abandon the normal procedure of grouping the sections in radius classes of equal width in favor of area classes of equal width. While this will evidently have the effect of grouping the resulting spheres in radius classes of unequal width, the inconvenience so occasioned is not great. Let the subdivision points, therefore, be defined by the relation

$$x_j^2 = jw, \qquad j = 0, 1, \cdots, n, \tag{4.5}$$

where w is defined by

$$b^2 = nw.$$

An approximate evaluation of $I_{i,j}$ may be obtained by using the trapezoidal rule‡ for approximate integration. Thus, if the interval (x_{j-1}, x_j) is divided into

‡ *Cf.*, for example, Smith, Salkover, and Justice, Calculus, p. 322.

m equal parts the rth subdivision point is, by virtue of (4.5),

$$w^{1/2}\left[(j-1)^{1/2}+\frac{r}{m}\left\{j^{1/2}-(j-1)^{1/2}\right\}\right], \qquad r=0,1,\cdots,m.$$

The square of this quantity, evaluated for substitution purposes, is

$$\frac{w}{m^2}\left[(m-r)^2(j-1)+2(m-r)r\{j(j-1)\}^{1/2}+r^2j\right].$$

If, in the latter expression, $\{j(j-1)\}^{1/2}$ is replaced by $j-\frac{1}{2}$ it reduces to

$$\frac{w}{m}\left[m(j-1)+r\right].$$

The error so introduced is negligible if $j\geqq 5$ and the practical approach of actually calculating the integrals associated with j less than 5 will show that the replacement is justified in these cases also. The approximate functional value of $(x_2^2-x_i^2)^{1/2}$ at the rth subdivision point is therefore seen to be

$$\frac{w^{1/2}}{m^{1/2}}\left[m(j-i-1)+r\right]^{1/2},$$

and the application of the trapezoidal rule yields

$$I_{i,j}\doteq\frac{w^{1/2}}{2m^{3/2}}\left[\{m(j-i-1)\}^{1/2}+2\sum_{r=1}^{m-1}\{m(j-i-1)+r\}^{1/2}+\{m(j-i)\}^{1/2}\right], \text{ if } i<j.$$

Thus the approximate evaluation of $I_{i,j}$ is, in effect, accomplished by simply adding square roots of successive integers. The most significant feature of this approximation is that it appears as a function of $j-i$. This implies that $a_{i,j}$ is a function of $j-i$, that $a_{i,n-k+1}$ is a function of $k+i$, and that $a_{n-k,n-k}$ is a constant. In view of the last remark we may choose c so that $a_{n-k,n-k}=1$, whereupon formula (3.9) reduces to

$$F_{n-k}=f_{n-k,k}. \tag{4.6}$$

Again it may be verified, if we recall formula§ (3.11), that $R_{i,k-1}$ is a function of $k+i$ alone for any fixed n. The significance of this fact becomes evident if we envisage the actual steps in the reduction of a given distribution of sectional radii. The *reduction factors* to be applied to the original frequencies $f_{n-1}, f_{n-2}, \cdots, f_1$ are, respectively, $R_{n-1,0}, R_{n-2,0}, \cdots, R_{1,0}$, and yield, through (3.10), the *reduced frequencies* $f_{n-1,1}, f_{n-2,1}, \cdots, f_{1,1}$. To these latter, with the first member deleted, we apply the reduction factors $R_{n-2,1}, \cdots, R_{1,1}$, which are respectively equal to the members of the first set with the last one, $R_{1,0}$, deleted. Similarly, if the last two members of the first set are deleted, we obtain the third set

§ Formula (3.11) reduces to $R_{i,k-1}=a_{i,n-k+1}$. However, since $R_{i,k-1}$ is readily seen to be independent of c, this simplification does not serve to reduce the labor of computation.

of reduction factors, *etc.* Thus the determination of $n-1$ reduction factors, requiring the calculation of n quantities of the form $I_{i,j}$, yields the complete set of reduction factors.

The use of successive values of m in the trapezoidal formula will reveal that if $m=3$ a satisfactory approximation to $I_{i,j}$ is obtained in every case except when $i=j-1$. In this instance it will be found desirable to use 16 for m. The first 15 reduction factors evaluated in this way will be found to be

$R_{n-k-1,k} = .835$	$R_{n-k-6,k} = .308$	$R_{n-k-11,k} = .227$
$R_{n-k-2,k} = .543$	$R_{n-k-7,k} = .285$	$R_{n-k-12,k} = .218$
$R_{n-k-3,k} = .438$	$R_{n-k-8,k} = .267$	$R_{n-k-13,k} = .209$
$R_{n-k-4,k} = .378$	$R_{n-k-9,k} = .251$	$R_{n-k-14,k} = .201$
$R_{n-k-5,k} = .338$	$R_{n-k-10,k} = .238$	$R_{n-k-15,k} = .195$

The retention of three figures in these values may be justified by employing exact integration to evaluate a few selected factors. The set here given is sufficient for a distribution of no more than sixteen classes; if distributions with more classes are to be handled additional factors may be easily and quickly calculated.

The actual reduction of section radius frequencies to the corresponding sphere radius frequencies may be accomplished in tabular form as illustrated by the following hypothetical example.

DISTRIBUTION OF RADII OF MINUTE SPHERICAL STRUCTURES DERIVED FROM A DISTRIBUTION OF AREAS OF THEIR RANDOM SECTIONS
(all figures rounded off to the nearest integer)

Section Area Classes (1000 sq. microns)	Equivalent Radius Classes (microns)	Mid-point	Section Frequency f_i	$R_{i,0}f_5$	$f_i-R_{i,0}f_5=f_{i,1}$	$R_{i,1}f_{4,1}$	$f_{i,1}-R_{i,1}f_{4,1}=f_{i,2}$	$R_{i,2}f_{3,2}$	$f_{i,2}-R_{i,2}f_{3i,2}=f_{i,3}$	$R_{i,3}f_{2i,3}$	$f_{i,3}-R_{i,3}f_{2,3}=f_{i,4}$	Sphere Frequency $F_i=f_{i,5-i}$
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)
40–50	113–126	120	*8*									8
30–40	98–113	106	38	7	*31*							31
20–30	80– 98	89	109	4	105	26	*79*					79
10–20	56– 80	68	124	4	120	17	103	66	*37*			37
0–10	0– 56	28	93	3	90	14	76	43	33	31	*2*	2

Since the distribution contains five classes, only the first four reduction factors are used, namely, .835, .543, .438, and .378. These, multiplied in order by the top member of column (4), constitute column (5). Column (6) is obtained by subtracting the members of column (5) from the corresponding members of column (4). Column (7), like column (5), is formed by multiplying the first three reduction factors in order by the top member of column (6). Column (8), like column (6), is the result of subtracting the members of column (7) from the corresponding members of column (6). Continuing in this way the triangular array is completed by column (12). Recalling formula (4.6), the sphere frequencies of column (13) are seen to be the top members of columns (4), (6), (8), (10), and (12).

The calculation of such statistical constants as the mean and the standard deviation must be performed from first principles, since the various short methods that have been devised assume class intervals of equal width.

5. Nonspherical structures. The practical statistician will seldom be presented with data derived from sections of spherical structures. The cells and other parts of plants and animals, for example, are not in general spherical. Nevertheless, many such structures may be regarded as approximations to spheres without significant sacrifice of accuracy. When this is the case, the applicability of the above results is immediate. Furthermore, it may be argued qualitatively that there is justification for applying them even when the structures are known to differ markedly from spheres. In support of this it may be pointed out that the average area of the sections of an ellipsoid of revolution, whose axes are in the ratio 3:2, is about 8% less than the average area of the sections of a sphere of equal volume. This is equivalent to a deviation of about 4% when reduced to linear measure.

When dealing with nonspherical structures the areas of the sections must be determined|| and arranged in classes of equal width. Corresponding to each class boundary the radius of a circle of equivalent area may be determined. These values will then serve as class boundaries for the radii of the spheres. The spheres should not be regarded as approximations in form to the original structures but rather as being equivalent to them in volume.

The derivation of analogous results for geometric figures other than the sphere is clearly possible. However in the absence of regular structures it would seem logical to apply the procedure outlined above and to make a critical appraisal of the conclusions.

6. Conclusion. The relationship between associated variates, derived in sections 3 and 4, is applicable to a variety of problems. The case of velocity magnitudes and their components has already been mentioned and will be developed in a subsequent paper. Another application is the derivation of a distribution of

|| Areas may be found by using a planimeter on camera lucida drawings, or they may be estimated from linear measurements made on the sections.

family sizes from data which is limited to the family order of birth of a group of children. The treatment required by each problem owes its individuality only to the fact that the method of application may be designed to simplify the computation. Such simplification is attained by a suitable choice of class boundaries for the two variates.

ON THE DENSEST PACKING OF CIRCLES

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1. Introduction. We show in this note that at most $A/\sqrt{12}$ circles of radius 1 can be placed in a convex polygon of area A and with angles not greater than $\frac{2}{3}\pi$,* such that no two of the circles overlap. This upper bound for the number of inscribed circles is the best possible one, but the restriction on the angles is essential. Somewhat similar results have been obtained, using an entirely different method, by A. Thue.† More recently, the problem was studied by L. Fejes‡ and R. Rado.§

Apart from a simple application of differential calculus, our method is elementary. The proof is based on two lemmas which have a certain interest in themselves.

2. The convex polygon $S(P)$. Let Σ be a set of points $P, P_1, P_2, \cdots$ in the plane π, such that the distance $\overline{P'P''}$ of any two different points P', P'' of Σ is at least 2. Hence, if $C(P), C(P_1), C(P_2), \cdots$ are the circles of radius 1 and centers $P, P_1, P_2, \cdots$, then no two of these circles overlap.

Let K be any circle in Π, say of radius r, and let K' be the concentric circle of radius $r+1$. If now $P', P'', \cdots, P^{(l)}$ are any points of Σ in K, then the circles $C(P'), C(P''), \cdots, C(P^{(l)})$ are contained in K'; their total area $l\pi$ is therefore not larger than the area $(r+1)^2\pi$ of K', and so $l \leqq (r+1)^2$. Hence *every circle contains at most a finite number of points of* Σ.

For every point P of Σ, denote by $S(P)$ the set of all points Q in Π for which

$$\overline{PQ} \leqq \overline{P_kQ} \qquad (k = 1, 2, \cdots). \tag{1}$$

The circle $C(P)$ is a subset of $S(P)$. For, if Q lies in $C(P)$, then

$$\overline{P_kQ} \geqq \overline{PP_k} - \overline{PQ} \geqq 2 - 1 = 1 \geqq \overline{PQ}, \quad \text{since} \quad \overline{PQ} \leqq 1,\ \overline{PP_k} \geqq 2.$$

Denote by Λ_k the locus of all points Q for which $\overline{PQ} = \overline{P_kQ}$. Evidently Λ_k is the line perpendicular to the line PP_k which intersects the segment PP_k at

* All angles in this paper are measured in radians. Moreover, we use the same symbols for the angles and their measures in radians.

† See his note read at the Scandinavian Mathematical Congress of 1892, and the paper, Norske Vid. Selsk. Skr. 1910, No. 1.

‡ Math. Z. 46, 1940, 83–85.

§ Dr. Rado was so kind as to give us the reference to Fejes's note, and he also informed us of his, as yet unpublished, results.

its center, say $P^{(k)}$. The points Q satisfying $\overline{PQ} \leqq \overline{P_kQ}$ form the semi-plane determined by Λ_k containing P, and this is a convex region. Therefore, from (1), $S(P)$ *is a closed convex region.* (Fig. 1.)

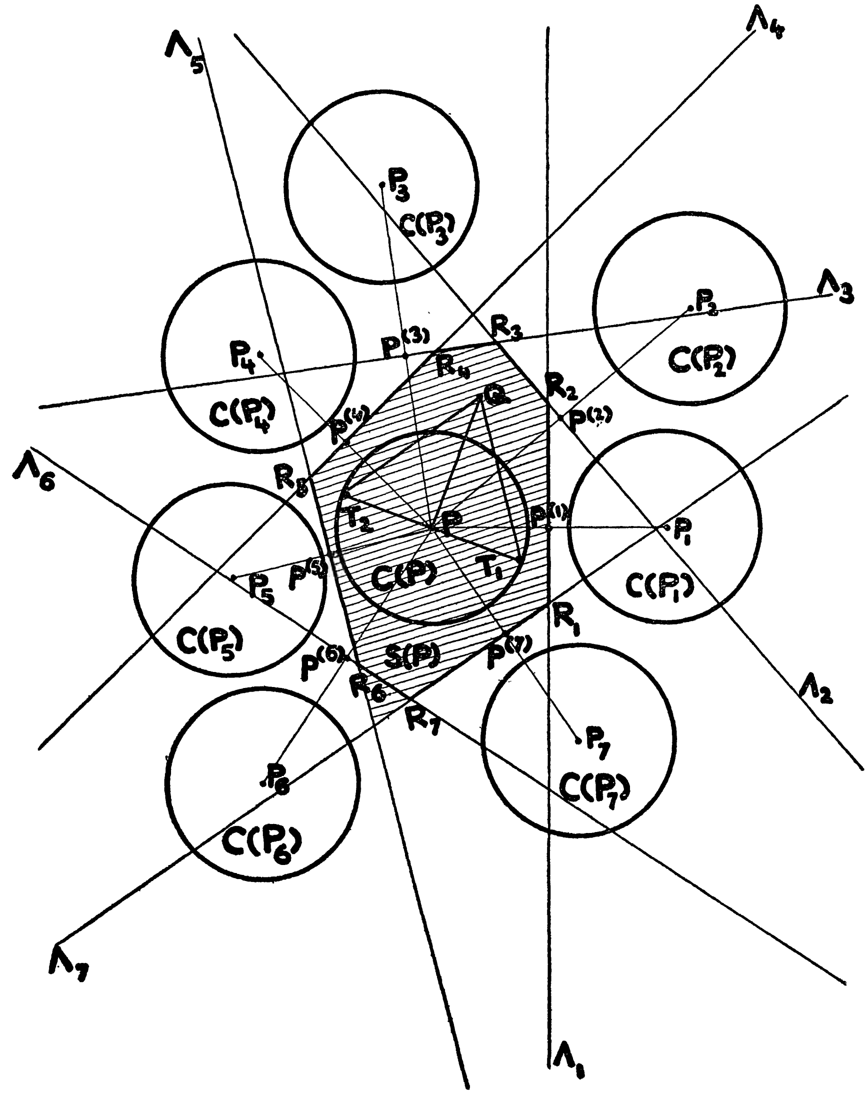

FIG. 1

We assume from now on that $S(P)$ is of *finite area*, $A(P)$ say. Let $Q \neq P$ be a point in $S(P)$, and denote by T_1 and T_2 the two endpoints of the diameter of $C(P)$ perpendicular to PQ. Since $S(P)$ is convex, and since the three points Q, T_1, T_2 belong to $S(P)$, the whole triangle T_1QT_2 is a subset of $S(P)$, and so its area is not greater than $A(P)$. Now the area of this triangle is

$$\tfrac{1}{2}\overline{T_1T_2} \times \overline{PQ} = \overline{PQ}, \quad \text{since} \quad \overline{T_1T_2} = 2,$$

whence

$$\overline{PQ} \leqq A(P) \quad \textit{for every point } Q \textit{ of } S(P). \tag{2}$$

By (2), $S(P)$ lies in the circle $C'(P)$ of center P and radius $A(P)$. Hence it has points in common only with those lines Λ_k which touch or pass through $C'(P)$. As $P^{(k)}$ is the point of Λ_k nearest to P, this requires that

$$\overline{PP^{(k)}} \leqq A(P), \quad \textit{i.e.}, \quad \text{that} \quad \overline{PP_k} = 2\overline{PP^{(k)}} \leqq 2A(P).$$

But since only a finite number of points P_k of Σ satisfy this inequality, the boundary of $S(P)$ meets only a finite number of the lines Λ_k. Therefore *$S(P)$ is a convex polygon*, say with the n sides $\Lambda_1, \Lambda_2, \cdots, \Lambda_n$, and the n vertices $R_1, R_2, \cdots, R_n$. We choose the notation such that these sides and vertices lie on the boundary of $S(P)$ in the order of their indices, and that R_k and R_{k+1} are the vertices on Λ_k, and so Λ_{k-1} and Λ_k are the sides through R_k. (The indices 0 and $n+1$ must be replaced by n and 1, respectively.)

3. A fundamental lemma. The lines from P to the n vertices $R_1, R_2, \cdots, R_n$ split $S(P)$ into the n triangles

$$R_1PR_2, \; R_2PR_3, \cdots, \; R_nPR_1,$$

say of areas

$$a_1, \; a_2, \cdots, \; a_n.$$

Let further the angles at P of these triangles be

$$\alpha_1, \; \alpha_2, \cdots, \; \alpha_n,$$

respectively. Then, evidently,

$$a_1 + a_2 + \cdots + a_n = A(P), \tag{3}$$

and

$$\alpha_1 + \alpha_2 + \cdots + \alpha_n = 2\pi. \tag{4}$$

In the next paragraphs, we prove the following

LEMMA 1: *For every index k,*

$$a_k \geqq \frac{\sqrt{3}}{\pi}\alpha_k, \tag{5}$$

with equality if and only if the two circles $C(P)$ and $C(P_k)$ touch each other, and are both touched by the circles $C(P_{k-1})$ and $C(P_{k+1})$.

For the proof of (5), put (Fig. 2)

$$x = \overline{PP^{(k)}} = \tfrac{1}{2}\overline{PP_k}, \quad \text{so that} \quad x \geqq 1.$$

Further denote by Γ the circle of center P and radius x; then Λ_k is a tangent of

Γ. The two lines from P to R_k and R_{k+1} cut off Γ a sector of angle α_k, hence of area $\frac{1}{2}x^2\alpha_k$. Since this sector lies entirely in the triangle R_kPR_{k+1} of area a_k, the

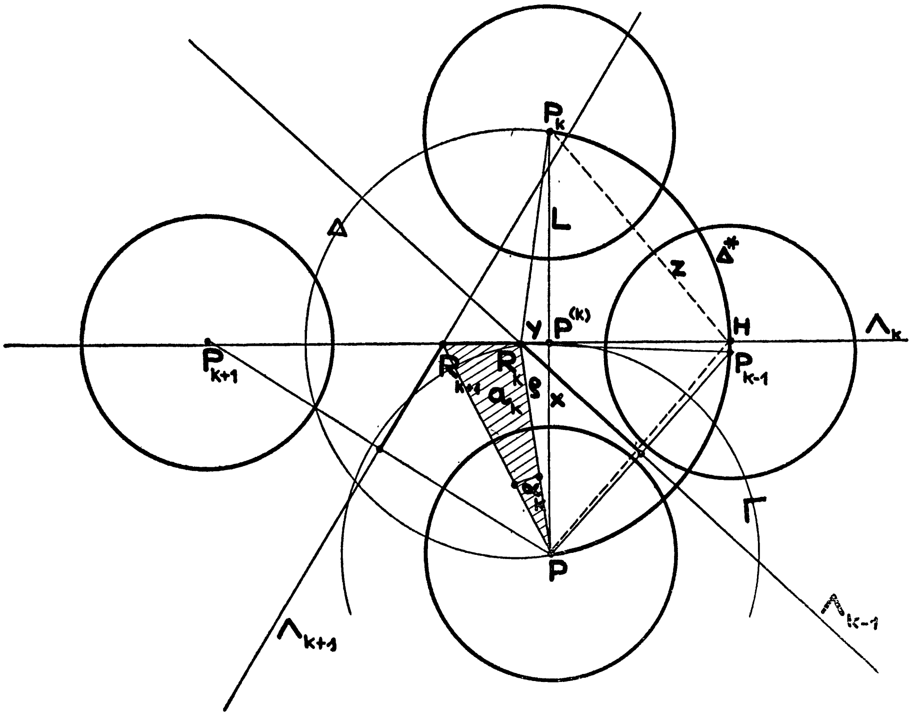

FIG. 2

inequality $a_k > \frac{1}{2}x^2\alpha_k$ holds. Hence if $\frac{1}{2}x^2 \geqq \sqrt{3}/\pi$, *i.e.*, if $x \geqq \sqrt[4]{12/\pi^2} > 1$, then (5) is satisfied with the sign "$>$" instead of "$\geqq$," and so the assertion is proved, since $C(P)$ and $C(P_k)$ obviously do not touch each other.

We may therefore exclude this case, and assume from now on that

$$1 \leqq x < \sqrt[4]{\frac{12}{\pi^2}} < \sqrt{\frac{4}{3}}\,. \tag{6}$$

We now prove the result:

The point $P^{(k)}$ is interior to the segment R_kR_{k+1}.

If this is not so, then R_k, R_{k+1} are two distinct points of the line Λ_k, lying on the same side of $P^{(k)}$. Hence one of these points, say R_k, is nearer than the other one to $P^{(k)}$; and R_k possibly coincides with $P^{(k)}$.

By our notation, Λ_{k-1} and Λ_k meet at R_k. Since all points of Λ_{k-1} are equidistant from P and P_{k-1}, this implies that $\overline{PR_k} = \overline{P_{k-1}R_k}$, and so P and P_{k-1} lie on the circle Δ of center R_k and radius $\rho = \overline{PR_k}$. This circle contains also P_k, and is divided into two arcs by the line L joining P and P_k. Let Δ^* be the arc which

meets Λ_k at a point, H say, separated from R_{k+1} by R_k; then Δ^* is separated from R_{k+1} by L. Since the only common point of the line Λ_{k-1} and the angle PR_kR_{k+1} is the vertex R_k of this angle, it follows that the image P_{k-1} of P in Λ_{k-1} is separated from R_{k+1} by both the line PR_k and its image R_kP_k in $\Lambda_k = R_kR_{k+1}$; hence P_{k-1} lies on Δ^*. On putting

$$z = \overline{PH} = \overline{P_kH},$$

it is evident that for every point Q on Δ^*,

$$\min(\overline{PQ}, \overline{P_kQ}) \leqq z,$$

and so, in particular,

$$\min(\overline{PP}_{k-1}, \overline{P_kP}_{k-1}) \leqq z.$$

Therefore, by the definition of Σ,

$$z \geqq 2. \tag{7}$$

Next put

$$\overline{R_kP^{(k)}} = y,$$

so that

$$\rho^2 = x^2 + y^2 \quad \text{and} \quad \overline{P^{(k)}H} = \overline{R_kH} - \overline{R_kP^{(k)}} = \rho - y = \sqrt{x^2 + y^2} - y \leqq x,$$

since $\sqrt{x^2+y^2} \leqq x+y$. Hence

$$z^2 = \overline{PP^{(k)}}^2 + \overline{P^{(k)}H}^2 \leqq x^2 + x^2 = 2x^2,$$

whence, from (6),

$$z \leqq \sqrt{2}\, x < \sqrt{2}\sqrt{\tfrac{4}{3}} < 2,$$

contrary to (7). This contradiction proves the result.

From the above, the line $PP^{(k)}$ divides the triangle R_kPR_{k+1} into the two triangles $R_kPP^{(k)}$, say of area b_k, and $P^{(k)}PR_{k+1}$, say of area c_k, so that

$$b_k + c_k = a_k. \tag{8}$$

The line $PP^{(k)}$ splits α_k into two angles $\beta_k = R_kPP^{(k)}$ and $\gamma_k = P^{(k)}PR_{k+1}$ satisfying

$$\beta_k + \gamma_k = \alpha_k. \tag{9}$$

By (8) and (9), Lemma 1 is proved if we can show that

$$b_k \geqq \frac{\sqrt{3}}{\pi}\beta_k, \tag{10}$$

and

$$c_k \geqq \frac{\sqrt{3}}{\pi}\gamma_k, \tag{11}$$

with equality if and only if $C(P)$ and $C(P_k)$ touch each other, and are touched

by $C(P_{k-1})$ in the case of the inequality (10), and by $C(P_{k+1})$ in inequality (11).

It suffices to prove the assertion (10), since (11) can be treated likewise.

As before, we denote by Δ the circle of center R_k and radius $\rho = \overline{PR_k}$ (Fig. 3); then P, P_{k-1} and P_k lie on this circle. The line L joining P and P_k divides Δ into two unequal arcs (since $P^{(k)}$ and R_k are distinct); let Δ^* be the larger one, *i.e.*

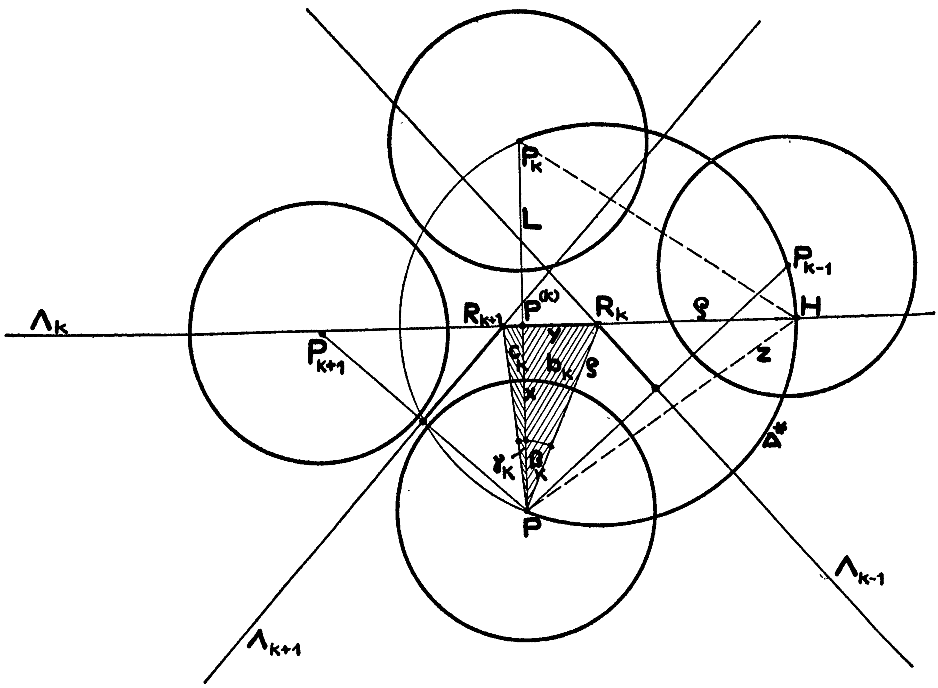

FIG. 3

that arc which is separated from R_{k+1} by L. Then, by an argument already used we see that P_{k-1} lies on Δ^*.

Denote again by H the point where Δ^* and Λ_k intersect, and put

$$y = \overline{R_kP^{(k)}}, \qquad z = \overline{PH} = \overline{P_kH}.$$

Then $\rho = \sqrt{x^2+y^2}$, and we find as in §7 that

$$z \geqq 2, \tag{12}$$

but now

$$\begin{aligned} \overline{P^{(k)}H} &= \overline{R_kH} + \overline{P^{(k)}R_k} = \rho + y = \sqrt{x^2+y^2} + y, \\ z^2 &= \overline{PH}^2 = \overline{PP^{(k)}}^2 + \overline{P^{(k)}H}^2 = x^2 + (\sqrt{x^2+y^2} + y)^2 \\ &= 2(x^2 + y^2 + y\sqrt{x^2+y^2}). \end{aligned} \tag{13}$$

From (12) and (13),

$$x^2 + y^2 + y\sqrt{x^2+y^2} \geqq 2,$$

whence, on solving for y,

$$y \geqq \frac{2 - x^2}{\sqrt{4 - x^2}}.$$

Since further

$$b_k = \tfrac{1}{2}xy, \qquad \beta_k = \sin^{-1}\frac{y}{\sqrt{x^2 + y^2}},$$

the inequality (10) can be written as

$$F(x, y) \geqq 0, \quad \text{where} \quad F(x, y) = \pi xy - \sqrt{12}\sin^{-1}\frac{y}{\sqrt{x^2 + y^2}}.$$

Hence the assertion (10) is proved if we can show that

$$F(x, y) > 0, \tag{14}$$

if

$$1 \leqq x < \sqrt{\tfrac{4}{3}} \quad \text{and} \quad y \geqq \frac{2 - x^2}{\sqrt{4 - x^2}}, \quad \text{and either} \quad x > 1 \quad \text{or}$$
$$y > \frac{2 - x^2}{\sqrt{4 - x^2}}, \quad \text{or both.} \tag{15}$$

For in the excluded case, we have

$$x = 1, \qquad y = \frac{2 - x^2}{\sqrt{4 - x^2}} = \sqrt{\tfrac{1}{3}}, \qquad z = 2, \qquad F(x, y) = 0,$$

and so the equality sign holds in (10). Moreover, since $z = 2$, every point $Q \neq H$ on Δ^* has a distance less than 2 from either P or P_k; therefore P_{k-1} must coincide with H, *i.e.*, the three circles $C(P)$, $C(P_{k-1})$, $C(P_k)$ touch in pairs.

When (15) is satisfied, then

$$\frac{x^2 + y^2}{x}\frac{\partial F(x, y)}{\partial y} = \pi(x^2 + y^2) - \sqrt{12} \geqq \pi\left(x^2 + \frac{(2 - x^2)^2}{4 - x^2}\right) - \sqrt{12}$$
$$= \frac{4\pi}{4 - x^2} - \sqrt{12} \geqq \frac{4\pi}{4 - 1} - \sqrt{12} > 0.$$

Hence $F(x, y)$ is a *strictly increasing* function of y. To prove (14), it therefore suffices to show that the function

$$f(x) = F\left(x, \frac{2 - x^2}{\sqrt{4 - x^2}}\right) = \pi\frac{x(2 - x^2)}{\sqrt{4 - x^2}} - \sqrt{12}\sin^{-1}(1 - \tfrac{1}{2}x^2)$$

is *strictly increasing*, since $f(1) = 0$. Now

$$f(x) = 2g(t), \quad \text{where} \quad t = 1 - \tfrac{1}{2}x^2, \qquad g(t) = \pi t\sqrt{\frac{1 - t}{1 + t}} - \sqrt{3}\sin^{-1} t,$$

and where, by (15), t satisfies the inequality

$$\tfrac{1}{3} = 1 - \tfrac{2}{3} < t < 1 - \tfrac{1}{2} = \tfrac{1}{2}.$$

On differentiating, we get

$$\frac{df(x)}{dx} = -2x\frac{dg(t)}{dt}, \qquad \frac{dg(t)}{dt} = \frac{1-t}{(1-t^2)^{3/2}}\left[\pi - (\pi t + \sqrt{3})(1+t)\right].$$

Now $3 < \pi < 10/3$ and $\sqrt{3} > 3/2$, hence

$$(\pi t + \sqrt{3})(1+t) \geqq (3\cdot\tfrac{1}{3} + \tfrac{3}{2})(1 + \tfrac{1}{3}) = \tfrac{10}{3} > \pi,$$

and therefore

$$\frac{dg(t)}{dt} < 0, \qquad \frac{df(x)}{dx} > 0,$$

as asserted. This concludes the proof of Lemma 1.

4. A second lemma. We now prove the following

LEMMA 2. *The convex polygon $S(P)$, defined in §2, is of area*

$$A(P) \geqq \sqrt{12}, \tag{16}$$

with equality if and only if $C(P)$ is touched by six circles $C(P_1), \cdots, C(P_6)$, whose centers form a regular hexagon of side 2 and center P.

From (3), (4), and Lemma 1, the inequality (16) follows at once. We see moreover, that the equality sign can hold only if

$$a_k = \frac{\sqrt{3}}{\pi}\alpha_k \qquad (k = 1, 2, \cdots, n),$$

i.e., if each circle $C(P_k)$ in the set

$$C(P_1), C(P_2), \cdots, C(P_n)$$

is touched by both $C(P_{k-1})$ and $C(P_{k+1})$, and itself touches $C(P)$. Hence

$$\overline{PP_1} = \overline{PP_2} = \cdots = \overline{PP_n} = \overline{P_1P_2} = \overline{P_2P_3} = \cdots = \overline{P_nP_1} = 2.$$

Now the regular hexagon, and no other regular polygon, has the property that its side is equal to the radius of the circumscribed circle; therefore n must be equal to 6, and the circles $C(P), C(P_1), \cdots, C(P_6)$ must be situated as asserted.

5. The theorem. We can now prove the following theorem.

Let Π_0 be a convex polygon of angles not greater than $\frac{2}{3}\pi$, hence of at most six sides. If A denotes the area of Π_0, then at most $A/\sqrt{12}$ circles of radius 1 can be placed in Π_0 such that no two of these circles overlap.

Proof (Fig. 4): Let $C_{01}, C_{02}, \cdots, C_{0l}$ be the circles placed in Π_0, and let $\Lambda_1, \Lambda_2, \cdots, \Lambda_n$ be the sides of Π_0. Denote further by Π_h the polygon symmetri-

cal to Π_0 in Λ_h, and by $C_{h1}, C_{h2}, \cdots, C_{hl}$ the circles symmetrical to $C_{01}, C_{02}, \cdots, C_{0l}$ in Λ_h; these new circles lie in Π_h. No two of the $m=(n+1)l$ circles

$$C_{01}, C_{02}, \cdots, C_{0l}, \cdots, C_{n1}, C_{n2}, \cdots, C_{nl} \tag{17}$$

overlap. This is obvious, from the hypothesis, for any two circles in the *same* polygon Π_h, and also for any two circles one of which lies in Π_0. To prove the assertion for two circles lying in two different polygons $\Pi_1, \Pi_2, \cdots, \Pi_n$, it is obviously sufficient to show that no two of these polygons, Π_h and Π_k say $(h\neq k)$, overlap. This is evident if Λ_h, Λ_k are parallel, since then Π_h, Π_k lie on opposite

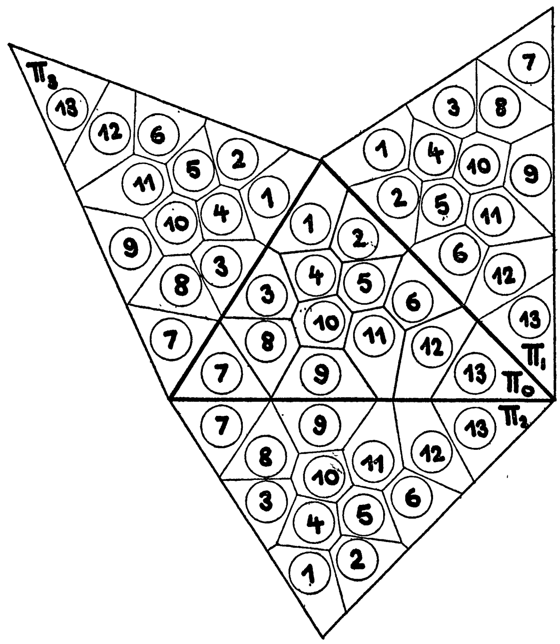

FIG. 4

sides of the strip determined by these lines. If Λ_h, Λ_k meet at a point, O say, we consider that angle ϕ of vertex O formed by these two lines which contains Π_0. Then $\phi\leqq\frac{2}{3}\pi$, from the hypothesis for Π_0, and so the images of ϕ in Λ_h and Λ_k are two non-overlapping angles of vertex O. Since one of these angles contains Π_h and the other contains Π_k, it follows that Π_h and Π_k do not overlap.

Denote now by Σ the set of the centers

$$P_{01}, P_{02}, \cdots, P_{0l}, \cdots, P_{n1}, P_{n2}, \cdots, P_{nl}$$

of the circles (17), so that Σ has the properties stated in §1. In accordance with the definition in §1, we form the convex regions

$$S(P_{01}), \cdots, S(P_{0l}), \cdots, S(P_{n1}), \cdots, S(P_{nl});$$

these m regions together fill out the plane, and no two of them overlap.

We show next that all l polygons

$$S(P_{01}),\ S(P_{02}),\ \cdots,\ S(P_{0l}) \tag{18}$$

lie in Π_0. For consider, *e.g.* $S(P_{01})$, and take any side Λ_h of Π_0. If P_{h1} is the point symmetrical to P_{01} in Λ_h, then every point Q of $S(P_{01})$ is at least as near to P_{01} as to P_{h1}, hence lies on the same side of Λ_h as P_{01}. This is true for all indices $h=1, 2, \cdots, n$, and so Q lies in Π_0. We deduce then that the total area of the l polygons (18) is just equal to the area A of Π_0.

By Lemma 2, each polygon $S(P_{0i})$ is at least of area $\sqrt{12}$. This implies that

$$l\sqrt{12} \leqq A, \qquad l \leqq \frac{A}{\sqrt{12}},$$

as asserted. Moreover, our proof shows that *the equality*

$$l = \frac{A}{\sqrt{12}}$$

holds, if and only if Π_0 *is the sum set of a finite number of regular hexagons of side* 2.

We finally remark that the Theorem is untrue for polygons with angles greater than $\frac{2}{3}\pi$. For instance, the regular heptagon circumscribed to a circle of radius 1 is of area less than $\sqrt{12}$.

AN APPLICATION OF A THEOREM OF SYLVESTER

L. R. WILCOX, Illinois Institute of Technology

In this paper a little used theorem on determinants due to Sylvester is employed to prove a theorem in the geometry of curves which generalizes the well known result that the tangent plane to a developable surface is the same at all points of a fixed generator.*

1. The theorem of Sylvester. Let a matrix $(a_{ij};\ i, j=0, \cdots, m)$ be given, and define

$$A_h \equiv \begin{vmatrix} a_{00} & \cdots & a_{0h} \\ \cdot & \cdot\ \cdot\ \cdot & \cdot \\ a_{h0} & \cdots & a_{hh} \end{vmatrix} \qquad (h = 0, \cdots, m),\ A \equiv A_m,$$

$$b_{rs}^{(h)} \equiv \begin{vmatrix} a_{00} & \cdots & a_{0h} & a_{0s} \\ \cdot & \cdot\ \cdot\ \cdot & \cdot & \cdot \\ a_{h0} & \cdots & a_{hh} & a_{hs} \\ a_{r0} & \cdots & a_{rh} & a_{rs} \end{vmatrix}, \qquad (r, s = h+1, \cdots, m),$$

$$B^{(h)} \equiv \left| b_{rs}^{(h)} \right| \qquad (h = 0, \cdots, m-1).$$

* E. P. Lane, Projective Differential Geometry of Curves and Surfaces, Chicago, 1932, pp. 37–38.

Sylvester's theorem states that for $0 \leqq h \leqq m-1$

$$A_h^{m-h-1} \cdot A = B^{(h)}. \tag{1}$$

In particular, if $h=m-2$, we have

$$\begin{vmatrix} a_{00} & \cdots & a_{0,m-2} \\ \cdot & \cdots & \cdot \\ a_{m-2,0} & \cdots & a_{m-2,m-2} \end{vmatrix} \cdot A = \begin{vmatrix} b_{m-1,m-1}^{(m-2)} & b_{m-1,m}^{(m-2)} \\ & \\ b_{m,m-1}^{(m-2)} & b_{mm}^{(m-2)} \end{vmatrix}. \tag{2}$$

It is this case with which we shall be concerned. The proof will be omitted; it is made inductively with the help of Laplace's expansion.†

2. The theorem on developable surfaces. Let us consider briefly a simple problem in three-dimensional projective geometry. Let a curve C be generated by a point $P(t)$ whose homogeneous coordinates are

$$x_i = f_i(t) \qquad (i = 0, \cdots, 3).$$

The plane $x_0=0$ and the line $x_0=x_1=0$ are called S_2 and S_1 respectively. Let us consider along with C the family $T_1(t)$ of its tangent lines and the family $T_2(t)$ of its osculating planes. Under suitable assumptions‡ the intersection $S_2 \cdot T_1(t)$ is a point $P^1(t)$, which generates a (plane) curve C^1. Each tangent $T_1^1(t)$ to C^1 cuts S_1 in a point $P^2(t)$. Now each intersection $S_1 \cdot T_2(t)$ is a point $Q^2(t)$. How are $P^2(t)$ and $Q^2(t)$ related?

The answer is obtained analytically. We give only the outline, since the details are simple computations. Points of the line $T_1(t)$ are given by linear combinations

$$\lambda f_i(t) + \mu f_i'(t),$$

whence $P^1(t)$ has coordinates

$$g_i(t) = \begin{vmatrix} f_0(t) & f_0'(t) \\ f_i(t) & f_i'(t) \end{vmatrix} \qquad (i = 0, \cdots, 3).$$

Similarly $P^2(t)$ has coordinates

$$h_i(t) = \begin{vmatrix} g_1(t) & g_1'(t) \\ g_i(t) & g_i'(t) \end{vmatrix} = \begin{vmatrix} \begin{vmatrix} f_0(t) & f_0''(t) \\ f_1(t) & f_1'(t) \end{vmatrix} & \begin{vmatrix} f_0(t) & f_0'(t) \\ f_1(t) & f_1'(t) \end{vmatrix} \\ \begin{vmatrix} f_0(t) & f_0'(t) \\ f_i(t) & f_i'(t) \end{vmatrix} & \begin{vmatrix} f_0(t) & f_0''(t) \\ f_i(t) & f_i''(t) \end{vmatrix} \end{vmatrix},$$

the second equality following by direct substitution and use of the formula for differentiation of determinants. Now the coordinates of $Q^2(t)$ are found to be

† G. Kowalewski, Determinanten-Theorie, Leipzig, 1909, pp. 83–88.

‡ The hypotheses in Section 3 are sufficient.

$$h_i^*(t) = \begin{vmatrix} f_0(t) & f_0'(t) & f_0''(t) \\ f_1(t) & f_1'(t) & f_1''(t) \\ f_i(t) & f_i'(t) & f_i''(t) \end{vmatrix}.$$

Applying (2) to $h_i^*(t)$ (using $m=2$), we have

$$f_0(t)h_i^*(t) = h_i(t),$$

whence $Q^2(t)=P^2(t)$, since their coordinates are proportional.

The result just obtained may be regarded as a proof of a theorem on developable surfaces. The tangents $T_1(t)$ generate the developable surface σ of C. Since $P^1(t)$ is on $T_1(t)$, C^1 is a plane curve on σ crossing all generators. The tangent plane to σ at $P^1(t)$ is the plane of $T_1(t)$ and $T_1^1(t)$, or, since $P^2(t)$ is on $T_1^1(t)$, it is the plane of $T_1(t)$ and $P^2(t)$. But $T_2(t)$ contains $T_1(t)$ and $Q^2(t)=P^2(t)$, whence $T_2(t)$ is the tangent plane. Since by a change of coordinate system, S_2 can be chosen arbitrarily, $P^1(t)$ is an arbitrary point of $T_1(t)$. Thus the tangent plane to σ is the same along a generator, being the osculating plane of C at the point of tangency of the generator.

3. The generalization. Let a curve C in n-dimensional projective space be generated by $P(t)$ with coordinates $f_i(t)$ $(i=0, \cdots, n)$. The values of t are supposed limited to a region in which the $f_i(t)$ are of class $C^{(n)}$ and all first principal minors of the Wronskian matrix

$$W(t) = (f_i^{(j)}(t);\, i, j = 0, \cdots, n)$$

are not zero. The meaning of this hypothesis will appear presently. Consider spaces S_k $(k=1, \cdots, n-1)$ given by

$$S_k\colon \qquad x_0 = x_1 = \cdots = x_{n-k-1} = 0.$$

The osculating k-spaces of C are denoted by $T_k(t)$ $(k=1, \cdots, n-1)$. We may define two sequences of curves, C^h, D^h $(h=0, \cdots, n-1)$ as follows. The hypothesis on $W(t)$ may be shown to be equivalent to the proposition that $S_{n-h}\cdot T_h(t)$ is a point $Q^h(t)$ and that $Q^h(t)$ is not in S_{n-h-1}. In fact, $Q^h(t)$ has coordinates $y_i(t)$, where

$$(3) \qquad y_i(t) = \begin{vmatrix} f_0(t) & f_0'(t) & \cdots f_0^{(h)}(t) \\ \cdot & \cdot & \cdot \\ f_{h-1}(t) & f_{h-1}'(t) & \cdots f_{h-1}^{(h)}(t) \\ f_i(t) & f_i'(t) & \cdots f_i^{(h)}(t) \end{vmatrix}.$$

Now D^h is the curve generated by $Q^h(t)$; $D^0=C$. The curves C^h are defined inductively. First define $C^0=C$. If C^h has been defined, is generated by $P^h(t)$ and is contained in S_{n-h} $(h=0, \cdots, n-2)$, then $S_{n-h-1}\cdot T_1{}^h(t)$ is a point $P^{h+1}(t)$

where $T_1^h(t)$ is the tangent line to C^h at $P^h(t)$. We shall establish by induction simultaneously the existence of the points $P^h(t)$ and the fact that $P^h(t)=Q^h(t)$.

The result is obvious for $h=0$. Let $h=0, \cdots, n-2$, and suppose the result true for h. The coordinates of $P^h(t)$ are proportional to those of $Q^h(t)$, namely those given by (3). Clearly $y_h(t)\neq 0$ by the hypothesis on $W(t)$, whence it follows that $T_1^h(t)\cdot S_{n-h-1}$ is a point $P^{h+1}(t)$, whose coordinates are

$$z_i(t) = \begin{vmatrix} y_h(t) & y_h'(t) \\ y_i(t) & y_i'(t) \end{vmatrix};$$

those of $Q^{h+1}(t)$ are

$$w_i(t) = \begin{vmatrix} f_0(t) & f_0'(t) & \cdots & f_0^{(h+1)}(t) \\ \cdot & \cdot & \cdot & \cdot \\ f_h(t) & f_h'(t) & \cdots & f_h^{(h+1)}(t) \\ f_i(t) & f_i'(t) & \cdots & f_i^{(h+1)}(t) \end{vmatrix}.$$

Applying Sylvester's theorem (2) to $w_i(t)$ (using $m=h+1$) we see that

$$\Delta w_i(t) = z_i(t),$$

where

$$\Delta = \begin{vmatrix} f_0(t) & \cdots & f_0^{(h-1)}(t) \\ f_{h-1}(t) & \cdots & f_{h-1}^{(h-1)}(t) \end{vmatrix}.$$

Since $\Delta\neq 0$, it follows that $P^{h+1}(t)=Q^{h+1}(t)$. Hence $C^h=D^h$ for $h=0, \cdots, n-1$.

We may sum up our results in the

THEOREM. *Let in a projective space S_n an increasing sequence of subspaces*

$$S_1 \subset S_2 \subset \cdots \subset S_{n-1} \subset S_n$$

be given. Let α_{hk} be the operation of passing from a curve C in S_k (but not in S_{k-1}) to a curve in S_h $(h<k)$ by intersecting S_h with the $(k-h)$-dimensional osculants of C. If C is a curve in S_n for which each $\alpha_{hn}C$ exists and is not in S_{h-1}, then

$$\alpha_{hh_1}\alpha_{h_1h_2}\cdots\alpha_{h_rn}C = \alpha_{hn}C, \tag{4}$$

where $h<h_1<\cdots<h_r<n$.

By our analysis,

$$\alpha_{hn}C = \alpha_{h,h+1}\alpha_{h+1,h+2}\cdots\alpha_{n-1,n}C,$$

and by induction the left member of (4) is shown to have the same value.

EDITED BY J. S. FRAME

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to J. S. Frame, Michigan State College, East Lansing, Michigan.

A NOTE ON SCALES OF NOTATION

H. D. LARSEN, University of New Mexico

If $1+i$ is taken as the base of a number system, the resulting representations bear some interesting relationships to the mathematics of finance. A number written in the scale of $1+i<2$ properly makes use of two digits, 0 and 1. The positional value of each digit is determined by the definition,

$$a_n a_{n-1} \cdots a_1 a_0 . a_{-1} a_{-2} \cdots a_{-m} = \sum_{j=-m}^{n} a_j(1+i)^j, \qquad a_j = 0 \quad \text{or} \quad 1.$$

As a matter of notation, let the index (k) above a digit indicate that the digit is repeated k times; thus, $1^{(2)}0^{(3)}.1^{(2)}=11000.11$. The following relationships involving the four fundamental compound interest functions result immediately:

$$(1) \qquad (1+i)^n = 10^{(n)};$$

$$(2) \qquad (1+i)^{-n} = .0^{(n-1)}1;$$

$$(3) \qquad s_{\overline{n}|} = 1^{(n)};$$

$$(4) \qquad a_{\overline{n}|} = .1^{(n)}.$$

The representation of numbers in the scale of $1+i$ effects a novel method of proving many of the theorems in the mathematics of finance, making use of the various properties of such a number system. In particular, it should be noted that a number in the scale of $1+i$ is multiplied by $(1+i)^n=10^{(n)}$ if the separatrix is moved n places to the right, and is multiplied by $(1+i)^{-n}=.0^{(n-1)}1$ if the separatrix is moved n places to the left. One example should suffice to illustrate the method of proof:

$$\begin{aligned} s_{\overline{n}|}(1+i)^m &= 1^{(n)} \times 10^{(m)} = 1^{(n)}0^{(m)} \\ &= 1^{(n+m)} - 1^{(m)} = s_{\overline{n+m}|} - s_{\overline{m}|}. \end{aligned}$$

This method of proof may have value in suggesting different proofs for certain theorems in the mathematics of finance. Thus, through this device the author developed an elegant proof for the formula,

$$a_{\overline{1}|} + a_{\overline{2}|} + \cdots + a_{\overline{n}|} = (n - a_{\overline{n}|})/i.$$

Further relationships might be mentioned. Circulating decimals in the scale

of $1+i$ are connected with the theory of perpetuities, while the usual algorithms for converting $s_{\overline{n}|}$ and $a_{\overline{n}|}$ from the scale of $1+i$ to the decimal scale are readily identified with the sinking-fund and amortization schedules. Additional relationships are left for the interested reader to discover for himself.

CLUB REPORTS 1942–43

Pi Mu Epsilon, Washington University

Due to unsettled conditions at Washington University the regular monthly meetings for the presentation of papers were dispensed with. The annual banquet and initiation of new members was held on December 13, 1942, when twenty-nine candidates were initiated. An address was heard on the subject:

Geophysical prospecting, by Professor S. H. Van Wambeck of the Electrical Engineering Department.

Officers for the year were: Director, James Embree; Vice-Director, Cecilia Lehman; Secretary, Kenyon Hammack; Treasurer, Professor Eugene Stephens; Corresponding Secretary, Dr. Jessica Young Stephens.

Kappa Mu Epsilon, Albion College

Each of the regular monthly meetings opened with a roll call, requiring those present to name in turn: a formula in calculus (November 10), a mathematician and his outstanding work (December 8), a theorem in geometry (January 12), a formula in trigonometry (March 9), and a formula in analytic geometry (April 13). Titles for the first three meetings were:

Number bases, by Richard Hadley.
The mil and its applications, by Alice Gibb.
Development of symbols of fundamental operations, by Betty Davis.

The topic following the initiation of new members on February 9 was:

Nomographic charts, by Dr. Ingalls.

At the March meeting several five minute talks were given by new members:

Early American arithmetics, by Virginia Tripp.
Mathematics in war, by Russell Miers.
Calculus of Newton, by Barbara Manley.
Napier's logarithms, by Betty Hossfeld.
Mathematics in war, by John Barcroft.
Algebraic congruences, by Helen Shephard.

In April, there was a round table discussion entitled

Science and Mathematics in War, including: *Mathematics*, by Dale Buerstetta; *Physics*, by Earl Dinger; *Chemistry*, by Howard Sherman.

The annual picnic was held in May. Officers for the year were: President, Clare Stanford; Vice-President, James Hollingsworth; Secretary-Treasurer, Elizabeth Davis.

DISCUSSIONS AND NOTES

Edited by Marie J. Weiss, Sophie Newcomb College, New Orleans, La.

The department of Discussions and Notes is open to all forms of activity in collegiate mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

RECIPROCAL EQUATIONS

W. V. Lovitt, Colorado College

The following is a simple proof of a formula given by Prof. L. E. Dickson.* The proof requires a knowledge of the theorem in finite differences that the rth difference of the polynomial

$$n^r + a_1 n^{r-1} + \cdots + a_r$$

is $r!$.

If we put $x+x^{-1}=z$ and $x^p+x^{-p}=V_p$, we have the well known recursion formula $V_{n+1}=V_n z - V_{n-1}$. We have†

$$\begin{aligned}
V_1 &= z\\
V_2 &= z^2 - 2\\
V_3 &= z^3 - 3z\\
V_4 &= z^4 - 4z^2 + 2\\
V_5 &= z^5 - 5z^3 + 5z\\
V_6 &= z^6 - 6z^4 + 9z^2 - 2\\
V_7 &= z^7 - 7z^5 + 14z^3 - 7z\\
V_8 &= z^8 - 8z^6 + 20z^4 - 16z^2 + 2\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots .
\end{aligned}$$

The recursion formula establishes that, neglecting signs, the first differences of the coefficients in any vertical column give the coefficients in the preceding column. All of the coefficients in the first column are $+1$. Hence, the coefficients of the $(r+1)$th terms of the V_n $(n>0)$ are a series whose rth differences are unity. Obviously

$$V_n = z^n - nz^{n-2} + \cdots + (-1)^r P_r z^{n-2r} + \cdots ,$$

where P_r stands for a polynomial in n of degree r.

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* L. E. Dickson, Elementary Theory of Equations, p. 83, John Wiley, 1941. For another form of the coefficients in this formula see an article by S. F. Bibb, this Monthly, May, 1943, p. 318.

† Lovitt, Elementary Theory of Equations, p. 88, Prentice Hall, 1939.

It remains only to construct these polynomials of degree r so that their rth differences are $+1$, and for the specified values of n the coefficients in column $r+1$ are reproduced. Since the rth difference of P_r is $+1$, we have

$$P_r = (n^r + a_1 n^{r-1} + \cdots + a_r) \div r!.$$

It remains to determine the coefficients a_i. This is accomplished by solving r linear equations in r unknowns.

Let us determine P_3. We have for

$$\begin{aligned} n = 6: &\quad 6^3 + 36a_1 + 6a_2 + a_3 = 3!(-2), \\ n = 7: &\quad 7^3 + 49a_1 + 7a_2 + a_3 = 3!(-7), \\ n = 8: &\quad 8^3 + 64a_1 + 8a_2 + a_3 = 3!(-16). \end{aligned}$$

Whence

$$P_3 = \frac{1}{3!}(n^3 - 9n^2 + 20n) = \frac{n(n-4)(n-5)}{3!}.$$

In like manner

$$P_2 = \frac{n(n-3)}{2!}.$$

To an observant student the pattern should now be clear. The numerator for P_r contains r factors the first of which is n. Omit the next r numbers namely $(n-1)$ $(n-2)$ $\cdots$ $(n-r)$. We have

$$V_n = z^n - nz^{n-2} + \frac{n(n-3)}{2} z^{n-4} - \frac{n(n-4)(n-5)}{3!} z^{n-6} + \cdots$$
$$+ (-1)^r \frac{1}{r!} n[n-(r+1)][n-(r+2)] \cdots [n-(2r-1)] z^{n-2r} + \cdots.$$

To obtain the coefficients in column $r+1$, use the coefficient of z^{n-2r} and begin the computation with $n=2r$. The proof is completed by induction. In this induction process one must take note of the following: the first difference of P_r is not P_{r-1} but

$$\frac{1}{(r-1)!}(n-1)[(n-1)-r][(n-1)-(r+1)] \cdots [(n-1)-(2r-3)]$$

which is, however, what P_{r-1} becomes if n is replaced by $n-1$. This will reproduce the preceding column (rth column) of coefficients if we start the computation with an integer n such that $n-1=2r$.

THE ROOTS OF A CAYLEY NUMBER

ROY DUBISCH, Montana State University

It has long been known that the three systems of the complex numbers, the quaternions, and the Cayley numbers possess many points of similarity in spite of the fact that the complex numbers form an associative and commutative system, the quaternions an associative but non-commutative system, and the Cayley numbers a non-associative and non-commutative system. Furthermore, these three algebras are unique among all algebras in that they are the only algebras for which the norm is a sum of squares and the norm of a product equals the product of the norms of the factors.* Thus, there is some justification in showing that the method of De Moivre for finding roots of complex numbers which has been extended by Brand† for finding roots of quaternions may also be used without any serious modification for finding roots of Cayley numbers.

Cayley defined his algebra in terms of eight units $1, i_1, \cdots, i_7$ such that $i_1^2=-1, \cdots, i_7^2=-1$,

$$i_1i_2 = i_3 = -i_2i_1, \qquad i_2i_3 = i_1 = -i_3i_2, \qquad i_3i_1 = i_2 = -i_1i_3,$$

and six similar sets of six relations with 1, 2, 3 replaced by 1, 4, 5; 6, 2, 4; 6, 5, 3; 7, 2, 5; 7, 3, 4; 1, 7, 6; respectively. We may then proceed as Brand did to write any Cayley number, $Z=a_0+a_1i_1+\cdots+a_7i_7$ as

$$Z = k(\cos\theta + \epsilon\sin\theta), \qquad (0 \leqq \theta < 2\pi),$$

where $k=\sqrt{a_0^2+\cdots+a_7^2}$,

$$\cos\theta = \frac{a_0}{k}, \qquad \sin\theta = \frac{\sqrt{a_1^2+\cdots+a_7^2}}{k},$$

and, in case $a_1^2+\cdots+a_7^2\neq 0$, ϵ is the unit vector

$$\epsilon = \frac{a_1i_1+\cdots+a_7i_7}{\sqrt{a_1^2+\cdots+a_7^2}}$$

while, in the contrary case, ϵ may be chosen at pleasure.

We may then verify that $\epsilon^2=-1$ so that one may proceed exactly as for the quaternion problem to obtain the

THEOREM: *A non-real Cayley number has exactly n nth roots. If Z is real and positive, it has just two square roots,* $\pm\sqrt{Z}$; *in all other cases a real Cayley number has infinitely many roots.*

Finally, we may note that the roots of a Cayley number may be computed by the use of formulas (4) and (5) of Brand's paper.

* See, for example, L. E. Dickson, Quaternions and their generalizations and a history of the eight square theorem, Annals of Mathematics, Vol. 20, 1919, pp. 155–171.

† This MONTHLY, Vol. 49, 1942, pp. 519–520.

RECENT PUBLICATIONS

EDITED BY VIRGIL SNYDER, Cornell University, and H. P. EVANS, University of Wisconsin

All books for review should be sent directly to the editors of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y., and not to any of the other editors or officers of the Association.

Plane and Spherical Trigonometry. Alternate Edition. By L. M. Kells, W. F. Kern, and J. R. Bland. New York & London, McGraw-Hill Book Company, Inc., 1943. 17+400 pages $2.00 (with tables $2.75).

Two editions of the same title and a smaller volume entitled *Spherical Trigonometry with Naval and Military Applications* by the same authors have been published and the reviews have appeared in this MONTHLY.*

All the good things that were written about those volumes are emphatically true about this one and will not be repeated here. The authors have naturally drawn upon their preceding work for many exercises and problems, but have added a substantial number of new and up-to-date ones in this edition. The reader will not fail to notice that the definitions of terms used in formulating the problems have been made with unusual care and clarity. The present volume also surpasses its predecessors, and I might add its competitors in the field, in the abundance of well drawn figures to accompany many of the problems to be solved by the students. Some teachers will welcome the early introduction to the use of the slide-rule and the more frequent reference to that technique in the rearrangement of the topics. In view of the apparent strength in the development of the numerical and computational aspect of trigonometry, I don't understand why sec. 28 on accuracy of computation in the previous edition was left out of this one. Even if that topic is discussed at length in the class room, most students want a correct and authoritative statement of it in their mathematics textbook.

The analytic aspect of trigonometry is on the whole well presented. Again from the abundant sets of equations, identities, and other types of problems the instructor can choose enough and of sufficient difficulty to condition the mental sinews of his best students. In common with practically all other trigonometry textbooks this one contains such statements as $\tan 90° = \infty$, which are not true, in preference to $\tan 90°$ does not exist, which is true. Also, geometrically minded instructors will regret that the authors did not make use of line values of trigonometric functions, at least in preparation of graphs.

These criticisms detract very little from the all around excellence of the book, which is undoubtedly one of the very best in the field.

GEORGE SAUTÉ

* First Edition and Second Edition reviewed by J. M. Feld in this MONTHLY, vol. 43, 1936, p. 39 and vol. 47, 1940, p. 703. *Spherical Trigonometry with Applications* was reviewed by G. L. Walker in vol. 50, 1943, p. 198.

Engineering Problems Illustrating Mathematics. A project of the mathematics division of the Society for the Promotion of Engineering Education. By J. W. Cell. New York and London, McGraw-Hill, 1943. 11+172 pages. $1.75.

Here is a little book that every teacher of mathematics from college algebra through calculus would do well to have on his desk. The book is a collection of problems comparable in difficulty with the standard problems in the ordinary American textbooks of mathematics, but couched in the language of engineering and physics. It is not intended that the book be put in the hands of all students. In fact the compilers take a most sensible view and state "This collection will be misused if there is any attempt to teach junior or senior engineering concepts in the freshman or sophomore mathematics class room." On the other hand, the collection will also be misused if it leads the student to believe that mathematics is studied only for its utility in solving engineering problems.

Almost all of the problems have been stated in such a way that no understanding of physical terms is necessary for the mathematics involved. After all, should not an engineer be able to slough off all superfluous structure and obtain a mathematical problem which he can attack unhampered by his physical interpretation of each symbol? This book will certainly help the student to develop such an ability. There is of course the danger that a student will be fooled into thinking he knows (or thinking he can fool someone else into thinking he knows) some engineering just because he can glibly use such terms as ohms, henrys, potential energy, or shear curve.

The accuracy of statement and numerical work were checked by sampling and found to be quite satisfactory. Many problems do not give all of the units involved but this is good engineering practice. The five parts are labeled Algebra, Trigonometry, Analytic Geometry, Differential Calculus, and Integral Calculus, respectively. Answers are given to all problems. The figures are well drawn and the printing is excellent. Thanks for a mathematics book printed in war time without a trace of war flavor.

J. F. RANDOLPH

Lessons in Elementary Analysis. By G. S. Mahajani. Third Edition. Poona, India. Aryabushan Press, 1942. 13+298 pages.

The second edition (1931) of this book was reviewed in the MONTHLY by H. R. Cooley in May 1938. The essential features of that edition have been retained. The following changes have been introduced, in addition to alterations in the exercises at the end of each chapter.

The proof of the second mean value theorem for integrals is now that of Landau. Notes on Frullani integrals and on the general form of the remainder in the Taylor series are added. The chapters on infinite integrals and on uniform convergence are rewritten as Chapters X (pp. 185–213) and XI (pp. 214–238). The excellence of the style and of the appearance of the book has been retained.

VIRGIL SNYDER

Calculus. By L. M. Kells. New York, Prentice-Hall, Inc., 1943. 7+509 pages. $3.75.

This is the kind of book any one who knows Professor Kells would expect from him. It has been written with care and with considerable pedagogical insight. There is an abundance of instructive illustrative examples which add much to the value of the book as a text for the student beginning the study of the Calculus. They give him an insight into the wide range of problems whose solution depends upon the methods of the Calculus and stimulate his interest.

In addition to the text of 361 pages there is a frontispiece showing Sir Isaac Newton experimenting with the solar spectrum, a table of integrals, four-place tables of common and natural logarithms, a table of trigonometric functions and one of values of e^x and e^{-x}.

There are a number of matters of detail which should be mentioned. On page 97 in discussing Newton's Method of approximating a root of an equation the author says that the tangent to the curve will cut the x-axis at a point whose abscissa is nearer a root than the abscissa of the point of contact. This statement requires some qualification. The statement on page 106 that certain inequality signs should be reversed if θ is negative is wrong. The reference in the footnote on page 152 to a hyperbola should be to an equilateral hyperbola. In the first line of page 162 the word *differential* should be replaced by the word *infinitesimal.*

In the proof of Rolle's Theorem on page 187 it is not necessary to assume the existence of a derivative at the ends of the interval. The same remark applies to the proof of the law of the mean. The subscript of y_l in the last line of page 191 should be deleted. The form in the heading of §74, page 194, should be $0\times\infty$, and not 0×0. In §75 there should be some explanation, not necessarily a proof, of the statement that $\lim_{x\to a} (\log y) = \log (\lim_{x\to a} y)$. In the inequalities at the top of page 228 the middle term should be x. In line 3, page 229, the statement should be "s represents the sum of the rectangles with their upper bases below or on the curve." As to line 15, page 230, it should be observed that the law of the mean does not require that the derivative of the function be continuous.

There is a little ambiguity in the definition of work in §100 since it is not explicitly stated that the motion is due exclusively to the force in question. There is a good discussion of the tangent plane to a surface on pages 359 and 360, but a poor definition of the envelope to a one-parameter family of plane curves on page 362. An equality sign is omitted in equation (10), page 374. It is by no means obvious as implied on page 375 that a double integral is equal to an iterated integral. Although no formal proof of this has a place in an introductory text attention should be called to the fact that a proof is necessary. On page 378 reference is made to the distance between two points when clearly it is the square of this distance that is meant.

There is no definition of the area of a curved surface in §156 although a formula is derived for this area. In the definition of convergence of an infinite series on page 400 the word "unique" should be deleted, since if there is a limit

it is unique. The symbol "$<$" in line 10 from the bottom of page 409 should be replaced by the symbol "$\leqq$," and in line 3, page 411, k should be restricted to being a positive integer. The error mentioned in line 6, page 423 should be designated by ϵ, and not by e. The student will have difficulty in seeing the reason for the statement on line 15, page 433, that neither of the values in question gives a maximum or a minimum value of $f(x)$, and the reason for the statement on lines 12 and 13, page 454, is not obvious.

I have here pointed out a number of things that seem to me to need correction. But many of these are due to obvious misprints and even those that are not can be easily corrected in a second printing, or in the class room by an alert teacher. As it stands the book is an unusually good one and when and if these changes are made it will be still better.

W. B. FITE

Grösse, Masszahl und Einheit. By Max Landolt. Zürich, Rascher & Cie., 1943. 85 pages.

This booklet is concerned with the theory of computations involving *physical magnitudes*, *measure numbers*, and *physical units*. Part I (pp. 11–34) consists mainly of a collection of typical numerical computations, such as might be included in a well-written college physics text. Part II (pp. 37–85) is a formalization of such computations in the notation of abstract groups. Magnitudes of the same kind are considered to combine "intensively" to form "intensive" groups (*e.g.*, 2 *gm.*$+$3 *gm.*$=$5 *gm.*). Magnitudes of different kinds are considered to combine "qualitatively" to form a "qualitative" group (*e.g.*, a certain mass$\times$a certain acceleration$=$a certain force). The theory is developed that physical computations consist of combinations of these two types of group-operations.

Part I shows clearly the practical significance of *units* as they are involved in physical computations. Part II begins with a brief discussion of the conceptual difficulties inherent in the operations of multiplying and dividing unlike quantities, and concludes with an interpretation of the group-operations as operations of arithmetic. Apart from the pedagogic value of Part I, the booklet is largely of theoretic interest.

C. C. TORRANCE

Navigational Trigonometry. By P. R. Rider and C. A. Hutchinson. New York, Macmillan Company, 1943. 232 pages. $2.00.

This book, a revision and expansion of an earlier text, is an outgrowth of the war need for more materials on surface and aerial navigation. The text is replete with definitions, problem materials, and methods of solution. It is not a book of theory but of practice. Its illustrations are sufficient and clear. With its practical problems, its five place tables of logarithmic haversines should make it a very usable text in navigational trigonometry.

WILLIAM MELCHER

Elementary Statistical Methods. By Helen M. Walker. New York, Henry Holt and Co., 1943. 25+368 pages. $2.50.

This is a textbook on statistics written for students with extremely meager mathematical backgrounds. It is quite generally believed by mathematicians that people who do not understand mathematics cannot be trusted to apply it and there is little doubt that an understanding of the derivations tends to produce a much more intelligent use of the results. However the fact remains that people with inadequate mathematical training are going to use statistics and the most we can hope is that these people will receive the best possible training consistent with their abilities. Thus the question is whether this textbook is a competent piece of work for what it attempts to do. In the opinion of the reviewer the answer is decidely in the affirmative. The author herself evidently has a clear understanding of statistical theory but she is also capable of taking the point of view of the students who cannot follow the various mathematical derivations.

The amount of mathematics contained in this book is about average for a first course in statistics. There is the difference, however, that the more difficult developments appear in the appendix. If the student is capable of reading the appendix, he is encouraged to do so. Otherwise he is advised to accept the results on authority. It seems likely that the mathematically trained students would attack the appendix with added zest after having seen the results and their applications.

The following list of topics treated in the book will give some idea of the scope: significant digits, class interval, frequency distribution, percentiles, averages, measures of variability, skewness, kurtosis, moments, the normal distribution, regression, correlation, sampling, randomness, and biased sampling. In the development of these topics the emphasis is placed on interpretation rather than on computation or methods of computing. The author makes the following comment. "Confronted with a set of raw data, the novice is tempted to ask himself, 'What measures have I learned to compute which I might make use of in respect to these data?' This is exactly the wrong way around. He ought to say, 'For what questions would I like to have these data provide the answers?'" Following this comment is a list of questions which might be asked of a specific set of data. The development of the methods of answering such questions constitutes the major portion of the book.

The reviewer feels that the mathematical theory of sampling has been somewhat slighted and that additional space in the appendix might well be devoted to this subject. One further criticism. The following statement on page 271 is misleading. "It can be proved mathematically that the sampling distribution of the mean is characteristically normal, and the frequency distribution of any observed set of means obtained from random samples of uniform size will tend toward the normal if enough samples are taken. Note that it is the *number of samples* which must be increased in order to produce this result, not the *number of cases in a single sample.*" The author refers to samples of fifty taken from a

large but finite parent population. Under such circumstances the normal distribution cannot be obtained as a limit by increasing the number of samples. Moreover the size of the samples does affect the degree of approximation to normality.

These criticisms however are minor. On the whole the book is very readable, it is intelligently written and should prove to be a useful text.

A. H. COPELAND

NEW BOOKS RECEIVED

Analytic Geometry. By K. B. Patterson and A. O. Hickson. New York, Crofts, 1944. 187 pages. $2.10.

An Introduction to Navigation and Nautical Astronomy. By W. G. Shute, W. W. Shirk, G. F. Porter, and Courtney Hemenway. New York, The Macmillan Company, 1944. 14+457 pages. $4.50.

Basic Air Navigation. By E. F. Blackburn. New York, McGraw-Hill Book Company, Inc., 1944. 7+300 pages. $3.00.

Basic Mathematics for War and Industry. By P. H. Daus, J. M. Gleason, and W. M. Whyburn. New York, The Macmillan Company, 1944. 11+277 pages. $2.00.

Compact Tables. Five Place Logarithms. Second Edition. By J. P. Ballantine. Seattle, Ballantine, 1944. $0.50.

Logarithms of Numbers. Trigonometric Functions. Compact Tables. By J. P. Ballantine. Seattle, Ballantine, 1944. $0.25.

Mathematics for Navigators. By Delwyn Hyatt and B. M. Dodson. New York and London, McGraw-Hill Book Company, Inc., 1944. 7+106 pages. $1.25.

"Student's" Collected Papers. Edited by E. S. Pearson and John Wishart with a foreword by Launce McMullen. London, University College, Biometrika office, Cambridge University Press, 1943. 14+224 pages. 15s.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND H. S. M. COXETER

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to H. S. M. Coxeter, 24 Strathearn Boulevard, Toronto 10, Canada.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 621. *Proposed by C. D. Olds, Purdue University*

Show directly (*i.e.*, without substituting θ into the value given by the standard algebraic proofs) that

$$\frac{1}{a+}\;\frac{1}{a+}\;\cdots = e^{-\theta},$$

where a is a positive number and $\theta = \text{arg sinh } (a/2)$.

E 622. *Proposed by V. Thébault, San Sebastián, Spain*

Find the smallest four-digit number such that the sum of products of pairs of digits is equal to the sum of products of sets of three.

E 623. *Proposed by N. A. Court, University of Oklahoma*

The circumsphere of a tetrahedron $ABCD$ meets four "cevians" LA, LB, LC, LD in the points A', B', C', D'; $A''B''C''D''$ is a tetrahedron homothetic to the tetrahedron $A'B'C'D'$ with respect to the point L. If P and Q are any two points in space, show that the four spheres $A'A''PQ$, $B'B''PQ$, $C'C''PQ$, $D'D''PQ$ are coaxal.

E 624. *Proposed by D. H. Browne, Buffalo, N. Y.*

Show that the integer nearest to $n!/e$ is a multiple of $n-1$.

E 625. *Proposed by C. A. B. Smith, Trinity College, Cambridge, England*

Let X_i, Y_i, Z_i be the cofactors of x_i, y_i, z_i in the general third-order determinant

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = D.$$

(*Cf.* E 481 [1942, 257–260].) Prove that

$$\begin{vmatrix} X_2X_3 & Y_2Y_3 & Z_2Z_3 \\ X_3X_1 & Y_3Y_1 & Z_3Z_1 \\ X_1X_2 & Y_1Y_2 & Z_1Z_2 \end{vmatrix} = -D^2 \begin{vmatrix} x_2x_3 & y_2y_3 & z_2z_3 \\ x_3x_1 & y_3y_1 & z_3z_1 \\ x_1x_2 & y_1y_2 & z_1z_2 \end{vmatrix}.$$

SOLUTIONS

Placing the Last Digit First

E 584 [1943, 454]. *Proposed by C. O. Oakley, Haverford College*

(a) Find the smallest integer N such that, if the units digit is transposed from right to left, a number M is obtained where $M = 5N$.

(b) Find the largest integer which satisfies these conditions (without complete repetition of digits).

Solution by R. K. Allen, Montpelier, Vermont. This question has been completely answered by W. B. Chadwick [1941, 251–253]. The smallest number is $(10^6-1)/7=142857$, and the largest is

$$9(10^{42}-1)/49 = 183673469387755102040816326530612244897959,$$

these being the repetends of the decimal equivalents of $1/7$ and $9/49$, respectively.

Also solved by W. E. Buker, J. E. Eaton, Irving Kaplansky, S. Karlin, Walter Penney, Milton Schiffenbauer, E. P. Starke, and the proposer.

A Quartic Equation

E 588 [1943, 512]. *Proposed by J. Rosenbaum, Bloomfield, Connecticut*

Solve the equation

$$x^4 - 2cx^3 + (2c^2 - r^2)x^2 + 2cr^2x - c^2r^2 = 0.$$

When will two of the roots be rational? Can all four roots be rational?

Solution by E. P. Starke, Rutgers University. The proposed equation can be written at once

$$(1) \qquad (x^2 - cx + c^2)^2 = (r^2 + c^2)(x - c)^2.$$

Put

$$(2) \qquad s = \sqrt{r^2 + c^2} = (x^2 - cx + c^2)/(x - c),$$

which is rational whenever x is. Then $x^2-cx+c^2=sx-sc$, and

$$(3) \qquad x = \tfrac{1}{2}(s + c) \pm \tfrac{1}{2}\sqrt{(s + c)(s - 3c)},$$

which is rational if and only if $(s+c)(s-3c)=k^2$ is a perfect square. Put $s+c=\rho u^2$, $s-3c=\rho(k/\rho u)^2=\rho v^2$. Then, since $r^2=s^2-c^2$,

$$s - c = \rho(r/\rho u)^2 = \rho w^2.$$

Elimination of s and c from these equations gives

$$(4) \qquad u^2 + v^2 = 2w^2,$$

for which the solutions in integers are known, *viz.*,

$$(5) \qquad u = \sigma(a^2 + 2ab - b^2), \qquad v = \sigma(a^2 - 2ab - b^2), \qquad w = \sigma(a^2 + b^2).$$

We have, then, by substitution,

$$(6) \qquad c = 2\rho\sigma^2 ab(a^2 - b^2), \qquad r = \rho\sigma^2(a^2 + b^2)(a^2 + 2ab - b^2).$$

These are integral values of c and r for which two roots of (1) are rational.

The other two roots of (1) are obtained by using, in place of (2),

$$(2') \qquad s = -(x^2 - cx + c^2)/(x - c).$$

The strictly analogous argument leads to

$$(3') \qquad x = \tfrac{1}{2}(-s+c) \pm \tfrac{1}{2}\sqrt{(s-c)(s+3c)};$$

$$s + c = \rho u^2, \qquad s - c = \rho w^2, \qquad s + 3c = \rho z^2;$$

$$(4') \qquad w^2 + z^2 = 2u^2;$$

$$(5') \qquad z = \tau(p^2 + 2pq - q^2), \qquad w = \tau(p^2 - 2pq - q^2), \qquad u = \tau(p^2 + q^2);$$

$$(6') \qquad c = 2\rho\tau^2 pq(p^2 - q^2), \qquad r = \rho\tau^2(p^2 + q^2)(p^2 - 2pq - q^2).$$

If all four roots are to be rational, we must have simultaneously

$$s + c = \rho u^2, \qquad s - c = \rho w^2, \qquad s + 3c = \rho z^2, \qquad s - 3c = \rho v^2,$$

which give both (4) and (4'). But these equations can be put in the form $2w^2 - u^2 = v^2$, $2u^2 - w^2 = z^2$, which imply $w^2 = u^2 = v^2 = z^2$. (See W. T. King, this MONTHLY, vol. 6, 1899, pp. 151–155: a discussion which can be radically simplified.) But then $c = 0$, and (1) has the four rational roots $x = 0, 0, r, -r$. No other set of four rational roots is possible.

Also solved by Hazel Schoonmaker Wilson and the proposer.

Two Generalizations of the Problème des Rencontres

E 589 [1943, 512]. *Proposed by Lloyd Dulmage, University of Manitoba*

Let ${}_2H_n$ denote the number of ways of putting n letters and n checks into n envelopes (respectively) so that every envelope contains either a wrong letter or a wrong check (or both), and let ${}_2J_n$ denote the number of ways when each envelope contains both a wrong letter and a wrong check. Show that

$${}_2H_n = \Delta^n(0!)^2, \qquad {}_2J_n = (\Delta^n 0!)^2,$$

and generalize to ${}_sH_n$, ${}_sJ_n$.

Solution by John Riordan, Bell Telephone Laboratories. In the general case, this is a problem of enumerating coincidences in sets of permutations of n elements (or packs of cards labelled 1 to n): $s+1$ permutations of n elements are placed under one another, and a count is made of positions in which elements are alike.

The total number of configurations is $n!^s$; for, each of the permutations except a standard one may be selected in $n!$ ways. If ${}_sH_n$ is the number with no coincidences, the number with i concidences is

$$\binom{n}{i} {}_sH_{n-i};$$

for, the positions of coincidence may be selected in $\binom{n}{i}$ ways, and the remainder must have no coincidences. Thus

$$n!^s = \sum_{i=0}^{n} \binom{n}{i} {}_sH_{n-i} = (1 + {}_sH)^n.$$

But

$$n!^s = E^n 0!^s = (1 + \Delta)^n 0!^s,$$

where E and Δ are the usual finite difference operators. Comparing coefficients,

$$_sH_n = \Delta^n 0!^s.$$

In particular, $_1H_n = \Delta^n 0!$. This is the well known solution for the Problème des Rencontres, the number of permutations on any one line having no coincidences with the standard. Hence the number of configurations in which each of the s lines has no coincidences with the standard (though they may coincide with one another) is

$$_sJ_n = ({}_1H_n)^s = (\Delta^n 0!)^s.$$

It may be noted that, according to Todhunter, the first result has been given by De Moivre, *Doctrine of Chances* (Problem XXXVI, 6th corollary); it also appears in Laplace's *Théorie Analytique des Probabilités* (National Edition, pp. 244–245). The numbers $_sH_n$ have interesting arithmetical properties, such as

$$_sH_{n+p} \equiv -\, {}_sH_n \pmod{p}, \qquad {}_{s+p-1}H_n \equiv {}_sH_n \pmod{p},$$

for any prime p. They also satisfy the recurrence relation

$$_sH_{n+1} = (-1)^{n+1} + (n+1)\sum_{i=0}^{s-1}\binom{n}{i}\Delta^i(n+1-i)^{s-1}{}_sH_{n-i};$$

e.g.,

$$\begin{aligned} {}_1H_{n+1} &= (-1)^{n+1} + (n+1)\,{}_1H_n, \\ {}_2H_{n+1} &= (-1)^{n+1} + (n+1)^2\,{}_2H_n + n(n+1)\,{}_2H_{n-1}. \end{aligned}$$

Two Sequences of Perfect Squares

E 592 [1943, 560]. *Proposed by V. Thébault, San Sebastián, Spain*

Show that the numbers

$$729,\ 71289,\ 7112889,\ 711128889, \cdots,$$

$$7744,\ 97970404,\ 997997004004,\ 9997999700040004, \cdots$$

are perfect squares. Generalize the latter sequence to an arbitrary scale of notation (with base greater than 4).

Solution by E. D. Schell, Arlington, Va. The general term of the first sequence is shown to be a square as follows:

$$\begin{aligned} 7\cdot 10^{2n} + (10^{2n-1} + \cdots + 10^{n+1}) + 2\cdot 10^n + 8(10^{n-1} + \cdots + 10) + 9 \\ = 7\cdot 10^{2n} + (10^{2n} - 10^{n+1})/9 + 2\cdot 10^n + 8(10^n - 10)/9 + 9 \\ = (64\cdot 10^{2n} + 16\cdot 10^n + 1)/9 \\ = \{(8\cdot 10^n + 1)/3\}^2. \end{aligned}$$

For the second, generalized to any radix $r>4$, we have

$$\begin{aligned}(r-1)(r^{4n-1}+\cdots+r^{3n+1})+(r-3)r^{3n}+(r-1)(r^{3n-1}+\cdots+r^{2n+1})\\ +(r-3)r^{2n}+4r^n+4\\ =r^{4n}-r^{3n+1}+(r-3)r^{3n}+r^{3n}-r^{2n+1}+(r-3)r^{2n}+4r^n+4\\ =r^{4n}+r^{2n}+4-2r^{3n}-4r^{2n}+4r^n\\ =(r^{2n}-r^n-2)^2.\end{aligned}$$

Also solved by D. H. Browne, Stanley Hughart, E. P. Starke, and the proposer.

The Monge Point of a Variable Tetrahedron

E 593 [1943, 560]. *Proposed by N. A. Court, University of Oklahoma*

A variable tetrahedron has three fixed vertices, a fixed circumsphere, and the sum of the squares of its edges is constant. Find the locus of its Monge point.

Solution by Howard Eves, Syracuse University. (Numbers in parentheses refer to articles in N. Altshiller Court's *Modern Pure Solid Geometry.*) Let A, B, C be the fixed vertices, V the variable vertex, O the center of the fixed circumsphere, S the centroid of triangle ABC, G the centroid and M the Monge point of the tetrahedron. Then, by (187), VS is constant, and the locus of V is a circle Σ, which lies on the circumsphere and has SO for axis. Therefore, by (170), the locus of G is a circle with radius one quarter that of Σ and likewise having SO for axis. Finally, by (230), the locus of M is a circle with radius one half that of Σ and again having SO for axis.

Also solved by J. S. Guérin, L. M. Kelly, and the proposer.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known text-books or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4120. *Proposed by F. J. Duarte, Caracas, Venezuela*

Given k and h as positive integers, show that

$$\sum_{n=0}^{h-1}(-1)^n\frac{(h-1)(h-2)\cdots(h-n)}{n!}\cdot\frac{k(k+1)\cdots(k+h-n-1)}{(h-n)!}=\binom{k}{h},$$

where for $n=0$ the summand is $k(k+1)\cdots(k+h-1)/h!$.

4121. *Proposed by Alfred Brauer, University of North Carolina*

In generalization of results of Sylvester and Mirimanoff the following theorem was proved, H. F. Baker, *Proc. London Math. Soc.*, (2) vol. 4, 1906, pp. 131–135.

Let d and m be relatively prime positive integers and $mm' \equiv 1 \bmod d$. Denote by $(m'x)$ the least positive residue of $m'x \bmod d$. Then

$$\frac{d^{\phi(m)}-1}{m} \equiv \sum_x \frac{(m'x)}{m-x} \bmod m,$$

where x runs over the positive integers less than m and relatively prime to m.

Recently, P. Kesava Menon proved the following theorems, *Proc. Indian Academy of Science*, Sect. A, vol. 17, 1943, pp. 107–113.

Let d, m, and n be integers such that I. $dn=m-1$; II. $dn=m+1$. Then we have

$$\text{I.}\qquad \frac{d^{\phi(m)}-1}{m} \equiv \frac{1}{d}\sum_x \frac{\{x/n\}}{x} \bmod m;$$

$$\text{II.}\qquad \frac{d^{\phi(m)}-1}{m} \equiv \frac{1}{d}\sum_x \frac{[x/n]}{x} \bmod m;$$

respectively, where x runs over the positive integers less than m and relatively prime to m, and where $\{a\}$ denotes the smallest integer greater than or equal to a, and $[a]$ denotes the greatest integer less than or equal to a.

Prove that these results are special cases of Baker's theorem.

4122. *Proposed by Otto Dunkel, Washington University*

Show that $\Delta^{n-r}0^n/(n-r)!$, where n and r are non-negative integers, $n \geqq r$ is a polynomial $f_r(n)$ in n of degree $2r$. The polynomial $f_r(x)$, $r \geqq 1$, has a positive integral value for all integral values of x, positive or negative, except for $x=0, 1, 2, \cdots, r-1, r$ for which values it vanishes. If r is an odd integer $\geqq 3$, $r-1$ and r are double roots.

4123. *Proposed by V. Thébault, San Sebastián, Spain*

For what values of m is the product $(2m+1)(10m+1)$ a square of an integer? *Application.* In what systems of numbers is a number of the form $aabb$ the square of a number of two digits bb?

SOLUTIONS

Partitions

4073 [1943, 125]. *Proposed by J. M. Feld, New York*

Let p_1 be the number of partitions of the positive integer N into an odd number of distinct positive integers and let p_2 be the number of partitions of N into an even number of distinct positive integers. For instance, for $N=8$, the

odd partitions included in p_1 are 8, 5+2+1, 4+3+1; and the even partitions included in p_2 are 7+1, 6+2, 5+3. Prove that: (1) If $N=n(3n\pm 1)/2$, where n is an odd integer, $p_1=p_2+1$. (2) If n is an even integer, $p_1=p_2-1$. (3) If N does not have the above form $p_1=p_2$.

Solution by R. C. Buck, Student, Harvard University. This is simply Euler's theorem on p_1 and p_2. (See Hardy & Wright, *Theory of Numbers*, p. 284, Theorem 358.)

This result follows at once from Euler's Identity

$$(1-x)(1-x^2)(1-x^3)\cdots = 1+\sum_1^\infty(-1)^n\left[x^{n(3n+1)/2}+x^{n(3n-1)/2}\right]$$

for the coefficient of x^N on the left is $p_2(N)-p_1(N)$, while on the right it is 0 unless N is of the form $n(3n\pm 1)/2$ when it is $(-1)^n$.

F. Franklin has given a diagrammatic proof of this, reproduced in Hardy-Wright, p. 285.

Solved also by the proposer in a similar manner.

Conic Inscribed in a Triangle

4043 [1942, 408]. *Proposed by H. F. Sandham, Trinity College, Dublin*

Prove that the angle in which the major auxiliary circle of a conic inscribed in a triangle cuts the nine point circle, is equal to the angle which the foci of the conic subtend at the inverse of one of them in the circumcircle. Complete this result and deduce that the minor auxiliary circle of the conic which has the Brocard points as foci, touches the nine point circle, and the major auxiliary circle cuts the latter in an angle which is the complement of three times the Brocard angle of the triangle.

Solution by the Proposer. Taking the radius of the circumcircle (O) as unity the origin at the circumcenter O, the Euler line as the axis of reals, and α, β, γ the complex numbers defining the vertices of the triangle, let s_1, s_2, s_3 denote the elementary symmetric functions of these numbers. If x, y denote a pair of isogonal conjugates,

$$x+y+s_3\bar{x}\bar{y}=s_1,\qquad \bar{x}+\bar{y}+xy/s_3=s_1. \tag{1}$$

The center C of the conic inscribed in the triangle with foci at x, y is $(x+y)/2$ and the radius of the major auxiliary circle (C) is $\rho=|1-x\bar{y}|/2$. The center N of the nine-point circle (N) is $s_1/2$ and its radius is $1/2$. Let θ denote the angle between the circles (N) and (C), then

$$\begin{aligned}|1-x\bar{y}|\cos\theta &= (1-x\bar{y})(1-\bar{x}y)/4+1/4-(x+y-s_1)(\bar{x}+\bar{y}-s_1)/4,\\ &= (1-x\bar{y})(1-\bar{x}y)/4+1/4-x\bar{x}y\bar{y}/4,\\ &= (1-x\bar{y}+1-\bar{x}y)/4.\end{aligned}$$

where (1) has been used in the reduction. Hence

$$1 - x\bar{y} = 2\rho e^{i\theta}, \tag{2}$$

and from this the first part of the problem follows.

Calculating the left member of (2) we find that

$$\begin{aligned} 1 - x\bar{y} &= \frac{x^3 - s_1x^2 + s_2x - s_3}{s_3(x\bar{x} - 1)}, \quad \text{or} \\ 2\rho(x\bar{x} - 1)e^{i\theta} &= (x - \alpha)(x - \beta)(x - \gamma)/\alpha\beta\gamma. \end{aligned} \tag{3}$$

Hence θ is the sum of the angles subtended at the vertices by the line joining the point x to O, and from this we have the theorem:

The angle between the pedal circle of a point and the nine-point circle is the complement of the sum of the angles each line joining the point to a vertex makes with an adjacent side, no side being taken twice.

From this the second part of the problem follows.

Editorial Note. The value of ρ may be derived for an arbitrary axis of reals through O as follows. Let y_1 be the orthogonal projection of y on the side $\beta\gamma$ of the triangle, then $4\rho^2 = (2y_1 - x - y)(2\bar{y}_1 - \bar{x} - \bar{y})$ which may be expressed in terms of the complex numbers β, γ; and for the other two sides there are similar expressions. Taking the sum of these gives

$$\begin{aligned} 4s_2\rho^2 = {} & 2(s_1^2 - s_2) - 2s_1(x + y) - (s_1s_2 - 3s_3)(\bar{x} + \bar{y}) + 3xy \\ & + (s_2^2 - 2s_1s_3)\bar{x}\bar{y} + s_2(x\bar{x} + y\bar{y}). \end{aligned} \tag{1}$$

Using the general equations

$$x + y + s_3\bar{x}\bar{y} - s_1 = 0, \qquad s_3(\bar{x} + \bar{y}) + xy - s_2 = 0,$$

we obtain after removing the factor s_2, which is zero only if the triangle is equilateral

$$4\rho^2 = (1 - x\bar{y})(1 - \bar{x}y). \tag{2}$$

This requires long and complicated calculations. Of the two conjugate imaginary factors on the right we may suppose that $1 - x\bar{y}$ is that one with a positive angle θ, $0 \leqq \theta \leqq \pi$. Then $2\rho e^{i\theta} = 1 - x\bar{y}$; and if we denote by y' the complex number for the inverse of the point y in (O) we have $y'\bar{y} = 1$ and then $2\rho e^{i\theta} = (y' - x)/y'$. This gives the theorem in the first part of the problem.

In the formula (3) of the above solution it should be noticed that, if x is inside (O), the quantity $x\bar{x} - 1$ is negative. This formula gives a theorem of the kind stated but it regards the *algebraic* sum of the angles subtended, and it requires considerable care to state the theorem precisely.

If $x \neq y$, it will be seen from the first theorem regarding θ that a necessary and sufficient condition for the tangency of (C) and (N) is that the focal axis xy passes through O. In the limit case $x = y$ we have (C) as the inscribed or an escribed circle. In the two solutions of 3857 [1940, 183] are other proofs of the

theorem that the minor auxiliary circle of the Brocard ellipse is tangent to (N), the nine-point circle of the corresponding triangle.

Limit Functions for Infinite Set of Functions

4058 [1942, 618]. *Proposed by R. P. Agnew, Cornell University*

Give an example of a sequence $f_n(x)$ of real continuous functions, defined over $-\infty < x < \infty$ and vanishing outside the interval $-1 \leqq x \leqq 1$, such that the dominating function $F(x)$ defined by

$$F(x) = \underset{n=1,3,2,\dots}{\text{l.u.b.}} \left| f_n(x) \right| \tag{1}$$

and the inferior and superior limit functions

$$\liminf_{n\to\infty} f_n(x), \qquad \limsup_{n\to\infty} f_n(x) \tag{2}$$

are all continuous and such that the members of the inequality (in which integration is over $-\infty < x < \infty$)

$$\begin{aligned} -\int F(x)dx \leqq \int \liminf_{n\to\infty} f_n(x)dx \leqq \liminf_{n\to\infty} \int f_n(x)dx \\ \leqq \limsup_{n\to\infty} \int f_n(x)dx \leqq \int \limsup_{n\to\infty} f_n(x)dx \leqq \int F(x)dx \end{aligned} \tag{3}$$

assume in order the values in the arithmetic progression $-5, -3, -1, 1, 3, 5$.

Remark: An elegant theorem on "integration of sequences" states that if $f_n(x)$ is a sequence of functions measurable over E_m (Euclidean space of m dimensions) and the dominating function (1) is integrable (Lebesgue), then the functions (2) are integrable and (3) holds. See Caratheodory, *Vorlesungen über Reele Funktionen*, Leipzig, 1927, p. 444. An example meeting the conditions of the problem shows that, even when all functions involved are continuous, the differences between successive members of (3) may all be equal to the constant 2.

Solution by B. H. Bissinger, Cornell University. For $k = 1, 2, 3$ let

$$g_k(x) = \left\{ \begin{array}{ll} 4kx, & 0 \leqq x \leqq \frac{1}{2} \\ -4k(x-1), & \frac{1}{2} \leqq x \leqq 1 \\ 0, & \text{elsewhere} \end{array} \right\}.$$

If we define

$$f_{2n-1}(x) = \left\{ \begin{array}{ll} g_3(x), & x \geqq 0 \\ -g_2(-x), & x \leqq 0 \end{array} \right\}, \qquad f_{2n}(x) = \left\{ \begin{array}{ll} -g_1(x), & x \geqq 0 \\ 0, & x \leqq 0 \end{array} \right\}$$

for $n = 1, 2, \cdots$, then for the sequence $f_n(x)$, the members of the inequality (3) assume the required values.

Solved also by C. M. Ablow and the proposer.

Editorial Note. The proposer gave the following example. Let $\phi(x)$ be the function which vanishes when $x<0$ and when $x>1$, and which has for its graph in the interval $0 \leqq x \leqq 1$ of the x, y plane a broken line running from $(0, 0)$ successively through $(1/3, 0)$, $(1/2, -3)$, $(2/3, 0)$, $(5/6, 6)$ to $(1, 0)$. Let $\psi(x) = -\phi(1-x)$ so that the graph $\psi(x)$ in the interval $0 \leqq x \leqq 1$ is the broken line joining in succession $(0, 0)$, $(1/6, -6)$, $(1/3, 0)$, $(1/2, 3)$, $(2/3, 0)$ and $(1, 0)$. Let $f_n(x) = \phi(x)$ when n is odd, and $f_n(x) = \psi(x)$ when n is even.

Ablow gave the sequence defined as follows;

$$
\begin{aligned}
f_0(x) &= -5(x-1),\ 0 \leqq x \leqq 1: \ = 5(x+1),\ -1 \leqq x \leqq 0; \ = 0,\ |x| > 1;\\
f_{2k-1}(x) &= -(x-1) - 2(x-1)\cos k\pi x, && 0 \leqq x \leqq 1,\\
&= (x+1) + 2(x+1)\cos k\pi x, && -1 \leqq x \leqq 0,\\
&= 0, && |x| > 1\\
f_{2k}(x) &= -f_{2k-1}(x), && k = 1, 2, 3, \cdots .
\end{aligned}
$$

He stated that

$$
\begin{aligned}
\limsup_{k\to\infty} f_{2k-1}(x) &= -3(x-1), \qquad 3(x+1), \qquad 0,\\
\limsup_{k\to\infty} f_{2k}(x) &= -(x-1), \qquad (x+1), \qquad 0
\end{aligned}
$$

for the respective intervals above. His proofs used the theory of point sets. It may appear simpler to consider first rational values of x, and then for irrational values consider the sequence of the convergents p_i/q_i when x is expressed as a simple continued fraction. We then have an easy proof for $\limsup f_{2k-1}(x)$ as stated above by taking $k = 2q_i$. For $\limsup f_{2k}(x)$ where $x = p/q$ is rational and p is an odd integer, we take k as an odd multiple of q. If x is irrational we consider only those convergents p_i/q_i of x for which p_i is an odd integer, and there is an infinite number of such. Here the limit points lie on the curve defined above. But this is not the case where $x = p/q$ is rational and where p is an even integer. Simple examples of such discontinuities in the limit curve in this case are $x = 0,\ 2/3$.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to B. W. Jones, White Hall, Cornell University, Ithaca, New York.

Assistant Professor A. V. Baez of Wagner College has been appointed acting instructor in physics at Stanford University.

Associate Professor H. A. Bender of the University of Akron has been appointed to an associate professorship at Rhode Island State College.

Professor Felix Bernstein of New York University has been appointed professor emeritus.

Assistant Professor H. F. Bright of Denison University has been appointed to an assistant professorship at the University of Rochester.

Dr. F. A. Butter, Jr., of the University of Wisconsin has been appointed lecturer in mathematics at Lawrence College.

Visiting Assistant Professor Nancy Cole of Kenyon College has been promoted to a visiting associate professorship.

Associate Professor J. D. Elder of Lynchburg College has been appointed to an assistant professorship in the physics department of Wabash College, Crawfordsville, Indiana.

Dr. R. W. Hamming of the University of Illinois has been appointed to an assistant professorship at the Speed Scientific School of the University of Louisville.

C. B. Helms of Antioch College is now teaching in the Naval Preflight School at the University of Pennsylvania.

Assistant Professor J. F. Heyda of Denison University has been appointed to an assistant professorship at St. Ambrose College, Davenport, Iowa.

Dr. D. H. Hyers of California Institute of Technology has been appointed to an associate professorship at the University of Southern California.

Assistant Professor R. N. Johanson of Hamilton College has been appointed to an assistant professorship at Boston University.

W. S. Krogdahl has been appointed to an adjunct professorship at the University of South Carolina.

Professor H. M. Mac Neille of Kenyon College is now overseas serving in a civilian capacity.

Professor E. J. McShane of the University of Virginia is on leave of absence to serve as head and mathematician of the Exterior Ballistics Branch of the Ballistic Research Laboratory at Aberdeen Proving Ground.

J. D. Novak of MacMurray College has been appointed to an associate professorship at the University of South Carolina.

Professor Irene Price of the State Teachers College, Oshkosh, Wisconsin, has been granted a leave of absence to serve as statistical analyst in the Army Air Forces.

M. A. Scheier of St. Bonaventure College has been appointed dean of the Division of Natural Sciences and Mathematics.

Dr. James Singer of Brooklyn College has been promoted to an assistant professorship.

Professor Emeritus Harris Hancock of the University of Cincinnati died on March 19, 1944. He was a charter member of the Mathematical Association.

SUMMER COURSES

The following institutions announce courses in mathematics for the summer of 1944:

Brown University. From June 12 to September 2 the program of advanced instruction and research in mechanics will be in its fourth year. Since this program is supported by the United States Office of Education, the Carnegie Corporation of New York, and the Rockefeller Foundation, no fees will be charged. The faculty will consist of Professors Stefan Bergman, Lipman Bers, Will Feller, D. L. Holl, Witold Hurewicz, Willy Prager, J. D. Tamarkin, Dr. R. K. Luneberg and one other still to be chosen. Courses in the following subjects are planned: introduction to partial differential equations, geometrical foundations of mechanics, practical analysis, differential and integral equations of mathematical physics, principles of mechanics, elasticity, plasticity, advanced elasticity, fluid dynamics, mathematical optics, and seminars and research in mechanics and other branches of applied mathematics. A special feature will be a course in mathematical optics under the leadership of Dr. R. K. Luneberg of the Spencer Lens Company whose lectures will be supplemented by Dr. Max Herzberger of the Eastman Kodak Company and Professor J. L. Synge.

The Catholic University of America. The following advanced courses will be offered: From June 13 to September 22. By Professor Finan: selected topics in algebra with applications to geometry. By Professor Ramler: functions of a complex variable. From July 3 to August 10. By Professor Rice: advanced calculus. By Professor Finan: theory of equations, selected topics in algebra and number-theory. By Professor Ramler: advanced Euclidean geometry, differential equations, synthetic projective geometry.

Columbia University. From July 3 to August 11 the following graduate courses will be offered: By Professor Kasner: survey of modern mathematics, differential geometry. By Professor Lorch: introduction to higher algebra. By Professor Murray: theory of functions of a complex variable. By Professor Ritt: differential equations.

Cornell University. From June 30 to October 21 in addition to the usual elementary courses, a course in spherical trigonometry and map projections, a course in differential equations, and certain courses continuing from the present term, the following courses will be offered: By Professor Flexner: topology. By Professor Hurwitz; partial differential equations of mathematical physics.

Northwestern University. From June 26 to August 26 the following advanced courses will be offered: definite integrals, solid analytic geometry, theory of statistics, history and teaching of mathematics, fundamental principles of alge-

bra, a seminar in analysis. Those who desire to do so may take any of these courses for six weeks instead of nine weeks, thus leaving on August 5.

The Ohio State University. From June 13 to September 1 the following advanced courses in mathematics will be offered: By Professor Bamforth: advanced calculus, Fourier series and spherical harmonics. By Professor Blumberg: fundamental ideas in algebra and geometry. By Dr. Mickle: solid analytic geometry. By Professor Rado: introduction to the theory of a complex variable, calculus of variations. By Professor Synge: advanced engineering mathematics II, vector analysis.

The State University of Iowa. From June 12 to August 4 the following graduate courses will be offered: By Professor Conkwright: differential equations. By Professor Knowler: elementary theory of statistics. By Professor Oberg: vector analysis. By Professor Price: modern secondary mathematics. By Professor Ward: matrices and determinants. By Professor Woods: pure geometry. There is an independent study unit for graduate students from August 7 to August 25.

The University of Chicago. The first term of the summer quarter is from June 20 to July 29 and the second term from July 31 to September 9. The following advanced courses will be offered: By Professor Albert: Galois theory. By Professor Barnard: metric differential geometry (second term only). By Professor Hartung: two courses in the teaching of mathematics. By Professor Reid: calculus of variations—multiple integrals, differential equations, integral and functional equations. By Professor Schilling: continuous groups.

The University of Illinois. From June 12 to August 5 the following graduate courses will be offered: By Professor Brahana: algebra. By Professor Trjitzinsky: complex variables. By Professor Bourgin: partial differential equations, differential geometry. By Dr. Welker: theory of statistics. From June 12 to September 30 courses will be offered in the following intermediate subjects: differential equations, advanced calculus, elementary statistics, fundamental concepts of mathematics, introduction to higher algebra, and introduction to higher analysis.

The University of Michigan. From July 3 to August 26 the following courses will be offered in addition to the standard courses in differential equations, theory of equations and determinants, and advanced calculus: By Professor Carver: introduction to air navigation. By Professor Craig: theory of statistics I. By Professor Dwyer: theory of statistics II. By Professor Fischer: finite differences. By Professor Hildebrandt: theory of functions of a complex variable, integral equations. By Professor Karpinski: teaching of algebra, history of geometry and trigonometry. By Professor Rainich: dynamics, topics in higher geometry. By Professor Thrall: advanced solid analytic geometry, modern algebra. By Professor Wilder: introduction to the foundations of mathematics, topology.

The University of North Carolina. The following advanced courses will be offered: From June 12 to July 20. By Professor Browne; theory of equations. By Professor Henderson: differential equations. By Professor Winsor: college geometry. From July 21 to August 29. By Professor Garner: history of mathe-

matics. By Professor Henderson: analytic geometry of space. By Professor Lasley: projective geometry.

University of Southern California. From July 1 to August 11 the following advanced courses will be offered: By Professor Ames: theory of probability and statistics. By Professor Hyers: modern higher algebra. By Professor Steed: vector analysis.

The University of Virginia. From July 1 to August 25 the following advanced courses will be offered: By Professor Hedlund: differential equations and applied mathematics. By Professor Whyburn: functions of real variables.

The University of Wyoming. The following advanced courses will be offered: From July 8 to August 5. By Professor Barr: astronomy, college geometry. By Professor Neubauer: advanced algebra. By Professor Varnum: advanced calculus. From August 7 to September 2. By Professor Barr: astronomy, differential equations. By Professor Neubauer: fundamental concepts.

WAR INFORMATION

EDITED BY C. V. NEWSOM

Send news reports upon the utilization of mathematicians or mathematics in war activities to C. V. Newsom, University of New Mexico, Albuquerque, New Mexico.

SALARIES AND TEACHING LOADS IN COLLEGE TRAINING PROGRAMS

Several studies have been made in recent months pertaining to salaries and teaching loads in institutions having Army and Navy College Training Programs. The conclusions reached in two of the studies have been summarized below.

The American Council on Education asked for information on salaries and teaching loads in sixty selected institutions having Army and Navy contracts. The colleges and universities were chosen so as to give a fair sample of participating institutions. Replies were received from 45 colleges and universities, and an analysis of the answers reveals that the following policies are being followed by the majority of institutions:

1. Increases in salary resulting from the change to a twelve-month year are being prorated on the basis of the increased number of months involved. Comparatively few institutions have attempted to develop a formula other than that of proportionate months, but adjustments have been made in individual cases on the basis of personal and professional factors involved.

2. No additional compensation is being provided as a result of heavier teaching loads during each term.

3. Comparatively few faculty members are being assigned exclusively to either the civilian or the training programs.

4. Every effort is being made to avoid any differentiation between salaries for teachers of trainees as compared to salaries for those who teach civilian students.

5. New appointments to the faculty are being made in so far as possible on the same basis as the regular scale of instructional salaries, although a few exceptions have been made in order to procure individuals of special and essential abilities.

6. Institutions generally have resisted any substantial increases in the teaching load, and where such increases have been necessary, every effort has been made to equalize them among all members of the teaching staff. Five institutions did report, however, that the teaching load for all members of the faculty had been increased to twenty hours, the figure originally urged by representatives of the Army and Navy contracting agencies.

7. Institutions are refraining from reducing salaries of those teaching courses with small enrollments due to shifting war demands. They have attempted to equalize total institutional responsibilities by shifting non-teaching activities to those with small instructional loads.

The Faculty Association of the Utah State Agricultural College considered the information made available by institutions replying to a questionnaire which had been widely distributed. The study was not exhaustive, but certain general trends were apparent in the data submitted. A summary follows.

In general, the number of students instructed is between 21 and 30 per section, with a few departments having classes numbering between 31 and 40. The normal peacetime load of English teachers appears to be about 12 hours; for physics and mathematics teachers, 15 hours appears to be the usual load, although there is considerable variation. At the present time, the teaching load of English teachers averages two or three hours more than normal, and the same increase seems to hold true for physics and mathematics instructors. The amount of money allowed for assistance in grading papers, and so on, varies considerably, with approximately 45% of all instructors receiving no assistance for their service courses, approximately 15% receiving assistance as needed, and the remainder receiving allotments with too great a variation to permit the formulation of a definite statement. In general, however, physics departments receive more funds for assistance than either English or mathematics departments. Most schools are attempting to keep each faculty member's teaching load at essentially a normal level, thus avoiding the problem of extra compensation; a few schools pay additional salary for extra hours which must be taught during the regular sessions. Institutions with an academic year of nine months pay additional salary to those who teach during the three intervening months. Thirty-seven per cent of the schools pay at the usual summer session rate during the summer school period. Most of the remainder have a salary scale above the usual summer session stipend. For teaching military personnel between the close of the summer session and the opening of the fall term, 80% grant extra compensation, usually at the regular monthly contractural rate.

EDUCATIONAL CREDIT FOR MILITARY EXPERIENCE

Considerable interest is being expressed by educators relative to plans for the educational placement of veterans returning from the present war. The importance of the problem follows partly from the fact that the Thomas Bill (S. 1509), or some similar measure, is quite certain to be passed by Congress; this Bill provides that many ex-service men possessing the necessary aptitudes and interests "shall be entitled to training at an approved educational or training institution for a period of one year, or for such lesser time as may be required to complete the course of instruction chosen by them. A further period of instruction not exceeding three additional years may be provided for persons of exceptional ability and skill." The levels at which this training must be given are indicated by Army statistics which show that 5.2% of all men in the Army are college graduates, 10.4% attended college but did not graduate, 26.1% received no education past high school graduation, 25.2% had some work in high school, and 33.1% had only grade-schooling or no formal education.

The American Council on Education, with the approval of regional accrediting associations and of the several committees concerned with programs of education in the armed forces, has sponsored a three-point approach to questions of educational credit and placement. The three points proposed as a basis for action are:

(1) *Credit for Military Training.* Institutions which allow credit for ROTC, physical training, hygiene, or free electives may well consider granting direct credit, in these terms, for military training, in proportion to the length and extent of military service and without further examination; not to exceed, however, the total amount of credit available in these fields. (For most individuals this will presumably not exceed one-half semester college credit or one semester high school credit.)

(2) *General Educational Placement.* At the time the individual is discharged from the armed forces, the United States Armed Forces Institute will make available to the educational institutions a "competence profile" of the returning service man (or woman), including his full military and previous educational record and also his Army Classification score and his scores in a battery of tests of general educational competence, to enable the school or college to effect an appropriate educational placement of the applicant in terms of his indicated educational maturity and the extent to which he has met the general educational requirements of the school or college.

(3) *Credit in Special Fields.* On the basis both of his general record and his achievement in the competence level battery, the returning student will be tested further in fields of special competence or training; and upon the scores of these detailed tests, which will also be supplied by the Armed Forces Institute, the receiving institution will be able to determine in terms of its own curricula what specific credit the candidate is entitled to receive in any special fields or subjects; and, as well, to plan and recommend a program of study for him.

In order to carry out such a program, testing materials designed to determine the educational equivalence of military experience are now under construction by a staff of specialists at the University of Chicago. It is planned to carefully verify, validate, and calibrate these materials in actual performance; they are to be reviewed by consultants nominated by the leading professional associations in the respective fields; and they are subject to final certification by the Advisory Committee of the Council as satisfactory for use in this program. In keeping with the type of experience to be evaluated, every effort is being made to construct the tests so that they will give a genuinely educated person an opportunity to demonstrate his competence, regardless of the manner in which it may have been acquired.

The following questions have been taken from a publication of the American Council on Education, and will serve to illustrate the types of materials and techniques employed. The items chosen are taken from that part of the general battery, previously mentioned, which is intended to measure the individual's general mathematical proficiency, at the high school level. Four sample items are given; a full test would include more than 50 items.

1. The current auditor's report shows the assessed valuation of Hillville to run around $1,000,000. Last year the public schools of Hillville spent approximately $25,000. This year $20,000 was requested to cover school costs. What was the reduction in the millage levied for the schools this year as compared with last year?

   1) 5 mills.
   2) 10 mills.
   3) 50 mills.
   4) 100 mills.
   5) Not given.

2. The approximate volume of a high round-top haystack may be determined by the following formula:

$$V - (.52M - .44W)WL.$$

In this formula W and L represent the stack's width and length. M is the "over" measurement obtained by throwing a rope over the stack and measuring the distance over the stack from a point on the ground on one side of the stack to the corresponding point on the ground on the opposite side. A stack of alfalfa which is 4 months old has an average width of 20 ft. and is 40 ft. long. Its "over" measurement is 40 ft. What is the approximate number of tons of alfalfa in the stack if alfalfa that has settled for more than 90 days runs around 480 cu. ft. per ton?

   1) 20.
   2) 30.
   3) 40.
   4) 50.
   5) 60.

3. Barrows bought two shares of stock at the same time, the par value of each being \$100. For one of the shares he paid \$70, and for the other he paid \$120. After he had had the stocks for a year, the company for whose stock he paid \$120 declared a 6% dividend, while the other company declared a 4% dividend. Which of the following statements about the comparative value of these two investments during the year concerned is true?

1) The actual rate of return is higher on the \$70 investment.
2) The actual rate of return is 2% higher on the \$120 investment.
3) The actual rates of return on these two investments are exactly equal.
4) The actual rate of return is between 0% and 2% higher on the \$120 investment.
5) The actual rate of return is over 2% higher on the \$120 investment.

4. Jones operates a sandwich stand. His sandwiches all sell for 10 cents. He also sells such things as soft drinks, coffee, candy bars, homemade soups, pie, cookies, etc. These are sold for either 5 cents or 10 cents. His typical sale is 15 cents, which necessitates his collecting a penny tax. He has plenty of dollar bills in his change drawer but is practically out of all coins. He sends his helper to the bank with a \$5 bill for change but neglects to tell what change to get. With which of the following combinations do you believe the helper should return?

1) 10 pennies, 21 nickels, 26 dimes, 5 quarters.
2) 125 pennies, 35 nickels, 20 dimes, 4 quarters.
3) 30 pennies, 20 nickels, 37 dimes.
4) 50 pennies, 10 nickels, 10 dimes, 6 quarters, 3 halves.
5) None of the above is better than any of the others.

FUTURE OF THE NAVY V-12 PROGRAM

The latest word from the Navy Department would indicate that no changes are contemplated in the V-12 Program other than those projected at the initiation of the project. This will probably be only that of anticipated attrition and a slight decrease in the number of basics admitted in the July first cycle.

The comparative stability of the V-12 Program has been partly explained by Dean J. W. Barker, Assistant to the Secretary of the Navy, in the words, "The Navy building program, of course, is a long-range program and we have the necessary places for those who will satisfactorily complete these programs on the commissioning schedule as estimated by us for the ships which are already projected and laid down on the ways. Consequently, our situation is slightly different from that of the Army in that our building program does have to be projected much into the future and our training program can be more closely tied to the actual construction program."*

NOTES ON THE TRAINING PROGRAMS

1. All applicants (A-12 and V-12) taking the two-hour "Army-Navy Qualifying Test" are examined over (a) the meaning and use of words, (b) scientific

matters of general knowledge, and (c) problems in mathematics.

2. "Men with inadequate preparation in mathematics and physics have represented the great bulk of the academic failures in the A.S.T. units."*—Lt. Col. T. D. Palmer, Jr., *Deputy Director, A.S.T.D.*

3. "The high schools of the country have been notified of (the difficulties encountered by V-12 students in their study of mathematics) and they have reported that they have been giving increased attention to the mathematics courses in the senior year in the high schools for the men preparing for these examinations."*—Dean J. W. Barker, *Assistant to the Secretary of the Navy.*

4. The Civilian Pilot Training Act of 1939 elapses on June 30, 1944. Through this legislation the Civil Aeronautics Administration War Training Service Program has been carried on in approximately 300 colleges. Many mathematicians have been employed as instructors. At a recent meeting of the "First National Clinic of Domestic Aviation Planning" held in Oklahoma City a resolution was passed, urging that it is to the national interest to preserve the existing organization of colleges and flight operators conducting Civil Aeronautics Administration War Training Service programs, and that the program should be continued after the cessation of hostilities as an Air Reserve Training Corps.

SELECTIVE SERVICE REGULATIONS CONCERNING STUDENTS

The important Selective Service Local Board Memorandum No. 115 entitled "Occupational Classification Other Than Agricultural" was amended April 4, 1944. The amendment involved special provisions applicable to registrants of ages 18 through 25; these are quoted below.

"Effective immediately, no registrant (whether a nonfather or a father) ages 18 through 25 (other than a registrant who has been found disqualified for any military service or who has been found qualified for limited service only) may be considered as a 'necessary men' entitled to be placed or retained in Class II-A or Class II-B unless: (a) There is filed with the local board a Form 42-A Special upon which the State Director of Selective Service in whose State the registrant's principal place of employment is located has endorsed a statement that, based upon the information in the Form 42-A Special, he recommends that the local board except the registrant from the general restriction against the occupational deferment of registrants ages 18 through 25. (b) An exception to the restriction against occupational deferment of registrants ages 18 through 25 is specifically authorized by the Director of Selective Service without a statement from a State Director of Selective Service, as provided in (a) above, and the local board determines that the registrant comes within the exception described by the Director of Selective Service."

Previous issues of Local Board Memorandum No. 115 have included in the list of exceptions, referred to in (b) above, "students who qualify for occupational deferment in accordance with the provisions of Activity and Occupational

* S.P.E.E. Conference, Chicago, Ill., Oct. 25, 1943.

Bulletin No. 33-6." Since the new Memorandum did not state such an exception, it was implied that students would now be subject to the provisions of the new regulation. Due to the uncertainty involved, however, Leonard Carmichael, Director of the National Roster, sent a clarifying telegram on April 8 to presidents of colleges and universities. His message stated, "Regret to advise that deferment of college students in fields of engineering, physics, chemistry, geology and geophysics under quota system has been rescinded by Selective Service Local Board Memorandum 115 issued in revised form April 4, 1944. Question of continued deferments for students who can graduate before July 1, 1944 and for pre-professional students in medicine and theology now being reconsidered. Will advise you in full detail as soon as final decisions become known to us."

On April 10, a Selective Service announcement sent to state directors said that temporary deferments are recommended for registrants in recognized colleges or universities who will graduate before July 1 and who are pursuing full-time courses in certain scientific and specialized fields. Mathematics was in the list of 24 specialized fields.

THE MATHEMATICAL ASSOCIATION OF AMERICA

NEW MEMBERS

The following forty-two persons have been elected to membership on applications duly certified:

CLAIRE F. ADLER, Ph.D. (New York Univ.) Asst. Prof., New York Univ., New York, N. Y.

K. J. ARNOLD, Ph.D. (Mass. Inst. of Tech.) Asst. Prof., Univ. of New Hampshire, Durham, N. H.

MRS. L. H. BAKER, JR., A.B. (Texas Christian) Jr. Chem. Engr., Humble Oil and Refining Co., Houston, Tex.

C. R. BLYTH. Student, Queen's Univ., Kingston, Ont., Can.

J. D. BURK, B.A. (Toronto) Asso. Prof., Univ. of Toronto, Toronto, Ont., Can.

L. VIRGINIA CARLTON, A.M. (Tulane) Asst. Prof., Wesleyan Coll., Macon, Ga.

K. L. CLARK, Ph.B. (Wisconsin) Mathematician, A. O. Smith Corp., Milwaukee, Wis.

L. C. DAMSGARD, A.B. (Union Coll., Nebr.) Asst. Prof., Math. and Astr., Pasadena Jr. Coll., Pasadena, Calif.

J. E. EATON, Ph.D. (Yale) Instr., Queens Coll., Flushing, N.Y.

WILL FELLER, Ph.D. (Göttingen) Asso. Prof., Brown Univ., Providence, R. I.

M. J. FORRAY, A.B. (New York Univ.) Instr., Washington Square Coll., New York Univ., New York, N. Y.

MARY A. GOINS, A.M. (Michigan) Asso. Prof., Western Coll., Oxford, Ohio

A. W. GOODMAN, A.M. (Cincinnati) Instr., Syracuse Univ., Syracuse, N. Y.

MRS. LEOTA C. HAYWARD, M.S. (Colorado State Coll.) Asst. Prof., Colorado State Coll. of A. and M. A., Fort Collins, Colo.

FRITZ HERZOG, Ph.D. (Columbia) Asst. Prof., Michigan State Coll., East Lansing, Mich.

N. L. JACOBSON, B.S. in Educ. (Oregon) Instr., Central Missouri State Teachers Coll., Warrensburg, Mo.

MAX KATZ. Accountant, Katz, Zuckerman and Co., New York, N. Y.

F. C. LEONE, M.S. (Georgetown Univ.) Instr., Purdue Univ., West Lafayette, Ind.

L. D. LEVINE, A.B. (Brooklyn) Instr., Carnegie Inst. of Tech., Pittsburgh, Pa.

LIONEL LONDON, B.S.C.E. (Illinois) Analyst, A. O. Smith Corp., Milwaukee, Wis.

S. S. MCNEARY, A.M. (Pennsylvania) Instr., Drexel Inst., Philadelphia, Pa.

W. B. MICHAEL, A.B. (U.C.L.A.) Instr., Pasadena Jr. Coll., Pasadena, Calif.

A. C. MOELLER, B.S. (Western Reserve) Instr., Denison Univ., Granville, Ohio

C. E. MULLAN, A.M. (Duquesne) Senior Engr., Duquesne Light Co., Pittsburgh, Pa.

P. F. NEMÉNYI, D.Sc. (Berlin) Acting Asst. Prof., State Coll. of Washington, Pullman, Wash.

J. C. OXTOBY, A.M. (California) Asso. Prof., Bryn Mawr Coll., Bryn Mawr, Pa.

BOB PARKER, A.M. (Texas Tech.) Instr., Texas Tech. Coll., Lubbock, Tex.

WALTER PRENOWITZ, Ph.D. (Columbia) Asst. Prof., Brooklyn Coll., Brooklyn, N. Y.

L. W. RAMSEY, B.S. (A. and M. Coll. of Texas) Asst. Prof., Math. and Engr., Texas Christian Univ., Fort Worth, Tex.

R. M. ROBINSON, Ph.D. (California) Asst. Prof., Univ. of California, Berkeley, Calif.

MARGARITA RODRÍGUEZ. Aux. Prof., Matanzas Inst., Havana, Cuba.

G. F. ROSE, B.S. (Case School) Research Engr., Westinghouse Elec. and Mfg. Co., East Pittsburgh, Pa.

G. S. ROSENBERG. Student, Illinois Inst. of Tech., Chicago, Ill.

M. J. SHEEHY, A.B. (U.C.L.A.) Asst. Physicist, Univ. of California Div. of War Research, U. S. Navy Radio and Sound Lab., San Diego, Calif.

R. H. SORGENFREY, Ph.D. (Texas) Instr., Univ. of California at Los Angeles, Los Angeles, Calif.

HILLEL SPITZ, A.B. (George Washington) Instr., Univ. of New Hampshire, Durham, N. H.

E. L. STANLEY, M.S. (Tennessee) Instr., Clemson Coll., Clemson, S. C.

L. W. STARK, A.B. (Univ. of Kansas City) Acting Prof., Acting Head of Dept., William Jewell Coll., Liberty, Mo.

J. S. STUBBE, M.S. (Brown) Instr., Univ. of Cincinnati, Cincinnati, Ohio

J. W. SWANK. Statistician, Southern California Edison Co., Ltd., Los Angeles, Calif.

F. A. VALENTINE, Ph.D. (Chicago) Asst. Prof., Univ. of California at Los Angeles, Los Angeles, Calif.

H. S. ZUCKERMAN, Ph.D. (California) Asst. Prof., Univ. of Washington, Seattle, Wash.

THE SIXTH ANNUAL MEETING OF THE NORTHERN CALIFORNIA SECTION

The sixth annual meeting of the Northern California Section of the Mathematical Association of America was held at the University of California at Berkeley, California, on Saturday, January 29, 1944. In the absence of Professor E. B. Roessler, Chairman of the Section, the Vice-Chairman, Professor Gabor Szegö, presided at the morning session. Professors Sophia H. Levy and F. R. Morris presided at the afternoon session. During the noon recess luncheon was served at the Men's Faculty Club.

The meeting was attended by sixty persons, including the following twenty-three members of the Association: H. M. Bacon, G. A. Baker, H. W. Becker, B. A. Bernstein, G. C. Evans, S. A. Francis, F. T. Frank, Emma V. Hesse, D. H. Lehmer, Sophia H. Levy, E. D. Miller, F. R. Morris, W. H. Myers, E. J. Phillips, R. M. Robinson, Ethel Spearman, Pauline Sperry, Ruth G. Sumner, Gabor Szegö, R. K. Wakerling, A. R. Williams, Frantisek Wolf, Clyde Wolfe.

The following officers were elected for the coming year: Chairman, Gabor Szegö, Stanford University; Vice-Chairman, Pauline Sperry, University of California; Secretary-Treasurer, H. M. Bacon, Stanford University. Mrs. Ruth G. Sumner, Oakland High School, was reelected to represent the Section as associate editor of the *California Journal of Secondary Education.*

The following program was presented:

1. *Euclidian metric invariants of conics by tensor algebra*, by Dr. T. C. Doyle, Stanford University, introduced by the Secretary.

The speaker illustrated the application of tensor algebra to the invariant theory of the conics and the reduction of their equations to canonical form. This was done by extending the euclidian metric group

$$x^i = a^i_r\bar{x}^r + a^i_0, \qquad (i = 1, 2),$$

to contain one more variable x^0, and then applying this group to the general equation

$$g_{rs}x^rx^s + g_{os}x^s + g_{oo} = 0$$

of the conic after a similar extension to three variables.

2. *Solutions of several problems in heat transfer*, by Professor L. M. K. Boelter, University of California, introduced by the Chairman.

Professor Boelter, who is Associate Dean of the College of Engineering, University of California, appeared on the program by invitation of the Section. He spoke for about an hour on certain topics relating to mathematics in engineering. He discussed three methods by which engineering problems are solved, namely: (1) application of criteria based upon accumulated experience; (2) extrapolation of data from model studies, wherein physical (as well as geometric) conditions of similarity have been met; (3) analytical or graphical solution of differential equations defining idealized systems which resemble, but never coin-

cide with, the real systems under investigation. The considerations which determine the choice of method were outlined. It was also remarked that the engineering student should be encouraged to become familiar with certain phases of mathematics such as differential equations, probability, statistics, and some of the less familiar functions which occur in the analysis of physical systems.

3. *Present trends in secondary school mathematics*, by Professor L. B. Kinney, Acting Dean of the School of Education, Stanford University, introduced by the Vice-Chairman.

Dean Kinney outlined those trends which have become apparent in the adjustments of the schools to the present emergency. He also commented upon the programs in the armed forces and in industry. It was remarked that the perceptible trends result in the main from the emphasis upon mere ability to perform mathematical operations, and from the need for mathematical competence in a large group of individuals previously illiterate mathematically. The following trends were noted: (1) an increasing emphasis upon mathematics for use; (2) the tendency for the organization of courses to emphasize the social and personal significance of the subject, as well as the systematic nature of the field; (3) the programs set up for special groups of students tend to be liberated from dominance by the general student; (4) there is increased utilization of the more varied approaches and activities in the mathematics classroom; (5) there are fewer topics in the typical mathematics course, but the topics included are more completely developed; (6) there promises to be a continued demand for mathematics as a part of a general education.

4. *Some problems connected with discordant permutations*, by Professor D. H. Lehmer, University of California.

The speaker discussed some recent advances in connection with several problems relating to discordant permutations. Among the topics discussed were the recently published results of Schöbe and Kaplansky on the *problèm des ménages*, the contribution of Narashimamurti to the problem of cable splicing, and the results of Kariwala, Riordan, and Becker on Latin configurations.

5. *An application of conformal mapping to aeronautical engineering*, by H. J. Shaw, Stanford University, introduced by the Secretary.

This paper was a summary of an application of conformal mapping which was given by T. Theodorsen and I. E. Garrick in their paper on *General Potential Theory of Arbitrary Wing Section* in the N. A. C. A. *Bulletin* for 1933. The problem considered was the determination of the operating characteristics of an airplane wing having a cross section of arbitrary shape. It was first noted that the problem can be reduced to that of mapping the boundary of the wing cross section conformally upon the boundary of a circle. The method of mapping a graphically given wing section of arbitrary shape is as follows. The wing section is subjected to the transformation $w=z+(1/z)$. This maps any practical wing section in the w plane upon a nearly circular curve S in the z plane. It can be

shown that if the points on the boundary of S are denoted by $e^{\Phi+i\Psi}$ and points on the circle by $e^{i\psi}$, then the mapping of the near circle upon the circle is given by the integral equation

$$\Psi = \psi + \frac{1}{2\pi}\int_0^{2\pi} \Phi(\psi) \cot \frac{t-\psi}{2}\, dt.$$

Numerical solutions of this equation for a number of points on the boundaries can be obtained by successive approximations. The process converges so rapidly that in practice the first approximation is usually sufficient.

6. *Graphical methods in exterior ballistics*, by Dr. L. H. Swinford, University of California, introduced by the Chairman.

In this address it was pointed out that the integration of the classical system of differential equations for the solution of the special problem of exterior ballistics cannot be performed by quadratures for an arbitrary or empirical resistance law. It was noted that the method of integration by successive approximations is one of the favored methods, and that this can well be carried out graphically. The graphical method may be preferable even in cases in which the principal equation is integrable by quadratures, since these quadratures are awkward. It was also remarked that the graphical method would seem to be preferable to the Euler-Otto method for the quadratic law of resistance.

7. *Solution of cubic and quartic equations by the method of identities*, by V. F. Ivanoff, San Francisco.

This paper was read by title.

H. M. BACON, *Secretary*

CALENDAR OF FUTURE MEETINGS

Twenty-Seventh Summer Meeting, Wellesley, Mass., August 12–14, 1944.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENEY MOUNTAIN
ILLINOIS
INDIANA, Indianapolis, November 10, 1944
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA, St. Paul, May 6, 1944
MISSOURI
NEBRASKA, Lincoln, May 6, 1944
NORTHERN CALIFORNIA, San Francisco, January 27, 1945
OHIO
OKLAHOMA
PHILADELPHIA, Philadelphia, November, 1944
ROCKY MOUNTAIN
SOUTHEASTERN
SOUTHERN CALIFORNIA, Los Angeles, March 10, 1945
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE
WISCONSIN, Milwaukee, May, 1944

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NOTICE OF CHANGE OF ADDRESS by members of the Association, as well as correspondence regarding the business of the Association, should be sent to the SECRETARY-TREASURER, W. B. CARVER, McGraw Hall, Cornell University, Ithaca, N. Y. Change of address must reach the Secretary-Treasurer five weeks before the change becomes effective.

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THE NATURE OF MATHEMATICAL PROOF*

R. L. WILDER, University of Michigan

1. Introduction. In presuming to come before you with such a title as "The nature of mathematical proof," let me assure you that I am not doing so with the idea of presenting any new or startling facts. I do this because I think it is good for us, as mathematical specialists of one sort or another, speaking in terminologies that frequently render us obscure even to one another, to pause and reflect now and then on just what we are doing and how we are doing it. For certainly we put a great part of our time and energy into the act of proof. We ask one another, "Do you think anyone will ever prove the Fermat Theorem?" or, "Do you think anyone will ever prove the continuum hypothesis?" A host of mathematical questions would receive answers if only we were able to find proofs for the theorems which hold the keys to their solutions.

In our teaching of such seasoned parts of mathematics as classical geometry, we continually ask our students to prove theorems. We see frequently in our journals articles which embody only new proofs of old theorems, which are justified either by increased simplicity or by the requirement of fewer logical hypotheses. And we have observed the study of proof methods which has now gone on for many years in mathematical logic, particularly among those who have been styled the Hilbert school of thought, or have come under its influence. Also we are aware that proofs have been published which have not found general acceptance, not because any error was made in the logical reasoning, but because some principle which may have theretofore been found quite innocuous, or, indeed, passed unnoticed, is suddenly revealed in all its power. This is particularly the case when the principle renders possible the proof of some theorem that a respectable body of mathematicians have already decided should not be true.

I suppose there is considerable ground for the logistic thesis as originally stated by Russell, to wit, that "Pure mathematics is the class of all propositions of the form 'p implies q,' where p and q are propositions containing one or more variables, the same in the two propositions, and neither p nor q contains any constants except logical constants." One of the chief of these constants is implication. And Russell asserts that in addition to these logical constants, mathematics uses a notion which is not a constituent of the propositions which it considers, namely the notion of *truth*.† If there is any more prominent notion in the popular mind concerning the nature of mathematics, other than that it deals with "figures," it is probably that the results of mathematics are certain and absolutely true.

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* Except for the introduction of a series of headings, this is an address delivered by invitation to the meeting of the Association at Chicago on November 28, 1943. At the suggestion of the editor, an appendix is attached embodying a brief exposition of some of the concepts employed; readers unfamiliar with the latter may wish to read this appendix first.

† Russell, B., The Principles of Mathematics, vol. 1, Cambridge, 1903. For a critique by Russell himself of the material quoted above, see the Introduction to the Second Edition, New York, 1938.

Now it is chiefly the following aspects of mathematics that I should like to discuss: (1) The nature of this process or procedure that we call "proof," and (2) the relation of the end result of the process to "truth." Specifically, what position does proof occupy in mathematics, and how does it influence the so-called truth or certainty of our results?

2. Types of proof. I cannot attempt here any complete or exhaustive analysis of proof, even if I felt competent—for this you may consult the works, either classical or current, of Russell and his followers, or of Hilbert and his followers, for example, according to your philosophical inclinations. As a matter of fact, I am using the word "proof" in a very broad sense, not in a narrow technical sense. For my purposes I want merely to recall some of the general outlines of proof as we apply it every day both in class and research. I am not interested in all the step-by-step processes, as, for example, substitution, which go to make up the atomic elements of proof.

2a. Mathematical induction; transfinite induction. One element which I cannot avoid, since it not only develops, like the amoeba, a long chain of descendants, but itself occupies a central position among our proof outlines, is the well-known syllogistic type of proof. Given p, and that p implies q, then q follows. From a certain point of view, our basic proof by mathematical induction seems to be nothing but an omega series of syllogisms, as pointed out by Poincaré:* The theorem is true of the number 1. Now if it is true of 1, it is true of 2. Hence it is true of the number 2. And so we proceed. But of course we don't "proceed" in reality—what we do, in order to avoid an infinite and unending series of syllogisms, is to use a property of the natural number sequence to knock over a row of syllogisms much as a child tips over a row of vertical blocks by setting them in the proper juxtapositions and then tipping over the first of them. Everything depends on having the proper juxtapositions, and it is a fundamental property of the sequence of natural numbers, as formulated for instance by Peano,† that the order type guarantees the mathematical induction process. In other words, we have introduced an infinite totality, and by properly ordering its terms, succeeded in applying the syllogism only once to prove a theorem about the infinite totality.

Modern mathematics has incorporated another extension of this mode of proof, namely, proof by transfinite induction. From the fact that the omega series is only an elementary type of well-ordered series, we generalize to the application of the induction method to any well-ordered sequence. All we need is that after having considered any part of the sequence, there be a first element in what is left, provided the latter is not empty. And in addition to the fact that if true for an element then true for its successor, the fact that if true for all predecessors of an element, then true for the element. For example, if for a given well-ordered set of elements $\{x_\alpha\}$ we have defined a function $f(x_1)$, and have

* Poincaré, H., La Science et l'Hypothèse, Paris, 1912, Chap. I, §V.

† Peano, G., Formulaire de Mathematiques, vol. II, Turin, 1897. See §2, p. 1.

defined $f(x_\beta)$ whenever $f(x_\alpha)$ for $\alpha<\beta$ has been defined, then we can assert that $f(x)$ is defined for *all* x_α of the sequence.

But notice the difference in application from that of ordinary mathematical induction. In the latter case the sequence x_α is ordinally similar to the natural number sequence, $1, 2, \cdots, n, \cdots$, and is probably hypothesized to begin with, as in the theorem that for every n in the natural number sequence, $1+2+\cdots+n=n(n+1)/2$. And the sequence of natural numbers is unquestionably the most reputable of infinite totalities. But transfinite induction, on the other hand, is applied to infinite totalities which for various reasons are not in such favor, and which frequently are only timidly mentioned by those who do favor them, as for example in the case of the set of all ordinals of the second class. And, moreover, the arrangement of the totality in a well-ordered sequence for purposes of induction is usually imposed at some step in the process of proof.

We find, for instance, an eminent contemporary avoiding a proof which involves the notion of the totality of all the ordinal numbers of the second class, with the explanation: "I regard the intuition of the totality of transfinite numbers of the second class as defective, and I believe that there is a certain degree of illusion in the clarity which the usage of the notations for the 'small' transfinite numbers of the second class casts on our intuition of this totality. We are led thus to inexact intuitions, and this fact as well as other analogous facts compel us to be very cautious in the introduction of transfinite numbers of the second class by the logical path."* And on the part of those who do not take so radical a view we find instance after instance, as in the case of a well-known publication of Kuratowski,† of papers devoted either partially or completely to the devising of means of eliminating transfinite numbers from mathematical proof. And while many of these latter are careful to include words of explanation to the effect that of course they understand that transfinite numbers are only an extension of the arithmetic of integers, they are plainly uneasy about the "existence" of these numbers, and hazy about where to draw the line of compromise between elegance of proof and elimination of these numbers. It seems rather clear that *in order to have mathematics we must have the infinite; but just HOW infinite, that is the question*! To incorporate into our logical apparatus of proof the infinite totality of natural numbers is quite proper, but to assume the existence (whatever that may mean) of the infinite totality of second class ordinals is to risk rejection of one's proof by part of the mathematical public, as well as an invitation to a still larger part of that public to try to find a proof which makes no use of transfinite numbers—or at least not *all* the numbers of the second class.

2b. Proof by example. Let me next mention proof by example. We use it both constructively and destructively. The usual constructive use to which we

* Lusin, N., Leçons sur les Ensembles Analytiques, Paris, 1930, p. 27.

† Kuratowski, C., Une méthode d'élimination des nombres transfinis des raissonements mathématiques, Fundamenta Mathematicae, vol. 3, 1922, pp. 76–108; in footnote 1, citations will be found to other papers of a similar nature.

put it is that of transferring the question of the consistency of a set of postulates to some other domain, as Hilbert did when he reduced the question of the consistency of his geometry postulates to that of the arithmetic of real numbers. In the case of postulate systems whose generality allows of finite interpretation in the world of our senses, we usually argue that the law of contradiction disallows the existence of contradictory propositions—but the method is generally admitted to be no guarantee of consistency when the degree of infinitude demanded by the system drives us to the use of strictly mental concepts—as, for example, transfinite ordinals. The destructive use of the method of proof by example is that of proving the impossibility of proving a theorem by furnishing an "exception to the rule." Thus, to our forebears, it seemed to be the rule that all continuous functions defined over an interval had a derivative at least somewhere, until Weierstrass gave his famous example. We continually settle problems by this method—perhaps the famous four-color problem will be solved this way. But, of course, we are here again faced with the question as to the validity of an example, and the distressing lengths to which some authors go in giving a rigorous formulation of an example are no doubt due in good part to the desire to leave no question open as to its validity. Unfortunately the existence question rears its head again, if the example reposes in the domain of the not generally accepted; but at least in this case the example seems to have a more general effect, inasmuch as the scoffers don't usually resume the search for the proof of the theorem whose impossibility has apparently been proved within domains or by methods which they reject. I haven't heard lately of anyone trying to prove that every bounded real function is measurable!

2c. Reductio ad absurdum. Let us get on to a much more popular type of proof—that of reductio ad absurdum. In good repute among many formalists and logicians both for its elegance and its incisive character, its use has been limited by those of an intuitionist leaning because of its dependence on the law of the excluded middle. Of course, in one respect, namely, in proofs of nonexistence, its use is approved by all, no matter of what school of thought. If the assumption of the existence of a mathematical entity leads to contradiction, no one wants to maintain its existence. But one notes in recent years an increasing use of various constructive methods for existence proofs. To ask for proof of existence by construction is not sufficient basis to label one an intuitionist, since many have come to feel that unless the proof of existence by reductio ad absurdum offers very great advantages by reason of its simplicity or avoidance of complicated construction, there is not sufficient justification for its use; it is better to know how to build up the entity desired than to show, for example, that the class of which it is an element is not empty. It is somewhat amusing, I think, that some of us become so used to the "contra-positive" type of argument that we automatically take an argument that is ideally suited to the constructive form of proof, and twist it into a reductio ad absurdum. For example, consider the classic theorem that there exists an infinity of prime num-

bers. As you know, Euclid showed how, given a certain finite set of prime numbers, one can construct a prime number not in the set, and it is no task to go a step further and make this number unique. Hence it seems hardly necessary to argue that if it be assumed that the set of all prime numbers is finite, say k in number, then by Euclid's method there can be shown to exist a $(k+1)$th prime and hence a contradiction, inasmuch as Euclid's method itself provides a valid procedure for constructing an infinite series of primes by recursion (although I am aware I am using the word "constructive" here in a sense which many may not admit—about which more presently). Of course, whether one wants to give up the extremely elegant proofs of the fundamental theorem of algebra which use the contra-positive method, to replace them by the intuitionistic method of proof* is another question, and one that I presume can only be decided by an appeal to one's innate convictions on the foundations.

2d. Constructive methods. Let us look more closely at the constructive method, however. What do we mean by construction? I presume that one would like to stipulate that a proof by construction should, theoretically at least, enable the one who cares to carry through the steps of the proof in any particular case to construct the entity whose existence is proved. Suppose, for example, that this entity is a real number. Do we mean that the proof is to enable us to start writing down the number, digit by digit, so that independent workers, by following the steps provided in the proof, would generate the same series of digits no matter how far they carried the process? For instance, by employing the Cantor diagonal process on the well-known ordering of the algebraic numbers in an omega series, one can constructively or effectively define a unique transcendental number, corresponding not only to every natural number n, but to every transfinite ordinal of the second class which may be constructively defined in a certain sense into which I will not go here; and, indeed, in such a way that if one specifies, for example, the transfinite ordinal ω^2, one can start writing down within a few minutes the corresponding transcendental number, digit by digit.

But the trouble with this definition of constructive proof of existence of real numbers is that one can easily give definitions that do not furnish a means to start writing out a series of digits, but that plainly do fix a certain real number. For example, consider the number N which is equal to

$$1+(-1)^{\rho}/(10)^{\rho},$$

where ρ is the number of the first decimal place in the decimal development of π where a sequence of digits 0123456789 commences; or, if there is no such number ρ, then $N=1$. If we know that ρ exists and is even, we can begin to write the number $1.0000 \cdots$ until we get to the ρth place, where there is a 1. But if ρ is odd, then we must begin to write $0.999 \cdots$ (to ρ digits). But we

* See Brouwer, L. E. J., and de Loor, B., Intuitionistischen Beweis des Fundamentalsatzes der Algebra, K. Akad. der Wetenschappen, Proceedings of the Section of Science, vol. 27, 1924, pp. 186–188.

don't know whether ρ exists or not, and in the absence of such information we can't even begin to write down the proper digits for N. Nevertheless we have to admit that the definition fixes a number N, and we notice that it fixes it to any required degree of approximation, even though we can't start writing down its digits. Evidently, then, we may have to decide that existence of a real number is proved constructively when for any given positive number ϵ the number is determinate within an error of at most ϵ.

But granted that this definition is satisfactory, we run into another difficulty: Are we going to have to set up a new criterion for what we mean by proof by construction every time we prove the existence of a new mathematical entity? That is, is our definition to be, in a sense, a function of the thing whose existence is to be proved? As another example, consider the Borel Theorem for a finite interval of the real number system. Let G denote a set of open intervals covering a closed interval ab. We may define what would certainly seem to be a constructive method for selecting a finite set of intervals of G which cover ab. All we need do is first find that point, call it a_1, which is the least upper bound of the points x such that both a and x are simultaneously covered by an interval of the collection G.* Then find that point, which we call a_2, which is the least upper bound of the points x_1 such that both a_1 and x_1 are simultaneously covered by an interval of the collection G. And so on. A non-existence argument by the contra-positive method is easily given to show that we finally reach an a_k (k a natural number) which is the first to exceed b. Then to get the finite set of intervals of G covering ab we first select an interval of G, say I_k, which covers a_{k-1} and b; this is possible because of the definition of a_k. Then we select an interval I_{k-1} which covers a_{k-2} and overlaps with I_k—this is possible because of the definition of a_{k-1}. Then we select an interval I_{k-2} of G which covers a_{k-2} and overlaps with I_{k-1}, and so on. The set of intervals $I_1, \cdots, I_k$, where I_1 is finally selected to cover a and overlap with I_2, is a finite set of intervals of G covering ab.

I said that this would seem to be a constructive proof. But I am aware that although only a finite number of selections of intervals of G are required, two independent workers, given the same data concerning the collection G, could hardly be expected to produce the *same* covering of ab by intervals of G. So if this is to be a constructive definition of a covering, another definition of what we mean by constructive proof is needed. But in providing such a definition, we would again be making our definition of constructive proof a function of the sort of thing whose existence is to be proved. And finally we may conclude that we cannot say in general just what we do mean by constructive proof, except that for a real number it may mean one thing, for another mathematical entity it may mean another thing, etc.†

* We assume here that the intervals of G are actually *given* (analytically defined), in such a way as to enable us to find this number, a_1. Interesting examples of this sort may be given.

† See also §8 of the article by K. Menger, Some applications of point-set methods, Annals of Mathematics, vol. 32, 1931, pp. 739–760.

2e. Principles not universally accepted. Of course, the fact that a proof of existence does not use the reductio ad absurdum argument is no guarantee that constructive methods have been employed. This is particularly the case when use is made of the Zermelo choice axiom or the continuum hypothesis.* And it is admitted that the existence of many mathematical entities cannot be proved without the use of one or the other, or both of these principles. And there are many among us, not all intuitionists, who refuse to recognize such proofs. I have noticed particularly that those who don't *need* such aids in their research are quite prone to pass laws against the use of such powerful tools as the choice axiom, the continuum hypothesis, transfinite induction, *etc.*—the number-theorists, for example (shades of Kronecker!). Be it to Brouwer's everlasting credit that he was not a number-theorist!

On the other hand, I think it is safe to say that even the most formal of the formalists considers it good ethics to post a sign somewhere if, for example, he uses the Zermelo choice axiom in a proof. Topologists know of one book† in which the author's use of the choice axiom is to be so frequent that he just puts the sign in the introduction and then proceeds to use the axiom freely—sort of a blanket license. And in Sierpinski's books‡ one finds classifications of theorems according to whether their proofs depend on the choice axiom, the continuum hypothesis, etc. And many an editor who had not the least tinge of intuitionism has accepted for publication an article giving the first proof of a reputable theorem without the use of the choice axiom or the continuum hypothesis. Most of us are aware of theorems that we would like to see proved without the use of these hypotheses, if only for esthetic reasons. But the *demand* for such proofs before admission to mathematical distinction is frequently unreasonable, I feel, in view of the very indefiniteness or scantiness of the hypothesis with which one has to work. Indeed, it seems clear that if we are going to have certian parts of mathematics at all, we shall have to have such principles. Certain mathematical problems are simply unsolvable without the continuum hypothesis. One may, like the intuitionist, solve such problems by the alternate procedure of tossing them out of the realm of mathematics—a cure by extermination, as it were—but most of us dislike to go to such lengths.

But I think that for my purposes I have covered enough of the materials that enter into proof in mathematics.

3. Mathematical dogmatism. Now we are probably quite familiar with the fact that mathematical proof is a function of the time. History shows this conclusively—Euclid would probably have complained of the lack of rigor displayed by his predecessors; Weierstrass felt it necessary to reorganize the foundations

* Cf. Sierpinski, W., Les exemples effectifs et l'axiome du choix, Fundamenta Mathematicae, vol. 2, 1921, pp. 112–118.

† Moore, R. L., Foundations of point set theory, Amer. Math. Soc. Coll. Pub., vol. XIII, New York, 1932.

‡ Lecons sur les nombres transfinis, Paris, 1928 (Chaps. VI and XII); Hypothèse du continu, Warsaw, 1934.

of analysis; and so on. The present is a time when it seems appropriate to reflect on the new and still uncertain elements that have come into mathematical proof. One may berate those who make their proof methods dependent on some particular mathematical philosophy, such as intuitionism, but at the same time be taking considerable license when, in order to prove a much desired result, he resorts to methods that are both novel and uncertain. We may, if we like, hide behind the excuse that a proof which uses a hypothesis which so far as anyone knows has never led to contradiction, is at least a clue to a possible mathematical fact, and as such it may encourage us to establish the fact on firmer ground. This recalls Weyl's observation to the effect that giving a non-constructive existence proof is like informing the world that somewhere there exists buried treasure but not stating where it lies!* But more to the point is the fact that there is certainly some worth in a proof which shows that the theorem is in the same category with the not generally accepted hypothesis, as, for example, when it is shown that certain theorems are equivalent to the continuum hypothesis, or just as it was shown that the choice axiom, well-ordering theorem and comparability principles are all equivalent. Nevertheless, these facts are no ground for dogmatism.

The present division of mathematical thought into schools, insofar as it degenerates into dogmatism, is not, in my opinion, healthy. On the other hand, insofar as it leads to dispassionate discussion of fundamental principles, it is a decidedly healthy development. I cannot refrain, in this connection, from expressing my protest against what I like to call, variously, the "mathematical dogmatist" or "the mathematical fascist." He may have no religious philosophy or affiliation in the ordinary sense, yet if you venture to doubt the validity of some one of his favorite proof-methods, you may find yourself in danger of physical violence. In his eyes, you are a mathematical free-thinker, an anarchist, and paradoxically although he may be both of these in his political beliefs, he won't tolerate them in mathematics. It is as though he were the mathematical prototype of the arch-reactionary whose political and economic emotions stem from his possession of an unusually large share of the world's goods; the mathematical reactionary is motivated by his possession of a large body of mathematical theory whose foundations are in danger of attack by the mathematical revolutionist.

In the case of some individuals, it is almost as though mathematics had become a kind of religious fanaticism, rather than a labor of love. They give one a feeling that we have in mathematics the intuitionist, formalist and logistic "Theologies." Insofar as the leaders of these systems become dogmatic, just so far do they become the mathematical analogues of the Dalai Lama or the Pope of Rome. And although I am not concerned here with dogmatism in religion, I think I am justified in protesting its presence in mathematics. Of course, analogy must not be pushed too far—we don't venture to predict the creation

* Weyl, H., Philosophie der Mathematik und Naturwissenschaft (Teil I), Handbuch der Philosophie, Abt. IIA, Munich and Berlin, 1926, p. 41.

of an annual collection to be called "Kronecker's pence" to be presented to Brouwer. But the claim to possess mathematical infallibility, the attempt to set up formalistic or intuitionistic proof rituals—any of these acts and their like, are not, I venture to say, exemplifications of the truly mathematical spirit. On the other hand, to set forth these ideas as suggestions for the cure of mathematical ills or the improvement of proof-methods in the direction of greater rigor, is, it seems to me, to proceed in a truly scientific way.

As mathematicians, dealing with a subject that must be kept scrupulously abstract, which has no material connections except in the manner of what we call "applied mathematics," we must ever be on guard against dogmatism. It is natural for the layman to regard our works with awe, to think of us as the possessors of absolute truth, since we have been singularly successful in avoiding contradiction in applied mathematics. But we must not allow this veneration with which we are regarded to tempt us to set up our own mathematical cults or political philosophies. Rather we need to practice democracy, not sneering at new ideas as dangerous to our pet mathematical theories, but regarding them as possible improvements on the existing system. I think that the worth of new ideas can safely be left, in the long run, to the judgment of the mathematical electorate. A mathematical idea that never "takes" is probably not worth taking—although we must remember that here, as in other human activities, the long range point of view should prevail. What is unpopular today may become the fashion of tomorrow—which is only further admonition to be receptive to new ideas.

And I want to point out here what seems to me to be a heartening fact, although I realize that I may be sorely trying some of my listeners, personal philosophies: Namely, that no matter if we do use, frequently, questionable methods of proof, or even make outright errors in proof, we are usually gathering mathematical fruit. That the calculus of Newton and Leibnitz had virtually no basis at all from the viewpoint of modern standards did not invalidate the calculus as mathematics. All it needed was bolstering up. Even now we argue about such items as Duhamel's Theorem or what we mean by the differential, but no one suggests that we throw the calculus out of our mathematical libraries, for we recognize intuitively that it is a body of acceptable mathematics—still susceptible to improvement, perhaps. It is not that we consider that the theorems of the calculus have been elegantly and conclusively proved in the theory of functions. To put the matter bluntly, it is a case of our *knowing mathematics when we see it.* And we don't set out to prove a theorem in the first place unless *we think it is worth proving.*

4. Source of the theorem to be proved. And this brings me to a consideration of the source of the thing we set out to prove—the theorem. Where do we get it? I think most of us would say from our intuition. Of course, we do not use the word "intuition" here in the sense of the intuitionist—we mean that intuition which guides the selections of the mathematical electorate. As has often been remarked, the Intuitionist (with a capital *I*) seems to use this intuition to tear

down part of the mathematical edifice—whereas here I use the term more in accord with its usual philosophical meaning. It is the sense in which most of us who engage in mathematical research understand the term—that mental medium that generates the mathematical hunch. What often seems to us perhaps the great good fortune or luck of those analysts of an earlier day whose ignorance of convergence criteria did not prevent their constructing mathematics of probably eternal significance was, I think, *mathematical intuition*, not luck. Only we must again beware of dogmatism or professionalism. As every research mathematician knows, the individual intuition is an amiable and usually reliable guide, but on the other hand it is evidently only human and prone to error. It is the collective intuition that seems more to be relied upon, and we may, I believe, have faith in its long-run judgment as to what is valid mathematics. And this is as it should be, since it is the mathematical intuition that makes mathematics. Without it, we would have nothing to prove. I am quite willing to grant that, with proof or without proof, or with proof by what some of my contemporaries insist is no proof at all, the mathematical theorem is not necessarily a statement of fact or truth in the ordinary sense. But it may be a decidedly elegant and artistic piece of mathematical statement. And as such it may be adopted by the mathematical electorate into the body of mathematical literature.

5. The role of proof. Now, granted that the mathematical theorem comes from the intuition, what is the role of proof? It seems to me to be only *a testing process that we apply to these suggestions of our intuition.* We have various kinds of tests, some of them, like the syllogism, substitution, or finite selection, universally relied upon, but others, such as those making use of the choice axiom or the total second ordinal class, not generally admitted. But even though some of these types of proof fail of general acceptance, is there any reason why we should not take successful passing of such tests as evidence that the concept is a possible candidate for admission to the body of mathematical theory? Some of my contemporaries have even gone so far as to question the sanity of any individual who makes so bold as to entertain thoughts about the continuum hypothesis. I am reminded that an eminent contemporary, in an address before this Association not so long ago*—an address which I take great delight in reading every now and then because of its fresh wit and clarity—took occasion to deride the statement which Sierpinski makes about the continuum hypothesis in the introduction to his book to the effect that "There are some people, even among eminent scholars, who doubt the possibility of ever solving the problem of the continuum. Under these circumstances, the consequences of the hypothesis of the continuum can be considered practically as if they were true." Now, although I don't like the occurrence of the word "true" here, if the assumption of a principle, as say of the hypothesis of the continuum, leads to no contradiction, why not take it as valid mathematics? I don't say that we should make

* Bell, E. T., The place of rigor in mathematics, this MONTHLY, vol. 41, 1934, pp. 599–607.

absence of contradiction the sole criterion for mathematical admissibility, but absence of contradiction together with mathematical usefulness are, or ought to be, sufficient. And certainly no one can deny that the continuum hypothesis is mathematically fruitful. I note, parenthetically, that Gödel established its consistency with certain well-known axioms of set theory provided the latter are consistent—in fact he did much more than this in that he included the axiom of choice and the generalized continuum hypothesis.* One of our most orthodox methods of proof is that of reductio ad absurdum, and if the assumption of a hypothesis leads to no contradiction, why not admit this as evidence for its mathematical admissibility? And it seems to me that the continuum hypothesis has passed plenty of tests, aside from the result of Gödel which I have just mentioned, in its mathematical fruitfulness.

6. Conclusion. In conclusion, then, I wish to repeat my belief that what we call "proof" in mathematics is nothing but a testing of the products of our intuition. Obviously we don't possess, and probably will never possess, any standard of proof that is independent of time, the thing to be proved, or the person or school of thought using it. And under these conditions, the sensible thing to do seems to be to admit that there is no such thing, generally, as absolute truth in mathematics, whatever the public may think. Our intuition suggests certain results, and they seem mathematically desirable—and, moreover, prove to be generally liked by the mathematical public. We test them by what we call a proof, of course—if they didn't pass any such test, or, worse still, if their assumption leads to contradiction, we wouldn't pass them on to our colleagues for their judgment as to their worth. Naturally, we cannot assume a dogmatic attitude about them—we shouldn't, as far as that is concerned, be dogmatic even about arithmetic—but we can rely, I think, upon the combined evidence of our intuition and the test which we call "proof," even though the latter may be rejected by some of our colleagues who have somehow attained, possibly from psychic sources, a preferred position among the gods that they think rule over the mathematical universe.

I suppose I am an intuitionist in the sense that I think that a mathematician is to be judged by the quality and reliability of his intuition at least as much as by his ability to prove something. And I am inclined to agree with the statement which I have heard attributed to the late E. H. Moore, that "Sufficient unto the day is the rigor thereof." We certainly owe a lot to those mathematicians of an earlier day whose rigor was rather non-existent from present-day points of view. And if you'll pardon a personal interpolation, if my intuition tells me that a certain theorem is desirable, and seems free of contradiction, I'll prove it in any manner I best can; if my proof makes use of methods or assumptions that certain more dogmatic of my brethren choose to reject outright, I'll prove it just the same and hopefully present it to a reputable journal—

* Gödel, K., The consistency of the continuum hypothesis, Annals of Mathematics Studies, No. 3, Princeton, 1940.

especially if it seems mathematically important. At the same time, however, I won't talk about "truth," "absolute rigor," and their ilk—for all I know they are will-o'-the-wisps—ideal conceptions, possibly, but with no natural habitat in the mathematical world.

APPENDIX*

I. Totalities. We use the term *totality* in Section 2a to indicate a collection of things supposed already existent. Thus, the collection, N, of natural numbers (positive integers) is not a totality when conceived of as in some sense a variable finite collection beginning with 1, 2, 3 and increased at will by the addition of a natural number obtained, say, by adding 1 to the largest number already in the collection. Rather, the totality N contains *all* natural numbers—we recognize that 123,341,687,912,847,712,868 is an element of this totality although perhaps it was never before written down.

In like manner we speak of the totality R of real numbers. In this case each element of the totality—a real number—is itself a totality, say, an ordered totality of digits in the decimal system. Thus $\sqrt{2}$ is $1.414 \cdots$, each digit in the array being determined by a certain law (*e.g.*, the algorithm for finding the square root). We cannot expect, however, to know such a law for the determination of the digits of every real number.

Between totalities there can be set up an equivalence relation, namely, the relation of 1-1 *correspondence*: If A and B are two totalities and if there exists a function f on A such that for every element a of A, $f(a)=b$ is an element of B and such that (1) if $a_1 \neq a_2$, $f(a_1) \neq f(a_2)$, (2) for every element b of B there is an element a such that $f(a)=b$, then f is called a 1-1 correspondence between A and B,† and A and B are said to *have the same cardinal number*; in symbols, $A \sim B$. Clearly (1) $A \sim A$, (2) $A \sim B$ implies $B \sim A$, and (3) $A \sim B$, $B \sim C$ imply $A \sim C$. Consequently each totality A determines a class $\mathfrak{C}(A)$ consisting of all those totalities X such that $A \sim X$, and if $A \sim X$ then $\mathfrak{C}(X)=\mathfrak{C}(A)$.‡ We may, with Frege, use the concept of the class $\mathfrak{C}(X)$ to define number. From this point of view, if A is the totality of all states in the United States, then $\mathfrak{C}(A)$ is customarily written 48. We call $\mathfrak{C}(A)$ the *cardinal number* of A. The cardinal numbers of infinite totalities are called *transfinite*. The number $\mathfrak{C}(N)$ is denoted by $\aleph_0$ (aleph-null), and any totality T such that a 1-1 correspondence exists between N and T has consequently the cardinal number $\mathfrak{C}(T)=\aleph_0$. We also specify that $\mathfrak{C}(T)=\aleph_0$ by the statement that "T is denumerable." And if a set T is denumerable, its elements can be considered as forming a sequence $t_1, t_2, \cdots, t_n, \cdots$.

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* For more extended expositions of concepts that are of necessity but briefly described here, the reader is referred to the excellent books of Sierpinski cited above.

† It would be more appropriate to say "1-1 correspondence between the elements of A and the elements of B," but we shall use the abbreviated form despite its inexactness.

‡ We use $=$ here to denote identity.

The cardinal number $\mathfrak{C}(R)$ is denoted by c. It will be shown below that $\aleph_0$ and c are different. An elementary statement of the *continuum hypothesis*, which is first referred to in Section 1 above, is that if M is a totality of real numbers that is not denumerable, then the cardinal number of M must be c.

N.B. Although the word "totality" seems to express most exactly the ideas involved, we frequently use the more common and less cumbersome synonymous term "set" in what follows.

II. Ordered totalities. The totality N is an example of an *ordered* infinite totality, where by "order" we mean the natural order $1<2$, $2<3$, etc.—"order by magnitude." Similarly, R is an ordered totality, again order by magnitude in the usual sense. But whereas in this case certain parts of R—such as the totality of positive real numbers—have no smallest element, every part of N has a smallest element. The distinction is denoted by the term "well-ordered," N being well-ordered and R not well-ordered. Explicitly, a totality T is *well-ordered* with respect to a binary relation $<$ if (1) T is simply ordered (*i.e.*, if x, y, z denote elements of T, then (a) if x and y are not identical, $x<y$ or $y<x$, (b) if $x<y$ then x and y are not identical and (c) $x<y$, $y<z$ imply $x<z$), and (2) if T' is a part of T (possibly T itself), then T' has a first element.

Between well-ordered totalities there can be set up a special type of equivalence relation $\approx$: If W_1 and W_2 are well-ordered sets and there exists a 1-1 correspondence between them which preserves order (*i.e.*, if a, b are elements of W_1 and $a<b$ then $f(a)<f(b)$ in W_2, and conversely), then we say W_1 and W_2 *have the same ordinal number*, and write $W_1 \approx W_2$. And just as we defined cardinal number above we may define for any set A which is well-ordered an *ordinal number* $\mathfrak{O}(A)$, which is the class of all well-ordered sets X such that $A \approx X$. But it should be emphasized that whereas $\mathfrak{C}(A)$ is unique for a given set A, $\mathfrak{O}(A)$ is unique only for a given well-ordering of A. For this reason it is advisable to replace $\mathfrak{O}(A)$ by $\mathfrak{O}(A, <)$, where $<$ is a symbol indicating the given ordering. For example, consider (1) N and the order $<$ given by its natural ordering, and (2) N and the order $<'$ defined as follows: If a, b are both odd, or both even, then $<'$ is the same as $<$; but if a is odd and b is even, then $a<'b$ (thus $2<3$ but $3<'2$). Then N is well-ordered with respect to both $<$ and $<'$, but it is easily shown that $\mathfrak{O}(N, <)$ and $\mathfrak{O}(N, <')$ are different. Generally, then, the same set has many different well-orderings, and to each of these corresponds an ordinal number.

In particular, the ordinal numbers that may be obtained from the various well-orderings of N are called the *ordinal numbers of the second class*—thus $\mathfrak{O}(N, <)$ and $\mathfrak{O}(N, <')$ above are two of the ordinals of the second class. The number $\mathfrak{O}(N, <)$ is denoted by ω, and is called an "omega series." Note, however, that the ordinal number of a finite set is unique—there is only one type of well-ordering for a set of 3 elements, for example—and it is customary to lump all such finite ordinals into one class, called the *first class*. And ordinal numbers not in the first class are called *transfinite ordinals*.

III. The well-ordering theorem, choice axiom, comparability. It is natural to inquire into the possibility of well-ordering a totality which, as it comes to our attention, is either not ordered at all, or if ordered is not well-ordered (for example, R is simply ordered with respect to its order by magnitude, but not well-ordered). As stated in II, $\aleph_0 \neq c$. It will be noted that this is equivalent to asserting that there is no well-ordering $<''$ of R such that $\mathfrak{O}(R, <'') = \mathfrak{O}(N, <)$. The classical proof of this assertion is that of Cantor which (with minor modifications) is as follows:

Assume $\aleph_0 = c$, and therefore that R can be considered as ordered in an omega series $r_1, r_2, \cdots, r_n, \cdots$. Let us denote the decimal part of each r_n by $a_1^n a_2^n \cdots a_n^n \cdots$, and let r be the number $0.b_1 b_2 \cdots b_n \cdots$ where b_n is 1 in case a_n^n is not 1, and b_n is 2 if a_n^n is 1. Then r is recognized by its form to be a real number, yet differs from r_1 in the first decimal digit, and generally from r_n in the nth decimal digit—contradicting the fact that all elements of R were supposed to be in the series $r_1, r_2, \cdots, r_n, \cdots$.

Now no one has ever determined a well-ordering for R. Moreover, no one has ever determined all types of well-orderings even of the totality N. Thus, although we know the $\mathfrak{O}(N, <)$ and $\mathfrak{O}(N, <')$ above, as well as infinitely many other ordinal numbers derived from the well-orderings of N, it is fairly well agreed that it would be impossible to define all such—in other words, to define effectively all transfinite ordinals of the second class. It is not surprising, then, that a well-ordering of R, which in a sense would exhibit an "infinitely longer" sequence than any well-ordering of N, has never been found. And it is also not surprising that many mathematicians felt that no well-ordering of R could possibly exist.

Despite this, and despite the fact that one can demonstrate formally the existence of infinitely many other cardinal numbers greater, in a sense that we do not define here, than either $\aleph_0$ or c, Zermelo proved in his famous paper *Beweis, dass jede Menge wohlgeordnet werden kann,** that every totality has a well-ordering; that is, given any set A whatsoever, there exists an order relation $<_a$ such that A is well-ordered with respect to $<_a$. A close inspection of the proof reveals, as it did to Borel,† that the proof is based upon a principle that had never received strict formulation and yet had undoubtedly been employed time and again, the principle now known variously as the "choice axiom" or "Zermelo postulate." Suppose M is a set; let us designate some particular element m of M as a *representative element* of M and write $r(M) = m$. (For example, if M is the set N, we might arbitrarily let $r(N) = 6$. Or if M is the set of residents of Maine, we can select any particular resident of Maine as $r(M)$) The simplest form of the choice axiom is as follows: If $\mathfrak{M}$ is a collection of mutually exclusive sets M, then there exists a *representative set* $\mathfrak{R}(\mathfrak{M})$ whose elements are representative elements $r(M)$, there being exactly one such rep-

* Mathematische Annalen, vol. 59, 1904, pp. 514–516.

† Borel, E., Quelques remarques sur les principes de la theorie des ensembles, Mathematische Annalen, vol. 60, 1905, pp. 194–195.

resentative for each set M in $\mathfrak{M}$. Thus if $\mathfrak{M}$ is the set of all states in the United States, where each state is considered as the collection of all its residents, it being assumed that no person is a resident of more than one state, then $\mathfrak{R}(\mathfrak{M})$ would consist of 48 people. The point of the "axiom" is that we can assume that it is not necessary actually to *name* each person in $\mathfrak{R}(\mathfrak{M})$; we can just assume that such a set as $\mathfrak{R}(\mathfrak{M})$ exists without going to all the bother of "naming"; and that when sets are not given to us as collections of well "named" individuals, we can hardly be expected to name representative elements.

But as a result of Zermelo's proof it appears that if one wishes to reject the idea of any well-ordering of the set R of all real numbers, for example, he must, to be consistent, reject the choice axiom. It is now known that numerous other parts of mathematics must simultaneously be rejected in that event. In particular, it is known that the choice axiom, the well-ordering theorem of Zermelo, and the comparability principle* are all equivalent—each implies the other two. So, for example, if we reject the choice axiom, we reject the possibility of comparing in any manner the relative "magnitude" of two arbitrary cardinal numbers.

IV. The continuum hypothesis. Assuming the choice axiom, and hence that every set can be well-ordered, the transfinite cardinal numbers can be considers as "alephs." An *aleph* is the cardinal number of an infinite well-ordered set. For the set N we have already designated the cardinal number by the symbol $\aleph_0$. It is not difficult to show by transfinite induction (see Section 2a above) that if W_1 and W_2 are infinite well-ordered sets with order relations $<_1$, $<_2$ respectively, then one of them, say W_1, has the same ordinal number as part of W_2; and if this part is not all of W_2, we say that $\mathfrak{O}(W_1, <_1) < \mathfrak{O}(W_2, <_2)$.

Denoting the cardinal numbers of W_1, W_2 by $\aleph'$, $\aleph''$ respectively, suppose that by new choices of $<_1$, $<_2$ we have $\mathfrak{O}(W_2, <_2) < \mathfrak{O}(W_1, <_1)$; then we say that $\aleph' = \aleph''$; but if such new choices are impossible, then that $\aleph' < \aleph''$. Then the alephs turn out to be well-ordered with respect to this $<$. The so-called *problem of the continuum* is to determine the position of c as an aleph in this well-ordering.

Let R be well-ordered with respect to an order relation $<_r$. Then if $\mathfrak{O}(N, <)$ is any well-ordering of N, $\mathfrak{O}(N, <) < \mathfrak{O}(R, <_r)$ and it is impossible that $\mathfrak{O}(R, <_r) < \mathfrak{O}(N, <)$ since this would give $c = \aleph_0$.† Hence if c is denoted by $\aleph(c)$, we have $\aleph_0 < \aleph(c)$. The assumption that $\aleph(c) = \aleph_1$ is the usual statement of the continuum hypothesis.

* The comparability principle states that if two totalities are given, there always exists a 1-1-correspondence between one and part of the other.

† It can be shown that the equality as defined in IV is the same as that whose impossibility was proved in III.

CONCERNING SOME PERSPECTIVE TRIANGLES

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1. Introduction. Let $F_1=0$, $F_2=0$, $F_3=0$ be three equations in two variables each, and set

$$L_1 \equiv k_2F_3 - k_3F_2, \qquad L_2 \equiv k_3F_1 - k_1F_3, \qquad L_3 \equiv k_1F_2 - k_2F_1,$$

where k_1, k_2, k_3 are constants. Then $L_1=0$ is satisfied by the simultaneous solutions of $F_2=0$ and $F_3=0$ and also by the simultaneous solutions of $F_1+k_2F_3=0$ and $F_1+k_3F_2=0$. Similar remarks hold for $L_2=0$ and $L_3=0$. Also, since $k_1L_1+k_2L_2+k_3L_3\equiv 0$, it follows that $L_1=0$, $L_2=0$, $L_3=0$ have for simultaneous solutions all simultaneous solutions of any pair of $L_1=0$, $L_2=0$, $L_3=0$.

In each of the following articles we propose to select $F_1=0$, $F_2=0$, $F_3=0$ as equations, in special form, of three positions of some geometric entity, and then to interpret the above algebraic analysis for the resulting figure. This is a useful trick for obtaining a set of different but related theorems. In such a process we are often likely to end up with a projective generalization including all the other cases. At any rate we obtain, in the succeeding articles, some new and rich figures which will call forth some new geometric definitions. We will be able to mention but a few of the theorems suggested.

2. Isogonal lines. Take $F_1=0$, $F_2=0$, $F_3=0$ as cartesian equations, in perpendicular form, of three straight lines, and let the origin of coordinates lie inside triangle $F_1F_2F_3$.

LEMMA. *The lines $F_3+k_1F_2=0$ and $F_2+k_1F_3=0$ are a pair of isogonal lines with respect to angle F_2F_3.*

For let p designate a point on $F_3+k_1F_2=0$ and q a point on $F_2+k_1F_3=0$. Then

$$F_3(p)/F_2(p) = F_2(q)/F_3(q) = -k_1.$$

But, since $F_2=0$ and $F_3=0$ are in perpendicular form, this says that the distances of the points p and q from the lines $F_2=0$ and $F_3=0$ are inversely proportional. This proves the lemma.

THEOREM 1. *If at each vertex of a given triangle ABC a pair of isogonal lines be drawn, then the triangle $A'B'C'$, whose vertices are the points of intersection of pairs of these lines belonging to the same side of the given triangle, is perspective with the given triangle.*

Let $F_1=0$, $F_2=0$, $F_3=0$ be the equations, in perpendicular form, of the three sides of the given triangle. Let the three pairs of isogonal lines be $F_3+k_1F_2=0$ and $F_2+k_1F_3=0$, $F_1+k_2F_3=0$ and $F_3+k_2F_1=0$, $F_2+k_3F_1=0$ and $F_1+k_3F_2=0$. The theorem then follows by article 1.

COROLLARY 1.1. *The Morley triangle of a given triangle is in perspective with the given triangle.*

COROLLARY 1.2. *The altitudes of a triangle are concurrent.*

Reflect each vertex of the given triangle ABC in its opposite side, giving triangle $A'B'C'$. By Theorem 1 triangles ABC and $A'B'C'$ are in perspective.

THEOREM 2. *Given three concurrent cevians, AD, BE, CF, of a triangle ABC. Take any point A' on AD. Let the isogonals of BA' and CA' meet the cevians CF and BE in C' and B' respectively. Then AC' and AB' are isogonal lines for angle A.*

This is an easy converse of Theorem 1.

THEOREM 3. *If m, r, u are concurrent cevians of triangle ABC, then n, s, v, the isogonal lines of m, r, u, are also concurrent cevians of triangle ABC.*

Take A' in Theorem 2 as the intersection of s and u. (These points of concurrency are well known as *isogonal conjugate points* for triangle ABC.)

We state the following three theorems without proof as each is a result easily achieved by long but straightforward analytical geometry. We set $F_1 \equiv a_1x+a_2y-p$, $F_2 \equiv b_1x+b_2y-q$, $F_3 \equiv c_1x+c_2y-r$. (In the present article, of course, we have the restriction $a_1^2+a_2^2=b_1^2+b_2^2=c_1^2+c_2^2=1$.)

THEOREM 4. *If $Q:(\bar{x}, \bar{y})$ is the center of perspectivity, then*

$$\bar{x} = \frac{\begin{vmatrix} p & a_2 & k_1 \\ q & b_2 & k_2 \\ r & c_2 & k_3 \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 & k_1 \\ b_1 & b_2 & k_2 \\ c_1 & c_2 & k_3 \end{vmatrix}}, \qquad \bar{y} = \frac{\begin{vmatrix} a_1 & p & k_1 \\ b_1 & q & k_2 \\ c_1 & r & k_3 \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 & k_1 \\ b_1 & b_2 & k_2 \\ c_1 & c_2 & k_3 \end{vmatrix}}.$$

COROLLARY 4. *The infinitude of triangles determined by the values mk_1, mk_2, mk_3, m a parameter and the k's constants, has a common center of perspectivity with the given triangle.*

THEOREM 5. *The equation of the axis of perspectivity is*

$$\sum [k_1(k_2^2 + k_3^2) + k_2k_3(1 + k_1^2)]F_1 = 0,$$

reducing to

$$F_1 + F_2 + F_3 = 0$$

in the case where $k_1 = k_2 = k_3 = k \neq -1, 0$.

COROLLARY 5. *The infinitude of triangles determined by the values $k_1 = k_2 = k_3 = k$, where k is a parameter, has a common center of perspectivity with the given triangle, namely the incenter of the given triangle, and also a common axis of perspectivity with the given triangle.*

THEOREM 6. *The equations of the sides of triangle $A'B'C'$ are*

$$S_1 \equiv k_2k_3(k_1^2 - 1)F_1 + k_1(k_3 + k_1k_2)F_2 + k_1(k_2 + k_3k_1)F_3 = 0,$$

$$S_2 \equiv k_2(k_3 + k_2k_1)F_1 + k_3k_1(k_2^2 - 1)F_2 + k_2(k_1 + k_3k_2)F_3 = 0,$$

$$S_3 \equiv k_3(k_2 + k_1k_3)F_1 + k_3(k_1 + k_2k_3)F_2 + k_1k_2(k_3^2 - 1)F_3 = 0,$$

reducing to

$$S_1 \equiv (k - 1)F_1 + F_2 + F_3 = 0,$$

$$S_2 \equiv F_1 + (k - 1)F_2 + F_3 = 0,$$

$$S_3 \equiv F_1 + F_2 + (k - 1)F_3 = 0,$$

in the case where $k_1 = k_2 = k_3 = k \neq -1$, 0.

COROLLARY 6. *For $k_1 = k_2 = k_3 = 2$ we have A', B', C' collinear.*

The figure for this corollary is very interesting. The three pairs of isogonals form a hexagon about ABC, and the three pairs of opposite sides of this hexagon meet in three collinear points, lying on the axis of perspectivity of the given triangle and the triangle formed by the feet of its angle bisectors.

3. Isotomic points. Take $F_1 \equiv a_1u + a_2v + 1 = 0$, $F_2 \equiv b_1u + b_2v + 1 = 0$, $F_3 \equiv c_1u + c_2v + 1 = 0$, Plücker line equations, in non-homogeneous form, of three points A, B, C.

LEMMA. *The points $F_3 + k_1F_2 = 0$ and $F_2 + k_1F_3 = 0$ are a pair of isotomic points with respect to the segment F_2F_3.*

For the cartesian coordinates of points $F_2 = 0$ and $F_3 = 0$ are (b_1, b_2) and (c_1, c_2) respectively. Similarly, the cartesian coordinates of the points $F_3 + k_1F_2 = 0$ and $F_2 + k_1F_3 = 0$ are

$$\left(\frac{c_1 + k_1b_1}{1 + k_1}, \frac{c_2 + k_1b_2}{1 + k_1}\right) \text{ and } \left(\frac{b_1 + k_1c_1}{1 + k_1}, \frac{b_2 + k_1c_2}{1 + k_1}\right)$$

respectively. These points are obviously isotomic points for the segment F_2F_3.

THEOREM 1. *If on each side of a given triangle ABC a pair of isotomic points be taken, then the triangle $A'B'C'$, whose sides are the joins of these points belonging to the same vertex of the given triangle, is perspective with the given triangle.*

This is merely an interpretation of Article 1 where $F_1 = 0$, $F_2 = 0$, $F_3 = 0$ are the Plücker line equations, in non-homogeneous form, of the vertices of the given triangle. This theorem is a sort of dual of Theorem 1 Article 2.

As the corresponding duals of Theorems 2 and 3 Article 2 we have the following result.

THEOREM 2. *Let a line cut the sides BC, CA, AB of a triangle ABC in the points D, E, F respectively. Through F draw a line cutting AC and BC in R and M respectively. Take N on BC isotomic with respect to M, and S on AC isotomic with respect to R. Let EN and DS cut AB in U and V respectively. Then U and V are isotomic points on AB.*

THEOREM 3. *If M, R, U are collinear points on the sides of a triangle ABC, then N, S, V, the isotomic points of M, R, U, are also collinear points on the sides of triangle ABC.*

DEFINITION. We will call the two lines of collinearity of Theorem 3 *isotomic conjugate lines* of triangle ABC.

From Theorems 4 and 5 Article 2 we now obtain the following theorem.

THEOREM 4. *If $q:(\bar{u}, \bar{v})$ is the axis of perspectivity, then*

$$\bar{u} = -\frac{\begin{vmatrix} 1 & a_2 & k_1 \\ 1 & b_2 & k_2 \\ 1 & c_2 & k_3 \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 & k_1 \\ b_1 & b_2 & k_2 \\ c_1 & c_2 & k_3 \end{vmatrix}}, \qquad \bar{v} = -\frac{\begin{vmatrix} a_1 & 1 & k_1 \\ b_1 & 1 & k_2 \\ c_1 & 1 & k_3 \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 & k_1 \\ b_1 & b_2 & k_2 \\ c_1 & c_2 & k_3 \end{vmatrix}}.$$

COROLLARY 4. *The infinitude of triangles determined by the values mk_1, mk_2, mk_3, m a parameter and the k's constants, has a common axis of perspectivity with the given triangle.*

THEOREM 5. *The Plücker equation, in non-homogeneous form, of the center of perspectivity is*

$$\frac{\sum [k_1(k_2^2 + k_3^2) + k_2k_3(1 + k_1^2)]F_1}{\sum [k_1(k_2^2 + k_3^2) + k_2k_3(1 + k_1^2)]} = 0,$$

reducing to

$$(F_1 + F_2 + F_3)/3 = 0$$

in the case where $k_1 = k_2 = k_3 = k \neq -1, 0$.

COROLLARY 5. *The infinitude of triangles determined by the values $k_1 = k_2 = k_3 = k$, say, where k is a parameter, has a common center of perspectivity with the given triangle, namely the common centroid of all the triangles and of the given triangle, and also a common axis of perspectivity with the given triangle, namely the line at infinity.*

We note that Corollary 6 Article 2 becomes trivial here.

4. Isotomic lines. Take $F_1 \equiv y-ax-p=0$, $F_2 \equiv y-bx-q=0$, $F_3 \equiv y-cx-r=0$, cartesian equations, in slope-intercept form, of three straight lines.

DEFINITION. Given an angle BAC and a line L not through vertex A, cutting AB and AC in M and N respectively. Let U and V be a pair of isotomic points with respect to the segment MN. Then the lines AU and AV will be known as a pair of *isotomic lines* of the angle BAC with respect to the line L.

(The concept of isotomic lines is one worth introducing into College Geometry. Note that isotomic lines of angle BAC with respect to line L are also isotomic lines of angle BAC with respect to any line parallel to L and not passing through A. Also note that isogonal lines of angle BAC are isotomic lines of angle BAC with respect to any line not passing through A and perpendicular to the angle bisector of angle BAC. Any two lines through the vertex A are isotomic lines of angle BAC for some line L.)

LEMMA. *The lines $F_3+k_1F_2=0$ and $F_2+k_1F_3=0$ are a pair of isotomic lines of the angle F_2F_3 with respect to the y-axis.*

For we have

$$F_3 + k_1F_2 \equiv y\,(1 + k_1) - (c + k_1b)x - (r + k_1q) = 0.$$

Hence the y-intercept of $F_3+k_1F_2=0$ is $(r+k_1q)/(1+k_1)$. Similarly, the y-intercept of $F_2+k_1F_3=0$ is $(q+k_1r)/(1+k_1)$. This proves the lemma.

THEOREM 1. *Given a triangle ABC and a line L not through a vertex, and a pair of isotomic lines through each vertex with respect to the line L. Then the triangle $A'B'C'$, whose vertices are the points of intersection of pairs of these lines belonging to the same side of the given triangle, is perspective with the given triangle.*

This is merely an interpretation of Article 1 where $F_1=0$, $F_2=0$, $F_3=0$ are the cartesian equations, in slope-intercept form, of the sides of the given triangle, and where L is selected as the y-axis.

Theorems 4 and 5 of Article 2 now yield the following theorem.

THEOREM 2. *If $Q:(\bar{x}, \bar{y})$ is the center of perspectivity, then*

$$\bar{x} = \frac{\begin{vmatrix} p & 1 & k_1 \\ q & 1 & k_2 \\ r & 1 & k_3 \end{vmatrix}}{\begin{vmatrix} -a & 1 & k_1 \\ -b & 1 & k_2 \\ -c & 1 & k_3 \end{vmatrix}}, \qquad \bar{y} = \frac{\begin{vmatrix} -a & p & k_1 \\ -b & q & k_2 \\ -c & r & k_3 \end{vmatrix}}{\begin{vmatrix} -a & 1 & k_1 \\ -b & 1 & k_2 \\ -c & 1 & k_3 \end{vmatrix}}.$$

COROLLARY 2.1. *If $k_1:k_2:k_3=p:q:r$, then Q is at the origin; that is, the center of perspectivity lies on L.*

COROLLARY 2.2. *If $k_1=k_2=k_3=k$, then Q is at infinity in the direction of L.*

COROLLARY 2.3. *The infinitude of triangles determined by the values mk_1, mk_2, mk_3, where m is a parameter and the k's are constants, has a common center of perspectivity with the given triangle.*

THEOREM 3. *The cartesian equation, in slope-intercept form, of the axis of perspectivity is given by the equations of Theorem 5 Article 3.*

COROLLARY 3. *If $k_1=k_2=k_3=k$, then the y-intercept of the axis of perspectivity is $(p+q+r)/3$. That is, the axis of perspectivity cuts L in the centroid of the points where the sides of the triangle cut L.*

THEOREM 4. *Given a triangle ABC and a line L cutting the sides BC, CA, AB in D, E, F respectively. Take points V and W on L such that $FV/VE=FW/WD$. Let AV and BW meet in C'. Then CC' is parallel to L.*

By Corollary 2.2 we have the next theorem.

THEOREM 5. *If m, r, u are concurrent cevians of triangle ABC, then n, s, v, the isotomic lines of m, r, u for a given line L, are also concurrent cevians of triangle ABC.*

DEFINITION. We will call the two points of concurrency of Theorem 5 a pair of *isotomic conjugate points* of triangle ABC *for the line L.*

THEOREM 6. *If $k_1=k_2=k_3=2$, then A', B', C' are collinear.*

The figure for this theorem is very interesting.

5. Isogonal points. Without pausing to set up the analytical geometry we merely state that isotomic lines may be dualized by the following definition.

DEFINITION. Given a line segment BC and a point P not on BC. Select on BC two points M and N such that PM and PN are isogonal lines for angle BPC. Then M and N will be known as a pair of *isogonal points* of the segment BC with respect to the point P. (This is another concept worth introducing into College Geometry.)

The dual of Theorem 1, Article 4 then appears as follows.

THEOREM 1. *Given a triangle ABC and a point P not on any side, and a pair of isogonal points with respect to P on each side of the triangle. Then the triangle $A'B'C'$, whose sides are the joins of these points belonging to the same vertex of the given triangle, is perspective with the given triangle.*

6. Isoanharmonic lines. Let $F_1 \equiv a_1x+a_2y-1=0$, $F_2 \equiv b_1x+b_2y-1=0$, $F_3 \equiv c_1x+c_2y-1=0$, cartesian equations, in intercept form, of three straight lines.

DEFINITION. Given an angle BAC and a point P not on the sides of the angle. Let AM and AN be lines such that $A(PBMC)=A(PCNB)$. Then the lines AM and AN will be known as a pair of *isoanharmonic lines* of angle A with respect to point P.

LEMMA. *The lines $F_3+k_1F_2=0$, $F_2+k_1F_3=0$ are a pair of isoanharmonic lines of the angle F_2F_3 with respect to the origin.*

For we have

$$F_3 + k_1F_2 \equiv (c_1 + k_1b_1)x + (c_2 + k_1b_2)y - (1 + k_1) = 0.$$

Hence the x-intercept of $F_3+k_1F_2=0$ is $(1+k_1)/(c_1+k_1b_1)$. Similarly, the x-intercept of $F_2+k_1F_3=0$ is $(1+k_1)/(b_1+k_1c_1)$. Then we have

$$\left(0, \frac{1}{c_1}, \frac{1+k_1}{c_1+k_1b_1}, \frac{1}{b_1}\right) = \frac{1}{k_1},$$

$$\left(0, \frac{1}{b_1}, \frac{1+k_1}{b_1+k_1c_1}, \frac{1}{c_1}\right) = \frac{1}{k_1}.$$

Hence the lemma is proved.

THEOREM 1. *Given a triangle ABC and a point P not on any of the sides, and through each vertex a pair of isoanharmonic lines with respect to the point P. Then the triangle $A'B'C'$, whose vertices are the points of intersection of pairs of these lines belonging to the same side of the given triangle, is perspective with the given triangle.*

This is merely an interpretation of Article 1 where $F_1=0$, $F_2=0$, $F_3=0$ are the cartesian equations, in intercept form, of the sides of the given triangle, P being the origin of coordinates.

THEOREM 2. *$Q(\bar{x}, \bar{y})$ is the center of perspectivity, then*

$$\bar{x} = \frac{\begin{vmatrix} 1 & a_2 & k_1 \\ 1 & b_2 & k_2 \\ 1 & c_2 & k_3 \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 & k_1 \\ b_1 & b_2 & k_2 \\ c_1 & c_2 & k_3 \end{vmatrix}}, \qquad \bar{y} = \frac{\begin{vmatrix} a_1 & 1 & k_1 \\ b_1 & 1 & k_2 \\ c_1 & 1 & k_3 \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 & k_1 \\ b_1 & b_2 & k_2 \\ c_1 & c_2 & k_3 \end{vmatrix}}.$$

COROLLARY 2. *If we take $k_1=k_2=k_3=k$, then Q is at the origin; that is, Q coincides with P.*

THEOREM 3. *Article 6 is a projective generalization of Articles 2 and 4. In Article 2, P is at the incenter of triangle ABC; in article 4, P is at infinity in the direction of line L.*

7. Isoanharmonic points. We now dualize isoanharmonic lines by the

DEFINITION. Given a line segment BC and a line L not through B or C. Let M and N be points on BC such that $(LBMC)=(LCNB)$. Then the points M and N will be known as a pair of *isoanharmonic points* of the segment BC with respect to the line L.

Our principal theorem becomes the following.

THEOREM 1. *Given a triangle ABC and a line L not through any of its vertices, and on each side a pair of isoanharmonic points with respect to the line L. Then the triangle $A'B'C'$, whose sides are the joins of these points belonging to the same vertex of the given triangle, is perspective with the given triangle.*

THEOREM 2. *Article 7 is a projective generalization of articles 3 and 5.*

8. Isopotential circles. Take $F_1=0$, $F_2=0$, $F_3=0$ as cartesian equations, in standard form, of three circles.

DEFINITION. Given two circles whose equations, in standard form, are $F_2=0$ and $F_3=0$. Then the two circles $S_{23}\equiv F_3+k_1F_2=0$; $S_{32}\equiv F_2+k_1F_3=0$ (which are coaxal with the given circles) will be known as a pair of *isopotential circles.*

We note that circle S_{23} is the locus of points the ratio of whose powers with respect to circles F_2 and F_3 is $-k_1$, and S_{32} is the locus of points the ratio of whose powers with respect to circles F_3 and F_2 is also $-k_1$. This geometrically justifies the name *isopotential circles.*

Our principal theorem, now somewhat complex, assumes the following form.

THEOREM. *Given three circles F_1, F_2, F_3 and the circles*

S_{23}, S_{32}, *isopotentials of* F_2 *and* F_3,

S_{31}, S_{13}, *isopotentials of* F_3 *and* F_1,

S_{12}, S_{21}, *isopotentials of* F_1 *and* F_2.

Then there exist circles L_1, L_2, L_3 such that

L_1 *is coaxal with* F_2, F_3 *and* S_{12}, S_{13},

L_2 *is coaxal with* F_3, F_1 *and* S_{23}, S_{21},

L_3 *is coaxal with* F_1, F_2 *and* S_{31}, S_{32},

L_1, L_2, L_3 *are themselves coxal.*

COROLLARY. *If $k_1=k_2=k_3=-1$ we have the familiar situation of the three radical axes of three circles taken in pairs being concurrent.*

RATIONAL POINTS ON CUBIC CURVES AND SURFACES*

L. J. MORDELL, University of Manchester

1. Introduction. Let $f \equiv f(x_1, x_2, \cdots, x_n)$ be a cubic polynomial with rational coefficients in the n variables $x_1, x_2, \cdots, x_n$, where $n=2$ or 3, and where we suppose that f is irreducible and is not a function of less than n variables. We deal with the rational solutions of the equation $f=0$, *i.e.*, the rational points on the manifold $f=0$. On writing $x_1 = X_1/X_{n+1}, \cdots, x_n = X_n/X_{n+1}$, we may suppose that the X's are integers with $X_{n+1} \neq 0$, and then the equation $f=0$ becomes a homogeneous cubic equation

$$g \equiv g(X_1, X_2, \cdots, X_{n+1}) = 0$$

in $n+1$ variables. The problems of finding the integral solutions of $g=0$ with $X_{n+1} \neq 0$, and the rational ones of $f=0$ are identical.

2. Plane cubics. Suppose first that $n=2$. Write the equation as $f \equiv f(x, y) = 0$, thus defining a plane irreducible cubic curve. The curve has either one double point, (a, b) say, or none. In the first case all the rational points can be found very simply. It is easy to show that a, b are rational numbers. Then there is obviously a one-one correspondence between all the rational points on the curve and the lines $y-b=m(x-a)$, where m is any rational number. The intersection of this line with the curve gives x, y as rational functions of m with rational coefficients.

When the curve has no double point, no general criterion is known for the solvability of $f=0$, nor is any method known for finding even one solution, even though the equation has attracted the attention of mathematicians for at least three centuries and in some form or other has been the subject of innumerable papers.

When, however, a set of one or more rational points $P_1, P_2, P_3, \cdots$ is known on $f=0$, *e.g.*, by inspection, in general an infinity of other rational solutions can be found by a process known in principle for many centuries. Thus unless P_1 is a point of inflexion, the tangent at P_1 meets the curve in just one other point different from P_1, say $P_{1,1}$, and this is a rational point since it is determined by a rational equation of the first degree. Again, from two points P_1, P_2, neither of which lies on the tangent at the other, a third rational point is given by the remaining intersection, say $P_{1,2}$, of the chord P_{12} and the cubic. The chord and tangent process can be applied not only to those of the original set but also to all the points found in this way. In general, an infinity of rational points will be found though in exceptional cases the points so found form a finite polygon.

The process can also be described by simple algebra without any mention of geometry and was so known to Diophantos about 250 A. D. Suppose, *e.g.*, rational points $(p, \pm q)$, $q \neq 0$, are known on the curve

$$y^2 = ax^3 + bx^2 + cx + d.$$

* Lecture given to the Archimedeans, Cambridge University, England, October 20, 1943.

On transforming the origin to $(p, 0)$, we may suppose that $d=q^2$. We now seek for the particular solution with $y=q+(c/2q)x$. On substituting and dividing by x^2, we have a linear equation for x. Similarly, on assuming that

$$y = q + \frac{c}{2q}x + \lambda x^2,$$

where

$$2\lambda q + \frac{c^2}{4q^2} = b,$$

substituting and dividing by x^3, we have a linear equation in x. It is not difficult to derive this value from that found above on applying to it and its reflection in the x axis the chord and tangent process.

The known rational point may be at infinity. Thus for the cubic curve

$$y^3 = a^3x^3 + bx^2 + cx + d,$$

where b, c are not both zero, the assumption $y=ax+b/3a^2$ leads to a linear equation for the finite point of intersection of the curve and its asymptote $y=ax+b/3a^2$ when the point at infinity is not an inflexion.

3. The finite basis theorem. In 1901, Poincaré made the conjecture that all the rational points on a cubic curve without a double point could be deduced from a finite number by repeated application of the chord and tangent process. Probably this finite basis conjecture had been known long before. I proved it 22 years ago and give now a brief outline of the proof. It depends on the consideration of the rational points on the quartic curve

$$y^2 = ax^4 + bx^3 + cx^2 + dx + e.$$

Here, as for the cubic, on putting $e=q^2$, the assumption

$$y = q + \frac{d}{2q}x + \left(c - \frac{d^2}{4q^2}\right)\Big/ 2q$$

in general leads to a new rational point.

Suppose now that (p, q) is a rational solution of the cubic equation $f(x, y)=0$. Put $x=p+tu$, $y=q+t$. Then

$$t^2S_3 + tS_2 + S_1 = 0,$$

where S_1, S_2, S_3 are polynomials of degree 1, 2, 3 in u. Hence

$$2tS_3 + S_2 = v,$$

where

$$v^2 = S_2^2 - 4S_1S_3.$$

This is an equation of the form

$$v^2 = au^4 + bu^3 + cu^2 + du + e.$$

On putting $u=X/Y$, $v=Z/Y^2$, where X, Y, Z are integers without a common factor, and $a=A/F, \cdots, e=E/F$, where $A, \cdots, F$ are integers, it takes the form

$$FZ^2 = AX^4 + BX^3Y + CX^2Y^2 + DXY^3 + EY^4.$$

Let θ be a root of the equation

$$A\theta^4 + B\theta^3 + C\theta^2 + D\theta + E = 0.$$

Then

$$(X - \theta Y)(AX^3 + X^2Y(B + A\theta) + \cdots) = FZ^2.$$

Now from an equation of the form $UV=FW^2$, where U, V, W are integers and $(U, V)=1$, we find easily

$$U = \pm F_1P^2, \qquad V = \pm F_2Q^2,$$

where P, Q are any integers and F_1, F_2 belong to a finite set of integers. But the procedure is not so simple when the factors U, V are algebraic numbers. Though it is easy to define algebraic integers, their factorization leads to new principles and requires a deep study of the arithmetic theory of algebraic numbers. Then we can deduce an equation of the form

$$m(X - \theta Y) = (\alpha_0 + \beta_0\theta + \gamma_0\theta^2 + \delta_0\theta^3)(\alpha + \beta\theta + \gamma\theta^2 + \delta\theta^3)^2,$$

where α, β, γ, δ are any integers and α_0, β_0, γ_0, δ_0, m belong to a finite set of integers. Then if the original equation has an infinity of integral solutions, and if the set α_0, β_0, γ_0, δ_0, m occurs also for the particular solution X_0, Y_0, say, we deduce

$$n^2(X - \theta Y)(X_0 - \theta Y_0) = (A + B\theta + C\theta^2 + D\theta^3)^2,$$

where A, B, C, D are variable integers, and n is fixed.

On multiplying out and replacing θ^4, θ^5, θ^6 in terms of $1, \theta, \theta^2, \theta^3$ and equating terms on both sides, we deduce two equations between the unknowns A, B, C, D. Then it is shown that certain linear functions of A, B, C, D lead to new solutions x_1, y_1 of the equation, and we have

$$X = R_1(X_1, Y_1, Z_1), \qquad Y = R_2(X_1, Y_1, Z_1), \qquad Z = R_3(X_1, Y_1, Z_1),$$

where the R's are rational functions with rational coefficients. From these we can deduce

$$\max(|X_1|, |Y_1|) < k(\max(|X|, |Y|))^{1/2}$$

where k depends on X_0, Y_0. By continuing this process, called the method of infinite descent, with the solution X_1, Y_1, Z_1, we are led to a solution X_n, Y_n with max. $(|X_n|, |Y_n|) \leq k^2$, and there can be only a finite number of such solutions. Further, the relations between the solutions X, Y, Z; X_1, Y_1, Z_1, *etc.*, can be proved to be essentially in accordance with the tangent and chord process.

The finite basis theorem can be put in a very simple analytical form if the cubic be reduced, as it may when one solution is known, to that of the canonical form

$$y^2 = 4x^3 - g_2x - g_3.$$

Then $x = \wp(u)$, $y = \wp'(u)$ is the usual parametric representation by elliptic functions, and all the rational solutions are given by taking $u = n_1u_1 + n_2u_2 + \cdots + n_ru_r$, where $u_1, u_2, \cdots, u_r$ are a finite number of constants and $n_1, n_2, \cdots, n_r$ run through all integer values.

4. Cubic surfaces. We turn now to the cubic surface, say $f(x, y, z) = 0$, or $g(x, y, z, w) = 0$ in the homogeneous form. We exclude the cases when $f = 0$ is a cone, *i.e.*, when the surface has a triple point; and when $f = 0$ is a cylinder, *i.e.*, f is essentially a function of only two variables, since the problems then become those of the plane cubic. There are some simple cases when all the rational points are easily found, *e.g.*, when the surface has a rational double point. About 1756, Euler found a solution of the equation

$$x^3 + y^3 + z^3 + w^3 = 0,$$

a subject included in elementary books on number theory. As shown substantially by Hermite, a solution can be easily found for any cubic surface if, as in Euler's case, there exist on it two non-intersecting lines, the coefficients of whose equations are either rational numbers or conjugate quadratic numbers. For a one-one correspondence can be established between all the rational points P of the surface other than those on the two lines, and the rational points Q on any plane with rational coefficients, by imposing the conditions that the line PQ meets each of the lines.

Next, Kraft, Lagrange and Euler, about 1770, solved equations of the form

$$\prod_{\theta,\phi,\psi} (x + \theta y + \theta^2 z) = w^3,$$

where θ, ϕ, ψ are roots of a cubic equation with rational coefficients. A parametric solution in terms of three rational numbers A, B, C is given by taking

$$x + \theta y + \theta^2 z = (A + B\theta + C\theta^2)^3,$$
$$w = \prod (A + B\theta + C\theta^2).$$

Suppose next that a rational point P is known on the surface $f = 0$. The tangent plane at P meets the surface in general in an irreducible plane cubic curve with a double point at P and so we can at once find an infinity of rational points depending on a rational parameter. This result was given essentially by Libri in 1820, but was probably known before that time.

In 1826, Ryley found a parametric solution of the equation $x^3+y^3+z^3 = n$ by the aid of a most ingenious artifice whose significance seemed very obscure until fairly recently.

5. Recent developments. After Ryley's work, the subject lay dormant for many years. Then in 1930 another solution of Ryley's equations was given by Richmond who sought solutions in which x, y, z were proportional to cubic polynomials in a parameter t, and Ryley's solution was found to fit in with those so found. The success of Richmond's method depended on the existence of three rational solutions of $x^3+y^3+z^3=0$, namely the inflexion $(1, -1, 0)$, *etc.*

The last year has been fruitful in new advances by B. Segre and myself, accounts of which are appearing in the Journal of the London Mathematical Society. For the first time since the 120 years that elapsed since Ryley's work, a new type of equation was solved parametrically. This was

$$(x+y+z)^3 - dxyz = m, \qquad (d \neq 0)$$

which includes Ryley's as a special case as is clear on putting

$$x = Y+Z, \quad y = Z+X, \quad z = X+Y, \quad d = 24, \quad m = 8r.$$

I found a particular solution on taking $m = dyz^2$. Then

$$(x+y+z)^3 = dyz(x+z),$$

which in homogeneous coordinates is a cubic curve with a double point at $(-1, 0, 1)$. The curve meets the general line $x+z=py$ through the double point, where $(p+1)^3y^3 = pdy^2z$. Hence

$$\frac{y}{z} = \frac{dp}{(p+1)^3},$$

and so

$$\frac{x}{z} = \frac{dp^2}{(p+1)^3} - 1.$$

Then from $m = dyz^2$,

$$\frac{m}{z^3} = \frac{d^2p}{(p+1)^3}.$$

Since z is to be rational, $p = dmt^3$, where t is a new parameter. Hence

$$z = \frac{dmt^3+1}{dt}, \quad y = \frac{dmt^2}{(dmt^3+1)^2}, \quad x = \frac{d^2m^2t^5}{(dmt^3+1)^2} - \frac{dmt^3+1}{dt}$$

is a parametric solution.

Then B. Segre found a parametric solution of the equation

$$x^3+y^3+az^3 = b, \qquad (b \neq 0)$$

by noting that a solution could be deduced from one for

$$xy(x+y) + az^3 = b$$

by a well known transformation connecting the forms

$$x^3+y^3, \qquad XY(X+Y).$$

It is now possible to apply the geometric ideas of Richmond's paper, since the curve $xy(x+y)+az^3=0$ in homogeneous coordinates has three rational points of inflexion given by $z=0$, $xy(x+y)=0$.

Then Segre made a systematic study of the arithmetical properties of cubic surfaces, and found many new and important results of which I mention here only a few. Starting from Libri's result, he proved that every cubic surface contained either 0, 1, 3 or an infinity of rational points. Surfaces with no rational points exist (and this also holds for eight dimensional manifolds as shown by Hasse and also by myself), and a particular case,

$$x^3 + 2y^3 = 7(z^3 + 2w^3),$$

of this result is given by Segre. We may suppose that $(x, y, z, w) = 1$. Then since to mod 7,

$$x^3 \equiv 0, \pm 1, \qquad y^3 \equiv 0, \pm 1, \qquad x^3 + 2y^3 \equiv 0,$$

clearly $x \equiv y \equiv 0$. Then since $z^3+2w^3 \equiv 0$, $z \equiv w \equiv 0$. Hence $(x, y, z, w) \equiv 0$, a contradiction.

He showed next that surfaces with only 3 rational points, if any exist, could be reduced to a canonical form, substantially,

$$z^2 - (1 + px)^2 = ax^3 + bx^2y + cxy^2 + dy^3,$$

the three rational points being $(0, 0, \pm 1)$ and the point at infinity where their join meets the surface. He then established the existence of a parametric solution with 3 parameters and so every surface has 0, 1 or an infinity of rational solutions. He did this by considering the 18 intersections of the surface with suitably chosen quadric and cubic surfaces, showing that 17 of these intersections can be rationally assigned and so the eighteenth can be found by rational elimination processes.

Immediately afterwards, when I knew that the equation could be solved, I found a rational parametric solution by simple algebra, and then applied the same idea to the equation

$$z^2 - k(1 + px)^2 = ax^3 + bx^2y + cxy^2 + dy^3.$$

This is a particular case of the equation

$$z^2 = f(x, y),$$

where f is a cubic polynomial in x, y with rational coefficients. Segre showed that every cubic surface with only one rational point could be reduced to this form. He then applied my method to deduce a rational parametric solution of the equation and so finally proved that every cubic surface has either none or an infinity of rational solutions.

6. An illustration. I illustrate my method by finding a rational parametric solution of the equation

$$z^2 - k = ax^3 + by^3.$$

We may suppose that $ab \neq 0$ as otherwise the question reduces to that of the cubic curve, and also that $k \neq 0$ since otherwise a rational solution is obvious on putting $x = pz$, $y = qz$. On putting $x = X/a$, $z = Z/a$, we may take $a = 1$.

Let θ, ϕ, ψ be the roots of $\zeta^3 - b = 0$. Then

$$(z - \sqrt{k})(z + \sqrt{k}) = \prod_{\theta,\phi,\psi} (x + \theta y).$$

We can satisfy this by putting

$$z + \sqrt{k} = \prod_{\theta,\phi,\psi} (x_1 + \theta y_1 + \theta^2 z_1 + \sqrt{k}(x_2 + \theta y_2 + \theta^2 z_2)),$$

$$z - \sqrt{k} = \prod_{\theta,\phi,\psi} (x_1 + \theta y_1 + \theta^2 z_1 - \sqrt{k}(x_2 + \theta y_2 + \theta^2 z_2)),$$

where x_1, y_1, z_1, x_2, y_2, z_2 are rational numbers, and

$$x + \theta y = (x_1 + \theta y_1 + \theta^2 z_1)^2 - k(x_2 + \theta y_2 + \theta^2 z_2)^2.$$

I may remark that every cubic equation with only a finite positive number of solutions can be reduced to the form $z^2 = f(x, y)$. The existence of any solution of the homogeneous cubic leads at once by an obvious rational linear transformation to an equation of the form

$$z^2 L_1 + z L_2 + L_3 = 0,$$

where L_1, L_2, L_3 are homogeneous polynomials of the first, second, third degrees respectively in x, y, w. Now L_1, L_2 are not both identically zero since it has been already assumed that the surface is not a cylinder. Next L_1 is not identically zero since then there would be an infinity of integral solutions given by taking arbitrary x, y, w for which $L_2 \neq 0$, and so by an appropriate linear transformation we can write $L_1 = w$. Next we have an infinity of solutions on taking $w = 0$ unless this implies $L_2 = 0$, *i.e.*, L_2 is divisible by w. But putting $z - L_2/w$ for z, the cubic equation takes the required form

$$\left(\frac{z}{w}\right)^2 = f\left(\frac{x}{w}, \frac{y}{w}\right),$$

where f is a cubic polynomial in x/w, y/w. On putting in here $\theta^3 = b$, $\theta^4 = \theta b$, and equating the coefficient of θ^2 to zero, we obtain

$$y_1^2 + 2x_1 z_1 = k(y_2^2 + 2x_2 z_2).$$

Also from the equation for $z \pm \sqrt{k}$, by subtracting, we get

$$1 = k \prod_{\theta,\phi,\psi} (x_2 + \theta y_2 + \theta^2 z_2)$$
$$+ \sum_{\theta,\phi,\psi} (x_2 + \theta y_2 + \theta^2 z_2)(x_1 + \phi y_1 + \phi^2 z_1)(x_1 + \psi y_1 + \psi^2 z_1).$$

This becomes on multiplying out

$$1 = k(x_2^3 + by_2^3 + b^2z_2^3 - 3bx_2y_2z_2) + x_2(x_1^2 - 3by_1z_1) + y_2(3by_1^2 - 3bx_1z_1) + z_2(3b^2z_1^2 - 3bx_1y_1).$$

Obviously any rational solution of these two equations in the six variables $x_1, y_1, z_1, x_2, y_2, z_2$ gives rational values of x, y, z. For a particular one take $x_2=0$, whence $x_1=(ky_2^2-y_1^2)/2z_1$. Substituting in the other equation, we have

$$1 = kby_2^3 + kb^2z_2^3 + 3by_1^2y_2 - \frac{3b}{2}y_2(ky_2^2 - y_1^2) + 3b^2z_1^2z_2 - \frac{3by_1z_2}{2z_1}(ky_2^2 - y_1^2).$$

Put $y_2=1/Y_2$, $y_1=Y_1/Y_2$, $z_1=Z_1/Y_2$, $z_2=Z_2/Y_2$. Then

$$Y_2^3 = kb + kb^2Z_2^3 + 3bY_1^2 + 3b^2Z_1^2Z_2 - \frac{3b}{2}k + \frac{3b}{2}Y_1^2 - \frac{3b}{2}\frac{Y_1}{Z_1}Z_2(k - Y_1^2).$$

On putting $3bZ_2/2Z_1=\lambda^3$ and taking Z_1, λ to be parameters, we have the equation in Y_1, Y_2

$$Y_2^3 = \lambda^3Y_1^3 + \frac{9b}{2}Y_1^2 - k\lambda^3Y_1 + kb^2Z_2^3 + 3b^2Z_1^2Z_2 - \frac{kb}{2},$$

of which a particular solution is given on taking

$$Y_2 = \lambda Y_1 + \frac{3b}{2\lambda^2}\cdot$$

Hence we have found a new parametric solution of the equation.

RECENT PUBLICATIONS

EDITED BY VIRGIL SNYDER, Cornell University, and H. P. EVANS, University of Wisconsin

All books for review should be sent directly to the editors of this department, American Mathematical Monthly, 531 West 16th Street, New York 27, N. Y., and not to any of the other editors or officers of the Association.

Business Mathematics for College Students. By G. H. Whiteaker. New York, McGraw-Hill Book Company, Inc., 1943. 12+184 pages. $1.50.

This is a paper bound book consisting of 91 detachable lesson sheets. The work starts with the simple addition combinations, then runs through topics such as: addition, multiplication, subtraction, division, fractions, the 6 per cent method, notes, discounts, markups, equations, formulas, interest, progressions, finance, *etc.* The attempt is to "develop that something called *figure sense* which all successful businessmen seem to possess." There are many excellent problems and frequent review lessons. Tests are provided at appropriate intervals.

N. G. GUNDERSON

Compact Tables. Arranged and published by J. P. Ballantine. Seattle, University of Washington, 1943. 38 pages. $0.25.

Compact Tables. Second Edition. By J. P. Ballantine. Chicago, The Charles T. Powner Company, 1943. 46 pages. $0.50.

The first edition contains the following five-place tables: logarithms of numbers, natural trigonometric functions, logarithms of trigonometric functions. The wide margins at the bottoms of the pages are filled with a variety of odds and ends: values and logarithms of π and e and their reciprocals, a few weights and measures, formulas from plane and spherical trigonometry, plane analytical geometry, and differential and integral calculus. Entries are in large sharp print on serviceable paper. The booklet is equipped with tabs for ready reference (these, however, are not reinforced) and is bound in a stiff paper cover. The logarithms of the trigonometric functions are given by separate functions, 10° to the page; for example, page 26 contains log sines for angles between 20° and 30° and log cosines for angles between 60° and 70°. One noticeable feature of these logarithmic trigonometric tables is the absence of proportional parts; only differences are given. Instead, a nomogram, to be found at the back of the booklet, may be used (with the help of a straight-edge) for interpolation in seconds or decimal parts of a degree. It should also be noted that the table of logarithmic tangents goes only as far as 45°.

In the second edition the table of logarithmic tangents is extended to 90°, and two tables are added: namely, a five-place table of natural logarithms and a six-place table giving the conversion from degrees, minutes, and seconds to radians.

The reviewer made no attempt to check the entries.

R. H. Bruck

Method-pamphlets on the Milne Method of Numerical Integration of First-order Differential Equations and of Certain Equations of Second Order. Oakland, California, Marchant Calculating Machine Company, 1944. 4 pamphlets: MM-216, MM-216A, MM-260, MM-261. 10, 6, 11, 4 pages. No charge.

The topics dealt with in these pamphlets are as follows:

MM-216, Integration of $dy/dx=f(x, y)$ when starting values are known;
MM-216A, Integration of $d^2y/dx^2=g(x, y)$ when starting values are known;
MM-260 and MM-261, Obtaining starting values for use in MM-216 and MM-216A.

These pamphlets outline methods by W. E. Milne for numerical integration of ordinary equations in a form which may be useful to computors to whom calculating machines are available. The material includes sample work sheets and statements about the reliability of the results in terms of differences for the starting values. The pamphlets contain references to National Council Research Bulletin, No. 92, 1933, by A. A. Bennett, W. E. Milne, and H. Bateman and to papers by W. E. Milne in this Monthly 1933, 1941, 1942.

L. W. Cohen

"Student's" Collected Papers. Edited by E. S. Pearson and John Wishart, with a Foreword by Launce McMullen. London, Biometrika Office, University College, 1943. 14+224 pages. 15s.

This book is well described by the following summary which appears on the jacket:

"The death in 1937 of William Sealey Gosset dealt a final blow to the anonymity which had concealed the personality of 'Student,' the statistician. Under this pseudonym his work had become famous wherever biological experiments were undertaken and the modern theory of statistics, with its emphasis on the valid deductions that can be made from the 'small sample,' was studied. 'Student's' test and 'Student's' tables have become household words in this field. They are included in their original form in this book, which is a re-issue, with Foreword, of all the scientific papers published by him over a period of 30 years. A unity of purpose runs through the whole of his contributions, and the result is, in retrospect, a valuable addition to the structure of the Science of Statistics."

It was his second paper (1908), *On The Probable Error of a Mean*, that made "Student" famous. Apart from its own merits as a notable piece of research, it played an important role in stimulating further contributions. This paper may be said to mark the beginning of the modern theory dealing with small samples and tests of statistical hypotheses. In all, "Student" published 21 papers and a few miscellaneous contributions. Although 14 of these papers appeared in *Biometrika*, the remainder were scattered throughout several other journals. We are thus indebted to the editors for making possible a balanced appraisal of the work of a versatile pioneer by collecting these papers into one volume. The central theme of his contributions was the application of statistical method to the research and routine problems of industry and agriculture. It would be surprising if the work of a pioneer did not occasionally seem open to criticism to the reader of to-day. But any such criticism would be overwhelmingly outweighed by the simplicity and directness of his methods of attack and his clear grasp of practical issues. His work was intimately related to the historical development of the new statistics.

Americans and others who were not acquainted with Mr. Gosset but who admired "Student" will welcome the Foreword which gives an interesting biographical sketch of this remarkable man and the Frontispiece photograph of him in informal pose.

J. F. KENNEY

An Introduction to Navigation and Nautical Astronomy. By W. G. Shute, W. W. Shirk, G. F. Porter, and Courtenay Hamenway. New York, The Macmillan Company, 1944. 14+457 pages. $4.50.

Certain features of this text are perhaps best described in the words of the authors. The following quotations are taken from the preface:

"This text is written in a language that a beginner in navigation can under-

stand. Technical terms and phrases are not avoided but are carefully defined and explained. Before a new phase of the subject is introduced, preceding material is frequently summarized to show the correlation with what has gone before. Thus the student is constantly reviewing his previous work. Before the subject of celestial navigation is introduced, all necessary basic astronomy is carefully explained, with particular emphasis on its application from the navigator's point of view.

"This text is complete in itself · · · . The H.O. 211 Table is given in full in the Appendix, as well as complete tables of logarithms and trigonometric functions. Excerpts from *The American Nautical Almanac*, *The American Air Almanac*, and the H.O. 214 Tables are sufficient for the solution of all problems given. The student is shown how to work problems of position on ordinary graph paper, thus the difficulty and expense of obtaining special plotting sheets or charts is eliminated. Throughout the text, tables and diagrams from publications of the United States Coast and Geodetic Survey, the Hydrographic Office of the United States Navy, and the United States Coast Guard are reproduced and explained so that the student will become acquainted with these important aids to navigation."

The only solutions of spherical triangles presented in this book are those based on the special tables H.O. 211 and H.O. 214 and the formulas needed in these solutions are not derived, only stated. There is no denying that the exclusive use of these tables permits concentration of effort and consequent mastery of the method given. Many instructors feel, however, that time in the classroom can best be spent mastering underlying principles rather than details of manipulation. Hence, the omission of all derivations in connection with spherical trigonometry is regrettable.

While the tables in this book are adequate to solve all exercises in the text it does materially increase student interest for them to take observations and compute their own positions. This, however, requires current issues of the *Nautical Almanac* which, added to the price of this text, makes it rather expensive for many students.

R. H. BARDELL

NEW BOOKS RECEIVED

Ansayo sobre Algebra de Anillos. By H. E. Calcagno. Buenos Aires, 1942. 28 pages.

Calculus Refresher for Technical Men. By A. A. Klaf. New York, McGraw-Hill Book Company, Inc., 1944. 8+431 pages. $3.00.

The Education of T. C. Mits. By H. G. Lieber and L. R. Lieber. New York, W. W. Norton and Company, Inc., 1944. 230 pages. $2.50.

Graphical Solution. Second Edition. By C. O. Mackey. New York, John Wiley & Sons, Inc.; London, Chapman & Hall, Ltd., 1944. 6+152 pages. $2.50.

Riddles in Mathematics. A Book of Paradoxes. By E. P. Northrop. New York, D. Van Nostrand Company, Inc., 1944. 8+262 pages. $3.00.

DISCUSSIONS AND NOTES

EDITED BY MARIE J. WEISS, Sophie Newcomb College, New Orleans, La.

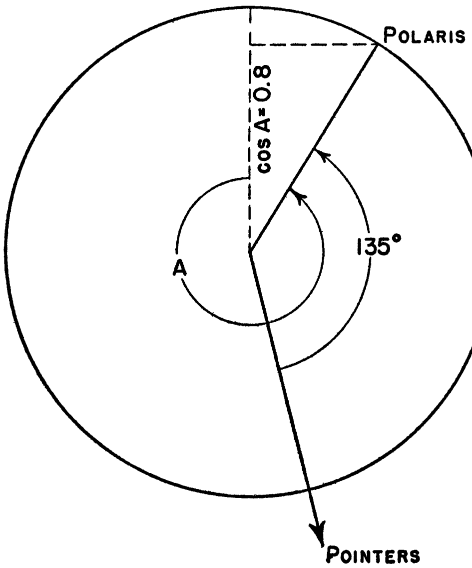
The Department of Discussions and Notes is open to all forms of activity in college mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

DETERMINATION OF LATITUDE IN AN EMERGENCY

JOEL BRENNER, Lawrence College

This article describes a method for determining latitude quickly and fairly accurately whenever a sextant, Polaris, the "pointers," and the horizon are simultaneously available. No tables or charts are needed.

Theory. The altitude of the elevated north pole is equal to the latitude of the observer. In 1944 Polaris is $1°\ 00'$ from the elevated pole. Hence the latitude of the observer is equal to the altitude of Polaris plus a correction equal to $-60' \cos A$, where A is the hour angle of Polaris (see figure). This angle can be found with sufficient accuracy by adding $138°\ 30'$ to the hour angle of the pointers.



Practice. To find A in the correction formula $60 \cos A$, one may estimate the hour angle of the pointers. Woodsmen calculate the time of night from the hour angle of the pointers, and their estimates have an error of less than $30^m = 7°\ 30'$. (This fact suggested the present method to me.) Knowing A, one must find $\cos A$ to one place of decimals. Successful students will find $\cos A$ from its graph. Or, it is easy to find the correction by construction, and without any tables. It is assumed that pencil and paper (or a nail and a piece of wood) are available.

Example. At sea, charts lost, clock stopped, tables illegible, sextant working, navigator takes altitude of Polaris, $44° 30' 20''$, I. C.—$0' 20''$, no dip. The refraction correction may be neglected.* After consultation with the crew, it is decided that the figure shows the approximate direction of the pointers. A circle is sketched and drawn free hand (or by the use of two holes in a card, or by the use of a coin); the hour angle of Polaris next estimated (an angle of $135° = 90° + 45°$ is easy to draw); and last the value of $\cos A = 0.8$ can be read to one place of decimals from the figure. The latitude equals $43° 42' \pm 4'$.

If good paper, sharp pencil, and useful work-table are available, and if the sextant circle be used for a protractor, the latitude may be found to $1'$ or less. An error in estimating the hour angle of the pointers is least significant when such hour angle is $135°$ or $315°$ and most significant when the angle is $225°$ or $45°$ ($\cos A$ changes fastest when $A = 90°$ or $270°$).

The clock should be started, and a note made of the time and position of the pointers. Difference in longitude made good can be estimated, again by use of the pointers.

A NOTE ON ENVELOPES

HOWARD EVES, Syracuse University

At a certain stage in calculus we teach the well known method of finding the envelope of a one-parameter family of curves given by cartesian equations. Occasionally a student wonders how to find the envelope of a one-parameter family of curves given by polar equations (without transforming to cartesian coördinates and then back again). An easily established but general theorem may be stated to the effect that *the same process used for a family of curves given in the cartesian coördinate system may be used for a family of curves given in any point coördinate system.* Inasmuch as many teachers of college mathematics are unaware of this fact and since it does not seem to be stated in any of the ready literature, it might be well to establish it.

Let $f(u, v, c) = 0$ be the one-parameter family of curves given in some (u, v) coördinate system. Imagine a superimposed (x, y) cartesian coördinate system, and let $u = u(x, y)$, $v = v(x, y)$ be the equations of the transformation of coördinates. Then the family of curves is given in the cartesian system by $F(x, y, c) \equiv f[u(x, y), v(x, y), c] = 0$. Since it is obvious that $\partial F/\partial c \equiv \partial f/\partial c$, the theorem is proved.

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* A good navigator knows the refraction corrections. For the observed altitudes 45°, 30°, 20°, 15°, the corrections are $-1'$, $-2'$, $-3'$, $-4'$, respectively.

ON THE CONCEPT OF DIVISOR

C. C. MacDuffee, University of Wisconsin

Most books on elementary number theory define divisibility of rational integers as follows:* "An integer a is said to be divisible by another integer b, not 0, if there is a third integer c such that

$$a = b \cdot c."$$

A perusal of the various books indicates that they fall into two groups: those which specifically put in the condition $b \neq 0$, and those which intend to do so but neglect it. All of them reach the conclusion that two integers not both 0 have a greatest common divisor, and a very few extend the theory by defining the greatest common divisor of 0 and 0 to be 0.

In the theory of ideals it is very clear that the greatest common divisor of 0 and 0 should be 0, and also that the least common multiple of a and 0 should be 0 for every a. It seems as if these conclusions should be obtainable by elementary considerations without the theory of ideals, and this note indicates that only courage to fly in the face of mathematics' most firmly fixed prejudice is required to bring it about.

If three numbers (rational integers) satisfy the relation

$$a = b \cdot c,$$

we shall say that $b|a$ (b divides a) and $c|a$. There are no reservations. Of course if $b=0$, c is not uniquely defined so that we do not talk about a quotient in this case, but 0 is divisible by 0.

If $b|a$, then a is a *multiple* of b.

THEOREM 1. *Every multiple of* 0 *is* 0.

THEOREM 2. *The number* 0 *is a multiple of every number.*

For $0 = 0 \cdot b$ for every number b.

THEOREM 3. *The only number which is a multiple of every number is* 0.

This is an immediate consequence of Theorems 1 and 2.

A *greatest common divisor* of two numbers a and b is a number d such that

1) $d|a$ and $d|b$,
2) if $d_1|a$ and $d_1|b$, then $d_1|d$.

A *least common multiple* of two numbers a and b is a number m such that

1) $a|m$ and $b|m$,
2) if $a|m_1$ and $b|m_1$, then $m|m_1$.

THEOREM 4. *Every two numbers a and b have a greatest common divisor d such that $d = p \cdot a + q \cdot b$, and $d = 0$ if and only if $a = b = 0$.*

Case I, a, b not both 0. That such a d exists and is not 0 is proved in every text-book.

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* An Introduction to the Theory of Numbers, G. H. Hardy and E. M. Wright, Oxford University Press, 1938, p. 1.

Case II, $a=b=0$. By Theorem 2, every number is a common divisor of 0 and 0. By Theorem 3, the only multiple of every number is $d=0$. Clearly

$$0 = p\cdot 0 + q\cdot 0$$

for every p and q.

THEOREM 5. *If d and d_1 are two greatest common divisors of a and b, then $d_1 = \pm d$.*

The proof of Case I is well known. The proof of Case II is obvious.

THEOREM 6. *Every two numbers a and b have a least common multiple m, and $m=0$ if and only if $ab=0$.*

Case I, $a\neq 0$, $b\neq 0$. It is well known that $m=ab/d$, and that $m\neq 0$.
Case II, $b=0$. The only multiple of 0 is 0, so that $m=0$.

THEOREM 7. $dm=ab$.

If neither a nor b is 0, this is well known. If $ab=0$, then $m=0$ so that it is still satisfied.

THEOREM 8. *The least common multiple of any infinite set of numbers is* 0.

For 0 is a common multiple, and it is the only one, for the set must contain integers of arbitrarily large absolute value.

In line with these considerations is the modern tendency* to define 0 as the order of a group of infinitely many elements. This definition can be justified as follows. Let a be an element of a group G. The period of a may be defined as the non-negative greatest common divisor of all integers n such that $a^n=1$. If there is no such integer n except 0, then clearly this greatest common divisor is 0, and it is natural to define the order of the cyclic subgroup generated by a to be the period of a, namely 0. With the convention that the order of a group of infinitely many elements is 0, it remains true for such groups as well as for finite groups that the order of the group is a multiple of the order of each subgroup, even though there be infinitely many subgroups of different orders, or one of order 0.

When this convention is adopted, the points of view of Steinitz and Albert regarding the characteristic of a field are reconciled. We may modify slightly the Steinitz definition and define the characteristic as the non-negative greatest common divisor d of all rational integers n such that $n\cdot i=0$ where i is the identity element of multiplication in the field. Thus two elements $s\cdot i$ and $t\cdot i$ are equal if and only if s and t are congruent modulo d, congruence modulo 0 being ordinary equality. But the greatest common divisor of all rational integers n such that $n\cdot i=0$ is the order of the (additive) cyclic group generated by the identity element of the field, and this is Albert's definition of characteristic.†

* Zassenhaus, Lehrbuch der Gruppentheorie, Berlin 1937, p. 3.

† A. A. Albert, Modern Higher Algebra, University of Chicago Press, 1937, p. 22.

A VECTORIAL DEVELOPMENT OF TWO DIFFERENTIATION FORMULAS

H. V. CRAIG, University of Texas

Geometrical background. Consider a unit circle with center at the origin. Let s be the arc length measured counterclockwise from the point (1, 0); and let ρ be the radius vector which traces out the circle. Since $\cos s$ and $\sin s$ are the coordinates of the vertex of ρ, we have $\rho=\rho(s)=\cos s\,\mathfrak{i}+\sin s\,\mathfrak{j}$, $\mathfrak{i}$ and $\mathfrak{j}$ denoting the unit vectors along the axes. Further, let $\Delta\rho$ be the chord vector $\rho(s+\Delta s)-\rho(s)$. Now since the limit of the ratio chord/arc is unity* and $\rho(s+\Delta s/2)$ is at all times normal to $\Delta\rho$ and approaches $\rho(s)$ as Δs approaches zero; it is tolerably clear that $D\rho$, the limit of $\Delta\rho/\Delta s$, is the unit vector perpendicular to $\rho(s)$ in the sense of increasing s. We are now ready to investigate $D \sin S$ and $D \cos S$.

Proof of the formulas. Having given that $D\rho$ is the unit vector perpendicular to ρ, we see that the slope of $D\rho$ is $-\cos s/\sin s$ and finally that $D\rho=-\sin s\,\mathfrak{i}+\cos s\,\mathfrak{j}$. But by the rules of elementary calculus, and they obviously apply in the present case, $D\rho=D\cos s\,\mathfrak{i}+D\sin s\,\mathfrak{j}$ and by equating corresponding components $D\cos s=-\sin s$, $D\sin s=\cos s$.

PROBLEMS AND SOLUTIONS

ELEMENTARY PROBLEMS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND H. S. M. COXETER

Send all communications concerning Elementary Problems and Solutions to H. S. M. Coxeter, 24 Southearn Boulevard, Toronto 10, Canada.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 626. *Proposed by W. E. Buker, Perry High School, Pittsburgh*

A ship at sea attempts to determine its position by plotting lines of position through observation of three stars. Due to inaccuracy of observation, these lines are not concurrent but form a triangle. What is the most probable position of the ship, assuming (a) that the errors are due to defective instruments and are of the same kind, (b) that the errors are due to inaccurate observation and may not all be of the same kind?

E 627. *Proposed by V. Thébault, San Sebastián, Spain*

Construct a convex enneagon, given the centers of exterior squares described on the sides.

* For an elementary argument, see M. S. Knebelman, An elementary limit, this MONTHLY, vol. 50, p. 507.

E 628. *Proposed by W. C. Rufus, Observatory of the University of Michigan*

Find the smallest positive integer, one-half of which is a square, one-third of which is a cube, and one-fifth of which is a fifth power.

E 629. *Proposed by H. S. M. Coxeter, University of Toronto*

Sum the series

$$\sum_{r=0}^{[n/2]}\left(-\frac{1}{4}\right)^r\binom{n-r}{r}.$$

E 630. *Proposed by R. A. Rosenbaum, U.S.N.R.*

For any point P of a given closed convex curve C, let P' be that point on the exterior normal to C at P for which $PP'=d$, a constant. The locus of P' is a curve C', parallel to C. Let s, s' be the respective lengths of C, C', and A, A' the areas within these curves. Show that

$$s' = s + \pi d,$$
$$A' = A + sd + \pi d^2.$$

(Cf. 4036 [1943, 397].)

SOLUTIONS

Intersections of Orthogonal Circles

E 591 [1943, 560]. *Proposed by C. J. Coe, University of Michigan*

Two given coplanar circles, (A_1) and (A_2), are cut orthogonally by a third circle, (B). Prove that a line joining either intersection point on (A_1) to either intersection point on (A_2) will pass through one of two points on the line of centers A_1A_2, these two points being the same for all choices of the orthogonal circle (B).

I. *Solution by J. S. Guérin, Catholic University of America.* The following equivalent theorem will be proved. If tangents be drawn to two circles (A_1) and (A_2) from any point on their radical axis, the line joining either point of contact on (A_1) to either point of contact on (A_2) passes through one of the centers of similitude of (A_1) and (A_2).

Let C_1 on (A_1) and C_2 on (A_2) be two such points of contact. If the line C_1C_2 intersects (A_2) in a second point D_2, the triangle $C_2A_2D_2$ is isosceles, and

$$\angle A_2D_2C_2 = \angle A_2C_2D_2.$$

But, since B lies on the radical axis of the two circles, the triangle BC_1C_2 is isosceles; thus

$$\angle BC_1C_1 = \angle BC_2C_1 = \tfrac{1}{2}\pi - \angle A_2C_2D_2 = \tfrac{1}{2}\pi - \angle A_2D_2C_2,$$

and

$$\angle BC_1C_2 + \angle A_2D_2C_2 = \tfrac{1}{2}\pi.$$

Hence A_2D_2 is perpendicular to BC_1, and the radii A_1C_1, A_2D_2 are parallel. In other words, the line $C_1C_2D_2$ passes through a center of similitude of the circles.

II. *Solution by Augustus Sisk, Maryville, Tenn.* (Numbers in parentheses refer to articles in N. Altshiller Court's *College Geometry.*) By (325), B lies on the radical axis of (A_1) and (A_2). By (331), the points of contact of two tangents drawn from B to the respective circles are antihomologous points. But lines joining antihomologous points on two circles pass through one of the centers of similitude of the circles.

Also solved by Howard Eves and L. M. Kelly.

Simultaneous Equations

E 594 [1943, 560]. *Proposed by F. J. Duarte, Caracas, Venezuela*

Solve the equations

$$x^2 = y^3, \qquad (x-3)y + (1-a-b)(3y-2x) = 0,$$

where

$$ab = -1, \qquad a^3 + b^3 = 1.$$

Solution by E. P. Starke, Rutgers University. For simplicity put $s=a+b$. Then

$$s(s^2+3) = s(s^2-3ab) = (a+b)(a^2-ab+b^2) = a^3+b^3 = 1.$$

That is, s is a root of $s^3+3s-1=0$, or $(s-1)^3+3s^2=0$. From the second given equation we get $y=2x(s-1)/(3s-x)$, which gives, upon substitution in the first equation,

$$8x^3(s-1)^3/(3s-x)^3 = x^2.$$

Hence $x=0$ (with $y=0$) is a double solution. If $x\neq 0$, we have

$$(3s-x)^3 = 8x(s-1)^3 = -24xs^2,$$

whence

$$(x-9s)(x^2+3s^2) = 0.$$

We thus obtain the solutions

$$\begin{aligned} x &= 9s, & y &= 3(1-s);\\ x &= is\sqrt{3}, & y &= \omega(s-1);\\ x &= -is\sqrt{3}, & y &= \omega^2(s-1); \end{aligned}$$

where $\omega=e^{2\pi i/3}$, $s=a-a^{-1}$, and $a^3=(\sqrt{5}+1)/2$.

Also solved by Hazel Schoonmaker Wilson and the proposer.

An Almost Equiangular Integral Triangle

E 595 [1943, 560]. *Proposed by H. T. R. Aude, Colgate University*

Find the smallest set of three different integers to represent the sides of a

triangle in which one angle is $60°$ and each of the other angles differs therefrom by not more than one minute.

Solution by C. L. Weaver, Boston, Mass. Let the sides of the triangle be $a<b<c$. Put $a=b-x$, $c=b+y$. By the law of cosines,

$$b^2 = a^2 + c^2 - ac,$$

or

$$b = \frac{x^2 + y^2 + xy}{x - y} = \frac{3y^2 + 3yz + z^2}{z}, \qquad z = x - y > 0.$$

By the law of sines,

$$\frac{b}{y} > \frac{\sin 60°}{\sin 60°\,1' - \sin 60°} = 5955.85.$$

Substituting the above expression for b, this gives

$$\left(\frac{y}{z}\right)^2 - 1984.28\frac{y}{z} + \frac{1}{3} > 0.$$

Therefore either $y/z>1984$ or $y/z<1/5952$. The latter possibility is ruled out since $3y^2$ must be a multiple of the integer z. Hence the least solution in positive integers is

$$z = 1, \qquad y = 1985, \qquad x = 1986,$$

whence

$$a = 11{,}824{,}645, \qquad b = 11{,}826{,}631, \qquad c = 11{,}828{,}616.$$

Also solved by R. A. Johnson, Joseph Rosenbaum, and Howard Eves.

Concurrent Circles

E 597 [1943, 633]. *Proposed by V. Thébault, San Sebastián, Spain*

Let P be any point in the plane of a triangle ABC. Let (U), (V), (W) denote the circles BCP, CAP, ABP, and (U'), (V'), (W') their images by reflection in the respective sides BC, CA, AB. Also let u, u' be the powers of A with respect to (U), (U'), and let v, v', w, w' be defined analogously. Show that the circles (U'), (V'), (W') are concurrent, and that

$$u + u' + v + v' + w + w' = a^2 + b^2 + c^2.$$

Solution by J. S. Guérin, Catholic University of America. Let P' denote the second point of intersection of (V') and (W'). Since

$$\sphericalangle APB = \sphericalangle AP'B \quad \text{and} \quad \sphericalangle APC = \sphericalangle AP'C,$$

it may readily be shown that $\sphericalangle BPC = \sphericalangle BP'C$. It follows that P' is on (U'). Thus (U'), (V'), (W') are concurrent in P'.

Let M denote the midpoint of BC, m_a the median AM, and R_u the radius of (U). We have

$$u + u' = AU^2 + AU'^2 - 2R_u^2 = 2AM^2 + 2UM^2 - 2R_u^2$$
$$= 2m_a^2 + 2(R_u^2 - BC^2/4) - 2R_u^2 = 2m_a^2 - a^2/2.$$

Similarly, $v+v' = 2m_b^2 - b^2/2$ and $w+w' = 2m_c^2 - c^2/2$. Hence

$$u + u' + v + v' + w + w' = 2(m_a^2 + m_b^2 + m_c^2) - (a^2 + b^2 + c^2)/2$$
$$= a^2 + b^2 + c^2.$$

Also solved by the proposer.

A Class of Convergent Sequences

E 598 [1943, 633]. *Proposed by H. S. Wall, Northwestern University*

Let $g_1, g_2, g_3, \cdots$ be any numbers such that

$$0 < g_p < 1, \qquad (1 - g_p)g_{p+1} > \tfrac{1}{4} \qquad (p = 1, 2, 3, \cdots).$$

Prove that

$$\lim_{p\to\infty} g_p = \tfrac{1}{2}.$$

Solution by Irving Kaplansky, Harvard University. For any real number x we have $x(1-x) \leqq \frac{1}{4}$. Hence

$$(1 - g_p)g_{p+1} > (1 - g_p)g_p,$$

i.e., the g's are monotone increasing. Since they are bounded they approach a limit z which must satisfy $(1-z)z \geqq \frac{1}{4}$. Hence $z = \frac{1}{2}$.

More generally, suppose $0 < g_p < 1$ and $(1-g_p)g_{p+1} > a > 0$. Then $a \leqq \frac{1}{4}$ and

$$\underline{\lim}\, g_p \leqq (1 + \sqrt{1 - 4a})/2, \qquad \overline{\lim}\, g_p \geqq (1 - \sqrt{1 - 4a}\ /2.$$

For, defining the sequence $\{x_p\}$ by

$$x_1 = 0, \qquad x_{p+1} = a/(1 - x_p) \qquad (p = 1, 2, \cdots),$$

so that $\lim x_p = (1 - \sqrt{1-4a})/2$, we easily prove by induction that

$$x_p < g_p < (1 + \sqrt{1 - 4a})/2.$$

In the extreme case when $a = \frac{1}{4}$, we find $x_p = (p-1)/2p$, so

$$(p - 1)/2p < g_p < \tfrac{1}{2}.$$

Also solved by J. M. Danskin, Jr., F. J. Duarte, N. J. Fine, J. B. Kelly, W. H. Myers, E. A. Nordhaus, William Scott, M. F. Smiley, E. P. Starke, J. E. Wilkins, Jr., and the proposer. One would-be solver believed he had disproved the existence of the proposer's sequence. But all the conditions are satisfied by the instance

$$g_p = (kp - 1)/2kp,$$

for any $k > 1$.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known text-books or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4124. *Proposed by T. W. Anderson, Jr., Princeton University*

Consider the set of n by n matrices whose entries are positive integers or zero. Let the sum of the entries of the ith row be r_i, $i=1, 2, \cdots, n$, and the sum of the entries of the jth column be $c_j, j=1, 2, \cdots, n$. For specified r_i and c_j, positive or zero integers, with

$$\sum_{i=1}^{n} r_i = \sum_{j=1}^{n} c_j$$

what are the minimum and maximum sums of entries in the main diagonal, *i.e.*, the minimum and maximum traces?

4125. *Proposed by Hüseyin Demir, Columbia University*

Prove that

$$\begin{vmatrix} \sin\theta_1 & -e^{-i\theta_1} & 0 & 0 & \cdots 0 & 0 \\ \sin\theta_2 & e^{i\theta_2} & -e^{-i\theta_2} & 0 & \cdots 0 & 0 \\ \sin\theta_3 & 0 & e^{i\theta_3} & -e^{-i\theta_3} & \cdots 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \sin\theta_n & 0 & 0 & 0 & \cdots 0 & e^{i\theta_n} \end{vmatrix} = \sin(\theta_1+\theta_2+\cdots+\theta_n).$$

4126. *Proposed by A. D. Wallace, University of Pennsylvania*

Let x, A, b denote respectively $(1, m)$, (m, n), $(1, n)$ matrices, an (i, j) matrix being one with i rows and j columns. If the matrix AA' is non-singular, show that the least square solution of $xA=b$ is the solution of an equation of the form $xA=b_0$. Determine b_0 and its geometric meaning.

4127. *Proposed by V. Thébault, San Sebastián, Spain*

The straight lines AG, BG, CG, DG drawn through the vertices and centroid G of the tetrahedron $ABCD$ meet its circumsphere again in A_1, B_1, C_1, D_1, and the planes perpendicular to the respective lines at these latter points determine the tetrahedron $A_2B_2C_2D_2$.

Show that (1). The two tetrahedrons have the same centroid and are hyperbolic. (2). The non-focal axis of the quadric surface with G as a focus inscribed in $A_2B_2C_2D_2$ is equal to the diameter of the Monge sphere of the Steiner ellipsoid inscribed in $ABCD$.

Note. The first part extends to the tetrahedron the properties of the triangle indicated by W. F. Beard, *Educational Times, Reprints*, 1917, p. 69.

SOLUTIONS

An Algebraic Identity and Partitions

4067 [1943, 65]. *Proposed by H. S. Wall, Northwestern University*

Prove that, if $0<x<1$,

$$\prod_{n=1}^{\infty}(1-x^{2n-1}) = 1\Big/\left[1+\sum_{k=1}^{\infty} x^{k(k+1)/2}/(1-x)(1-x^2)(1-x^3)\cdots(1-x^k)\right].$$

I. *Solution by R. C. Buck, Cambridge, Mass.* Consider the Euler identity

$$(1) \qquad \begin{aligned}&(1+x^2)(1+x^4)(1+x^6)\cdots\\ &= 1+\frac{x^2}{1-x^2}+\frac{x^2}{1-x^2}\,\frac{x^4}{1-x^4}+\frac{x^2}{1-x^2}\,\frac{x^4}{1-x^4}\,\frac{x^6}{1-x^6}+\cdots.\end{aligned}$$

(Hardy and Wright, *Theory of Numbers*, p. 275, Theorem 346, or p. 278, Theorem 348.) This holds for $-1<x<1$, hence for $0<x^2<1$. Replace x^2 by x, and we have

$$(2) \qquad \begin{aligned}&(1+x)(1+x^2)(1+x^3)(1+x^4)\cdots\\ &= 1+\frac{x}{1-x}+\frac{x}{1-x}\,\frac{x^2}{1-x^2}+\frac{x}{1-x}\,\frac{x^2}{1-x^2}\,\frac{x^3}{1-x^3}+\cdots.\end{aligned}$$

Now,

$$(1+x)(1+x^2)\cdots = \frac{1-x^2}{1-x}\cdot\frac{1-x^4}{1-x^2}\cdots = \frac{1}{1-x}\,\frac{1}{1-x^3}\,\frac{1}{1-x^5}\cdots$$

and combining this with (2), we have

$$(1-x)(1-x^3)(1-x^5)\cdots = \frac{1}{1+\dfrac{x}{1-x}+\dfrac{x}{1-x}\,\dfrac{x^2}{1-x^2}+\cdots}$$

which is the desired result.

II. *Solution by D. H. Lehmer, University of California.* This identity was known to Euler who gave an algebraic proof of it. The following "graphical" proof may be of interest. The proposed identity may be written

$$\frac{1}{(1-x)(1-x^3)(1-x^5)\cdots} = \sum_{k=0}^{\infty} \frac{x^{k(k+1)/2}}{(1-x)(1-x^2)\cdots(1-x^k)}.$$

Comparing the coefficient of x^n $(n>0)$ in the power series expansion of each side we see that the identity is equivalent to the following:

THEOREM A. *The number of partitions of n into odd parts is the sum, for all possible values of k, of the numbers of partitions of $n-(k+1)k/2$ into parts not exceeding k.*

This theorem follows at once from the following two theorems:

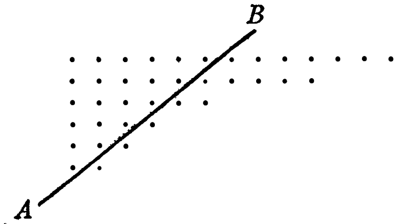
THEOREM B. *The number of partitions of n into odd parts is equal to the number of partitions of n into distinct parts.*

THEOREM C. *The number of partitions of $n-(k+1)k/2$ into parts $\leqq k$ is equal to the number of partitions of n into precisely k distinct parts.*

Theorem C may be proved graphically as follows. A partition, such as

$$13+10+6+4+3+2=38$$

may be represented graphically by the diagram



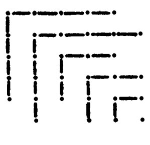
when it is read by rows. It is clear that partitions of n into precisely k distinct parts correspond to graphs of k rows, each of lesser length than row above it. Given such a graph we construct the line AB running in a northeasterly direction and passing just south of the first point of the last row. This line divides the graph into a "head" containing $(k+1)k/2$ points and a "tail." This tail, if read by diagonals parallel to AB, clearly gives a partition of $n-(k+1)k/2$ into parts $\leqq k$. In the example $k=6$ and the tail gives the partition

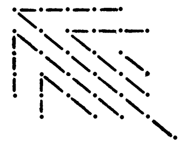
$$6+3+2+2+2+1+1=17$$

of $17=38-(6+1)6/2$ into parts $\leqq 6$. Hence to each partition of n into k distinct parts there corresponds a partition of $n-(k+1)k/2$ into parts $\leqq k$. To show that this correspondence is one to one, let any partition of the latter type be given. We construct a graph which, when read by columns, gives this partition. Then no row will contain more points than the row above it. We now shift the rth row bodily r units to the left $(r=1, 2, \cdots, k)$. The graph is now a tail to which we fix a head of $(k+1)k/2$ points to obtain a regular graph which,

when read by rows, gives a partition of n into k necessarily distinct parts. Hence the correspondence is one to one and Theorem C follows.

Theorem B is of course very well known. A graphical proof of it, not so well known, is due to Sylvester. It consists in reading in two ways a graph which is symmetrical with respect to its main diagonal. The two ways are indicated by the following figures.





$9+9+5+5+3+1+1=33$ $\qquad$ $11+8+7+5+2=33$

The correspondence thus established is easily seen to be one-to-one.

A Minimum Given by the First Brocard Triangle

4074 [1943, 125]. *Proposed by V. Thébault, San Sebastián, Spain*

On the sides of the given triangle ABC directly similar triangles ABC_1, etc., are constructed interiorly. Determine these latter triangles so that $(GA_1)^2+(GB_1)^2+(GC_1)^2$ is a minimum, where G is the centroid of ABC.

Editorial Note. The proposer gave the following remarks. The sum of the squares of the sides of triangle $A_1B_1C_1$ is equal to three times the sum of the squares of the distances of G to its vertices, so that it suffices to make the first sum a minimum. The vertices A_1, B_1, C_1 which give the desired minimum are the vertices of the first Brocard triangle for ABC, for which

$$\frac{\overline{B_1C_1}^2+\overline{C_1A_1}^2+\overline{A_1B_1}^2}{\overline{BC}^2+\overline{CA}^2+\overline{AB}^2}=\frac{1}{4}(1-3\tan^2 V),$$

where V is the Brocard angle for ABC.

A proof may be obtained by taking the centroid G as the origin for the complex coordinates t_1, t_2, t_3; u_1, u_2, u_3 of A, B, C; A_1, B_1, C_1 and setting $\lambda=e^{i\theta}$, $\theta=\angle BAC_1$, $0\leqq\theta<\pi$, and $r=AC_1/AB$. Then $u_3=(1-r\lambda)t_1+r\lambda t_2$, *etc.*, and we see at once that G is also the centroid of $A_1B_1C_1$. Denote by l_1 and m_1 the lengths of GA and GA_1. It then follows that

$$\sum m_j^2=(1-2r\cos\theta+2r^2)\sum l_j^2+r\sum(\lambda t_2\bar{t}_1+\bar{\lambda}t_1\bar{t}_2)-r^2\sum(t_2\bar{t}_1+t_1\bar{t}_2).$$

This may be transformed by using

$$\sum(t_2\bar{t}_1+t_1\bar{t}_2)=-\sum l_j^2=2\sum l_1l_2\cos(\theta_2-\theta_1),\qquad t_j=l_je^{i\theta_j},$$

$$\sum(\lambda t_2\bar{t}_1+\bar{\lambda}t_1\bar{t}_2)=-\cos\theta\sum l_j^2-4\sin\theta S,\ S=\text{area }ABC,\ \tan V=4S/3\sum l_j^2.$$

We then have

$$\frac{\sum m_i^2}{\sum l_i^2} = 3r^2 - \frac{3r\cos(\theta - V)}{\cos V} + 1$$

and from this we find that the minimum is given by $\theta = V$, $2r\cos V = 1$. Thus the triangles ABC_1, *etc.*, which give the minimum are isosceles with base angle V and the triangle $A_1B_1C_1$ for the minimum is therefore the first Brocard triangle for ABC.

A Transcendental Number

4078 [1943, 263]. *Proposed by Raphael Robinson, Univ. of California at Berkeley*

Show that the continued fraction

$$\frac{1}{2^{1!}} + \frac{1}{2^{2!}} + \frac{1}{2^{3!}} + \frac{1}{2^{4!}} + \cdots,$$

represents a transcendental number by using the following theorem of Liouville, *Journal de Mathématiques pures et appliquées*, 1851.

The partial quotients of a continued fraction representing the root x of an nth degree algebraic equation with rational coefficients never exceed the product of a certain constant by the $(n-2)$nd power of the denominator of the preceding convergent.

Solution by the Proposer. Let the kth convergent of the continued fraction

$$b_0 + \frac{1}{b_1} + \frac{1}{b_2} + \frac{1}{b_3} + \cdots$$

be A_k/B_k, where $b_0/1$ is considered as the zeroth convergent. Then $B_k = b_kB_{k-1} + B_{k-2} < (b_k+1)B_{k-1}$. By repeated application of this formula one obtains, since $B_0 = 1$,

$$B_k < (b_1 + 1)(b_2 + 1) \cdots (b_k + 1).$$

For the fraction in this problem

$$B_k < (2^{1!} + 1)(2^{2!} + 1) \cdots (2^{k!} + 1) < 2^{1!+1}\, 2^{2!+1} \cdots 2^{k!+1}$$
$$= 2^{k+1!+2!+\cdots+k!} < 2^{2k!}$$

if $k > 3$, since $k+1!+2!+\cdots+(k-1)!$ is the sum of k terms, none exceeding $(k-1)!$, and not all this great, and hence less than $k!$. Therefore $B_{2\lambda-1} < 2^{2(2\lambda-1)!}$, so that

$$b_{2\lambda} = 2^{(2\lambda)!} = 2^{2\lambda(2\lambda-1)!} > B_{2\lambda-1}^{\lambda} = B_{\lambda-1}^{\lambda-n+2} B_{2\lambda-1}^{n-2}$$

and $B_{2\lambda-1}^{\lambda-n+2}$ increases with λ beyond all limits, no matter what the value of n, so that the continued fraction does not satisfy an equation of the nth degree for any value of n, *i.e.*, is transcendental.

Editorial Note. The Liouville theorem stated in the problem results easily from the final inequality in the proof of Theorem 191, p. 160, in Hardy and Wright's *An Introduction to the Theory of Numbers*. Following this proof are two examples of transcendental numbers

$$(a) \qquad \sum_{i=1}^{\infty} 10^{-i!}, \qquad \frac{1}{10} + \frac{1}{10^{2!}} + \frac{1}{10^{3!}} + \cdots;$$

and it is stated that it is obvious that the base 10 may be replaced by other integers. The above problem was received before the appearance of this text in print.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to B. W. Jones, White Hall, Cornell University, Ithaca, New York.

Rutgers University has announced the creation of a Research Council with funds to strengthen and promote research throughout the university. Emphasis will be placed upon cooperative research between related departments and between outside organizations and university departments.

Professors G. D. Birkhoff and Harlow Shapley of Harvard University have been elected honorary members of the Mexican Mathematical Society. The former has also been elected corresponding member of the Mexican National Academy of Sciences.

Associate Professor Saunders Mac Lane of Harvard University has been elected a member of the American Academy of Arts and Sciences.

Assistant Professor A. E. Taylor and Professor W. M. Whyburn of the University of California at Los Angeles have been elected corresponding members of La Academia Nacional de Ciencias Exactas, Físicas y Naturales de Lima.

Associate Professor A. L. Underhill of the University of Minnesota represented the Mathematical Association at the inauguration of V. J. Flynn as president of the College of Saint Thomas on April 27, 1944.

The following promotions at Alfred University are announced: W. V. Nevins, III, to an assistant professorship, Assistant Professor L. R. Polan to an associate professorship, and Associate Professor L. L. Lowenstein to a professorship.

Dr. Leon Alaoglu, Professor W. L. Ayres and Assistant Professor J. W. T. Youngs of Purdue University are serving with the Army Air Forces. Professor Laurence Hadley is acting head of the department of mathematics.

Assistant Professor K. J. Arnold of the University of New Hampshire is now serving with the Statistical Research Group at Columbia University.

F. L. Celauro of Syracuse University has been appointed mathematician in the National Bureau of Standards in Washington.

Assistant Professor G. M. Ewing of the University of Missouri has been granted a leave of absence to serve in the Naval Ordnance Laboratory in Washington.

Professor F. A. Ficken of the University of Tennessee has been granted a leave of absence to serve in the Office of Scientific Research and Development in Washington.

N. J. Fine of Purdue University is now in the Research laboratory of the Lukas-Harold Corporation in Indianapolis.

Dr. R. H. Fox of the University of Illinois has been appointed to an assistant professorship at Syracuse University.

Dr. Michael Golomb of Purdue University has been promoted to an assistant professorship and is on leave for war research at Franklin Institute.

A. W. Goodman of Syracuse University is now serving with the Republic Aviation Corporation in Farmingdale, New York.

Dr. W. H. Gottschalk of the University of Virginia has been appointed to an assistant professorship at the University of Richmond.

Frank Hawthorne of Alliance College has been granted a leave of absence to serve as an associate physicist in the University of California Division of War Research in New York City.

Professor L. A. Hazeltine of Stevens Institute of Technology has resigned.

Associate Professor B. W. Jones of Cornell University has been promoted to a professorship.

Margaret E. Jones of Ohio State University has been promoted to an assistant professorship.

Dr. G. K. Kalisch of the University of Kansas has been appointed to an instructorship at Cornell University.

Associate Professor P. W. Ketchum of the University of Illinois has been given a leave of absence to engage in war research at Columbia University.

Professor J. F. Locke of Memphis State College has been granted a leave of absence to serve at the United States Naval Academy.

H. J. Miser of Illinois Institute of Technology has been appointed lecturer at Lawrence College, Appleton, Wisconsin.

Associate Professor L. F. Ollman of Hofstra College has been promoted to a professorship.

M. M. Resnikoff has been appointed to a professorship at State Teachers College, Minot, North Dakota.

Assistant Professor H. M. Schwartz of the University of Idaho has been appointed research fellow at the Bartol Research Foundation in Swarthmore, Pennsylvania.

Dr. M. E. Shanks of the University of Missouri has been granted a leave of absence to serve as a mathematician in the Aircraft Engine Research Laboratory at Cleveland.

Professor I. S. Sokolnikoff of the University of Wisconsin has been granted leave of absence to engage in war research.

Assistant Professor C. C. Torrance of Case School of Applied Science is on leave of absence serving with the Bureau of Ordnance of the Navy Department at Washington, D. C.

Professor H. S. Wall of Northwestern University has been appointed Visiting Lecturer at Illinois Institute of Technology.

A. H. Wheeler has been appointed to an affiliate professorship in the engineering staff for the A.S.T.P. program at Clark University.

Associate Professor A. S. Winsor of the University of North Carolina has been promoted to a professorship.

Dr. Clyde Wolfe of the Radiation Laboratory of the University of California died March 25, 1944.

WAR INFORMATION

EDITED BY C. V. NEWSOM

Send news reports upon the utilization of mathematicians or mathematics in war activities to C. V. Newsom, Oberlin College, Oberlin, Ohio.

THE TEACHER PROBLEM IN SECONDARY MATHEMATICS

Probably one of the most serious effects of the war on American education has been the withdrawal of approximately one-third of the trained teachers in the public schools to go into other fields of work. As a result, students below the

college level are being taught in many instances by persons who have the minimum qualifications to obtain emergency certificates. This means that a very large number of children are now receiving very poor instruction.

In many of the subject-matter courses of high school, the shortage of trained teachers has been partially offset by the tremendous decrease in the number of high school students. According to Circular No. 227 of the U. S. Office of Education, there were approximately 1,700,000 fewer students in high school in 1943 who were fifteen years of age and over than there were in 1940. In spite of this decrease, however, the actual number of students studying mathematics and physics is greater now than in previous years. In fact, the number of students registered in physics increased 11.6% from 1942 to 1943. In mathematics, the only other subject to show a positive increase, there were 3.5% more students in 1943 than in 1942. Such special courses as pre-induction mathematics and pre-flight mathematics have augmented the registration in mathematics. Of the special war-time courses upon a secondary level suggested by the Army and the U. S. Office of Education, it is interesting to observe that the largest enrollment of girls is to be found in pre-induction mathematics.

So the number of teachers in mathematics and physics now needed in high school is generally greater at the present time than in pre-war years. But, such teachers are continually receiving offers from governmental agencies and war industries. Teacher-training institutions have found it impossible to fill the rapidly increasing gap between the supply and the demand in view of the fact that they have suffered such a great decrease in enrollment. This latter situation in itself is critical since the supply of new well-trained teachers will be very small for a number of years. The U. S. Office of Education has made the comment that "the period of poor teaching will be prolonged after the war, until the returning teachers can bring their training up to date and normal size classes are graduated from teacher-training institutions."

The Engineering, Science, and Management War Training Program of the U. S. Office of Education has attempted to assist in the solution of the problem by sponsoring free, non-credit correspondence courses in mathematics and physics for those who might become high school teachers. These courses are being offered co-operatively by 19 colleges and universities. The response to this program up to the present time has been small, perhaps due to limited publicity.

A prospectus distributed by representatives of the ESMWT Program explaining the correspondence courses in physics and mathematics emphasizes that "the shortage of knowledge in mathematics and physics is a war emergency." The seriousness of the lack of competent instructors is portrayed in the following paragraph.

"The demand for persons capable of teaching high-school mathematics and physics has increased faster than the supply has been replenished. Both the armed forces and war industry require some competence in these fields, in consequence more young men and women than ever before must be taught mathematics and physics, especially in the high schools. The supply of persons capable of

teaching high-school mathematics and physics has been dwindling since before America's entry into the war. They have been drawn off into the services, industry, and war research. These are not tenuous estimates, but facts. They are of critical importance. Between the multiplying demand for instruction and the shrinking supply of instructors, particularly in smaller communities, there is a gap—a shortage potentially more serious than those in rubber, steel, and other strategic materials."

To register for an ESMWT correspondence course in mathematics, the registrant must indicate that he has completed high school or college courses in algebra and geometry. Moreover, he must obtain a statement from a responsible school official that he is now employed as a teacher of mathematics or physics, or that he will have reasonable opportunity for such employment upon the completion of the course. There are no fees of any kind, except that every student is expected to purchase his own textbooks. No academic credit is given for the work.

The course in mathematics, as outlined, includes topics in arithmetic and percentage, experimental geometry, intermediate algebra, logarithms, plane trigonometry, and spherical trigonometry. There are 48 lessons or assignments, the equivalent of six semester hours, or approximately nine quarter hours. It is expected that the material in the course can be covered in twenty-six weeks or less.

Persons desiring more information in regard to the ESMWT courses should apply to the ESMWT institutional representative in one of the following institutions: The Universities of Alabama, Arkansas, California, Chicago, Florida, Indiana, Iowa, Kansas, Louisiana, Minnesota, Nebraska, North Carolina, Tennessee, Texas, Utah, Washington, Wisconsin, or the State College of Oregon or Pennsylvania.

EASTERN CONFERENCE ON NAVY V-12 PROGRAMS

A conference attended by 32 representatives of 28 Eastern colleges and universities which have Navy V-12 Programs was held January 22, 1944. The meeting was sponsored by the American Council on Education. Certain items from the report of the conference appear below.

1. Commander Alvin C. Eurich, Officer in Charge of the Standards and Curriculum Section, Bureau of Naval Personnel, indicated that the Navy is not interested in the development of programs leading to degrees, but that the matter of degrees is the responsibility of the institutions. He referred to the statement on page 6 of the *Navy V-12 Bulletin No. 101*, concerning this matter. He indicated his hope, however, that the institutions with V-12 units might develop a fairly uniform policy with respect to the award of degrees for the completion of the Navy programs.

2. The conference unanimously adopted the following resolution:

"Resolved that it is the opinion of this group that a student who has satisfactorily completed the Navy V-12 curricula prescribed for deck candidates

should be regarded as having completed the equivalent of the four terms of the normal freshman and sophomore studies leading to the Bachelor's Degree. The college in which he may then seek enrollment should so adjust the curriculum of the junior and senior years that the student can meet the degree-requirements of the institution, with such modifications as may be necessary, within the limit of four additional terms.

"None of the prescribed courses of the Navy curriculum should be regarded as unacceptable for credit merely on the ground that the subject is not normally included within the institution's regular offerings."

3. After some discussion of required calculus in the third and fourth terms of the curriculum for deck candidates, the conference unanimously adopted the following resolution:

"That we request the Training Division of the Bureau of Naval Personnel to re-examine the curricula for deck candidates in the third and fourth terms, looking toward the elimination of required calculus, and to provide more effectively for differences in aptitude."

SOME DATA ON MATHEMATICS IN COLLEGES

Circular No. 228 of the U. S. Office of Education, shortly to be published, contains a statistical study of the effects of the war upon the colleges for the period, October 15, 1942–October 15, 1943. Circular No. 217 considered the period, October 15, 1941–October 15, 1942. The following material is from Circular No. 228. Figures on student registrations refer to nonmilitary enrollments estimated on the basis of returns from 561 institutions. The student tabulation under "all fields" covers engineering, biology, chemistry, economics, home economics, mathematics, physics and sociology.

Item	*All fields*		*Mathematics*	
	Men	*Women*	*Men*	*Women*
Teachers leaving their positions during 1943	8,239	4,293	442	145
Positions still unfilled on Oct. 15, 1943	1,846	785	110	120
Students on Oct. 15, 1943:				
Juniors	12,491	25,323	345	1,896
Seniors	13,119	22,982	363	1,538
Graduates	6,233	2,735	314	205
Students on Oct. 15, 1942:				
Juniors	40,310	21,190	2,065	1,412
Seniors	33,372	19,523	1,681	1,242
Graduates	6,313	3,976	465	255
Degrees conferred, 1942–'43:				
Bachelor's	92,273	93,467	1,071	1,277
Master's	9,987	7,840	68	74
Doctor's	3,025	429	42	9

MATHEMATICAL PUBLICATION DURING WARTIME

Available information indicates that the publication of mathematical research is generally being maintained at a high level throughout the world in spite of war-time handicaps. In this country, the quantity and the quality of the work being published is definitely comparable with that of the pre-war period. Editors of some of the journals have indicated, however, that a noticeable decrease in quantity may be expected in the near future. Most of the journals in this country can now publish acceptable papers with a minimum of delay.

As for foreign countries, the decrease in the amount of publication has, of course, been more radical. There are some interesting developments, however, which are not generally known in this country. For instance, the Russian mathematicians are very active, and are doing most outstanding work. It should be understood that this work is in pure mathematics to a very large extent, and even in its most abstract branches like number theory. The Stalin prize has repeatedly been given for such work. The Russians are also doing much in applied mathematics, but Russian applied mathematics has a character of its own. The Russians seem able to apply the most abstruse mathematical methods with great success, whereas in other countries, the so-called applied mathematics is often more in the field of engineering than in mathematics. Papers of an exceedingly mathematical character appear in Russian journals devoted to geophysics, engineering, and so on.

The mathematical work being done by some excellent Chinese mathematicians is outstanding. The new Chinese mathematics is exceedingly abstract, and is done under the most unfavorable circumstances. One of the younger Chinese mathematicians (S. S. Chern) has recently arrived in Princeton, and there is hope that some more will arrive.

In France, the mathematical production appears to have been reduced about 50%. The *Comptes Rendus* of Paris appear quite regularly; the same is true of some other French journals like the *Journal de Mathematiques*.

Mathematical production in Germany seems to be quite normal, and not very much reduced. In the case of Italy, very little information is available in regard to research activities.

The British mathematical output has been quite steady throughout the years of the war. Moreover, there appears to have been no essential change in its character.

NOTES ON POST-WAR EDUCATION

At present writing, Congress is still debating measures which provide for educational benefits for veterans. The Thomas Bill (S. 1509) and its companion measure, the Barden Bill (H.R. 3846), generally met the approval of educators in their broad provisions, especially the fact that educational benefits were to be administered on a national level by the U. S. Office of Education. Some

essential revisions of the two measures were needed, however, and recommendations to this effect were made on January 10, 1944, by representatives of 21 national associations of education. Before any serious consideration could be given to these recommendations, attention was focused in Washington on a substitute measure, the American Legion Omnibus Bill (S. 1617). This new bill appeared to be receiving much support in Washington, but was generally condemned by educators because of the fact that the educational program would be administered by the Veterans Administration. The Veterans Administration is almost exclusively concerned with vocational rehabilitation. As a result, educational agencies urged a compromise between the Thomas Bill and the Omnibus Bill. On March 13, a new Omnibus Bill (S. 1767) was introduced into the Senate; this incorporated many of the provisions of the Thomas Bill, and it supplanted both S. 1509 and S. 1617. In the House of Representatives, the Rankin Bill (H.R. 4357) was introduced on March 8. It is a companion bill to S. 1767 and replaced H.R. 3846, and the original Rankin Bill with the same number.

On March 13, 1944, the American Council on Education again sponsored a conference of the 21 educational associations to consider the situation brought about by the new Omnibus Bill. At this conference, the following statement of principles was unanimously adopted as being essential to any legislation which would provide educational benefits for discharged military personnel.

1. That the education of veterans under this act should be administered through the authorized educational agencies, federal, state, and local.

2. That responsibility for certification of eligibility of the individual in terms of military service and subsistence payments to individuals should rest with the Veterans Administration.

3. That in each state there shall be designated or created a duly authorized state educational agency which shall be broadly representative of the various levels and types of education within the state. The functions of such a state educational agency should be:

a. To furnish lists of approved educational or training institutions within the state.

b. To advise and assist the approved educational or training institutions furnishing training under this act.

c. To determine, subject to policies to be established on a national basis, the amount of payments to the educational or training institutions furnishing training under this act.

d. To provide educational and vocational guidance.

4. That the educational or training institution should determined the qualifications of the individual for study in such institution and for continuance in the courses.

5. That the individual should be free to select the institution in which he wishes to study, and, after counseling, to select the program of study which he desires to pursue.

It is generally believed in Washington that the termination of hostilities will not involve any great rush of veterans to college and university campuses. Such a point of view follows from such considerations as the following.

1. It is planned to demobilize men upon a gradual basis.

2. It is generally believed that a large number of younger men will be kept in uniform for a considerable period of time to be used for policing duties.

3. It is also believed by some persons that the drafting of 18-year-olds will be continued for a period of time. In this connection, two bills (Wadsworth H.R. 1906 and May H.R. 3947), providing for compulsory military training, have been introduced into Congress; these measures provide that their terms are to be placed in effect six months after the present war. The May Bill requires that the year of compulsory service shall begin at age 17, or at the completion of high school; the Wadsworth Bill leaves a three-year period, 18–21, in which the individual may select the year.

Educational measures for veterans being considered in Washington appear to be premised upon the assumption that approximately a million men will elect to continue their higher education after demobilization. A consultant to the U. S. Office of Education has estimated, however, that only about 150,000 will be able to qualify for a standard four-year college course. Great emphasis is being placed upon elaborate vocational rehabilitation measures.

THE MATHEMATICAL ASSOCIATION OF AMERICA

THE ANNUAL MEETING OF THE MICHIGAN SECTION

The annual meeting of the Michigan Section of the Mathematical Association of America was held Saturday, March 27, 1943, at the University of Michigan in Ann Arbor. This meeting also constituted the meeting of the mathematics section of the Michigan Academy of Science, Arts, and Letters. Morning and afternoon sessions were held, in addition to a luncheon-business meeting at noon. Professor Wayne Dancer of the University of Toledo, Chairman of the Section, presided at all the meetings.

Forty-three persons were in attendance, including the following thirty-three members of the Association: N. H. Anning, W. D. Baten, J. W. Bradshaw, J. B. Brandeberry, Angeline J. Brandt, Jack Britton, R. V. Churchill, C. J. Coe, A. H. Copeland, C. C. Craig, Wayne Dancer, P. S. Dwyer, Peter Field, C. H. Fischer, V. G. Grove, T. H. Hildebrandt, L. A. Hopkins, L. G. Johnson, L. S. Johnston, P. S. Jones, Wilfred Kaplan, L. C. Karpinski, C. E. Love, E. A. Nordhaus, J. K. Peterson, G. Y. Rainich, E. D. Rainville, L. J. Rouse, T. R. Running, B. M. Stewart, R. M. Thrall, E. T. Welmers, R. L. Wilder.

At the business meeting Professor G. C. Speeker of Michigan State College was elected Chairman, and Professor C. J. Coe of the University of Michigan was reelected Secretary-Treasurer for the ensuing year. The Secretary was directed to prepare a statement expressing the sorrow of the Section at the death of Professor E. R. Hedrick, and to send this statement to the Secretary of the Association for forwarding to the bereaved family.

The following program was presented:

1. *A Significance test for an index*, by Professor C. C. Craig, University of Michigan.

In a certain biological investigation, samples of N_1 values of x_1, N_2 values of x_2, and N_3 values of x_3, were supposed to be drawn from three normal universes with means α_1, α_2, and α_3, respectively, and with a common variance σ^2. The index $(\bar{x}_1-\bar{x}_3)/(\bar{x}_1-\bar{x}_2)=k$ was formed ($\bar{x}_1$, $\bar{x}_2$, and $\bar{x}_3$ being the means of the corresponding samples), and it was desired to test the hypothesis that $(\alpha_1-\alpha_3)/(\alpha_1-\alpha_2)=\kappa$. Professor Craig pointed out that this amounts to the test of the linear hypothesis

$$\alpha_1(1-\kappa)+\alpha_2-\alpha_3=0,$$

for which the Student-Fisher t-test is at once available. The test in this case was set up and numerically illustrated, and confidence limits for κ were calculated.

2. *A tribute to Earl Raymond Hedrick*, by Professor L. S. Johnston, University of Detroit.

Professor Johnston drew from his intimate personal relations with Professor Hedrick to sketch some details of the personality of that eminent mathematician.

3. *A simplified treatment of Riemann integration*, by Professor A. H. Copeland, University of Michigan.

The speaker presented a new definition of the Riemann integral, and an existence proof for it. The concept of uniform continuity of the integrand was not required. He showed how the definition is applied to physical problems, and brought out the fact that this treatment is both simpler and more rigorous than that usually offered.

4. *Reduction of differential equations to linear form*, by Professor E. A. Nordhaus, Michigan State College.

This paper dealt with the conditions under which an ordinary differential equation is reducible to linear form by a transformation of the dependent variable. An explicit expression for the transformation was given. This was then applied to first order equations to obtain a solution of the Bernoulli equation.

5. *Playing with a Pascal square*, by Professor W. C. Rufus, University of Michigan.

Due to the illness of Professor Rufus, his paper was presented by Professor

N. H. Anning. Professor Rufus had arranged the numbers of the Pascal triangle in a square, and found simple expressions for the numbers in any row, column, or diagonal, and for the ratio of any number to another at any given distance and direction from it. Application was made to expansions of a binomial and to the calculation of probabilities.

6. *A maximum problem*, by Dr. B. M. Stewart, Michigan State College.

Dr. Stewart's paper was published in this MONTHLY, vol. 49, 1942, pp. 454–456.

7. *Constrained vectors in the plane*, by L. G. Johnson, University of Michigan.

This speaker discussed the addition of vectors each of which is constrained to remain on a fixed line. In case two such lines intersected, he placed the terminal point of the first vector and the initial point of the second at the intersection, and defined the sum as the constrained vector running from the initial point of the first to the terminal point of the second. It was shown that this noncommutative addition enjoys certain of the usual properties of addition. Several interesting applications were made.

8. *Analyzing degrees of freedom into comparisons when the classes do not contain the same number of items*, by Professor W. D. Baten, Michigan State College.

Professor Baten showed how to analyze the degrees of freedom in analyses of variances when the classes do not contain the same number of observations. The "treatment sum of squares" was broken up into sums of squares with single degrees of freedom. Two proofs were offered, one based upon orthogonal linear transformations, and the other based upon the fact that a certain combination of quadratic forms each with a single degree of freedom is equivalent to a certain other quadratic form which is the sum of the squares pertaining to "treatment."

9. *New methods in air navigation*, by Professor H. C. Carver, University of Michigan, introduced by the Secretary.

The speaker outlined the basic principles of the graphical method of dead reckoning employed in air navigation, and applied them to problems of interception and radius of action. He also gave a brief discussion of numerical dead reckoning, in which he has developed new methods.

10. *A model of the celestial sphere*, by Professor L. S. Johnston, University of Detroit.

Professor Johnston exhibited a nicely constructed model of the celestial sphere, the various great and small circles of which were represented by circles of wire welded together. The model was about two feet in diameter. He gave some practical suggestions on the construction of such models and their use in the teaching of classes in spherical trigonometry and navigation.

C. J. COE, *Secretary*

THE MARCH MEETING OF THE SOUTHERN CALIFORNIA SECTION

The twenty-fourth regular meeting of the Southern California Section of the Mathematical Association of America was held at Los Angeles City College on Saturday, March 11, 1944. Professor D. C. Duncan, Chairman of the Section, presided at the morning and afternoon sessions.

The attendance was sixty-five, including the following thirty-five members of the Association: O. W. Albert, Harry Bateman, E. T. Bell, L. T. Black, Sister Myrtie Collier, P. H. Daus, R. P. Dilworth, D. C. Duncan, Harriet E. Glazier, Sister Miriam Hand, E. J. Hills, Frances C. Hinds, G. H. Hunt, C. G. Jaeger, Glenn James, E. M. Justin, G. R. Kaelin, G. R. Livingston, Ada A. McClellan, G. F. McEwen, P. M. Niersbach, W. B. Orange, W. T. Puckett, H. R. Pyle, E. C. Rex, J. M. Robb, G. E. F. Sherwood, R. H. Sorgenfrey, A. E. Taylor, B. P. Taylor, W. I. Thompson, V. C. Throckmorton, Morgan Ward, W. M. Whyburn, M. A. Zorn.

At the business meeting the following officers were elected for the next year: Chairman, P. G. Hoel, University of California at Los Angeles; Vice-Chairman, R. P. Dilworth, California Institute of Technology; Program Committee, Frances C. Hinds, Chairman, G. R. Kaelin, William Glenn, and the Secretary. It was decided to hold the next meeting at George Pepperdine College in Los Angeles on March 10, 1945.

The following papers were presented:

1. *Diffusion from a circular area having an initial concentration exceeding that outside by a constant*, by Professor G. F. McEwen, Scripps Institution of Oceanography.

This paper dealt with the problem of finding a solution, in a form suitable for numerical application, of the diffusion equation

$$\frac{\partial u}{\partial t} = a^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \quad \text{or} \quad \frac{\partial u}{\partial t} = a^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right)$$

subject to the boundary conditions

$$u = c \quad \text{for} \quad r < r_1, \quad \text{and} \quad u = 0 \quad \text{for} \quad r > r_1, \quad \text{when} \quad t = 0.$$

The formal solution was obtained in terms of a definite integral involving Bessel functions. Except in the particular cases corresponding to the center and circumference, it was necessary first to differentiate the integral, and to resort to approximate methods of integration, in order to obtain close approximation in terms of tabulated functions.

2. *Concerning the "width" of plane continua*, by Dr. R. H. Sorgenfrey, University of California at Los Angeles.

By use of the concept of congruent continua, there can be assigned to each plane continuum a non-negative number called its width. The measure of a continuum is not greater than the product of its diameter and its width. If a con-

tinuum has width zero, it can contain no triod nor can it separate the plane; it must therefore be irreducible between some two of its points. In order for there to exist in some bounded portion of the plane infinitely many disjoint continua congruent to a given continuum M, it is necessary and sufficient that M have width zero. An example is given of a non-locally connected continuum having width zero.

3. *The inversion of order of doubly infinite integrals*, by Professor Morgan Ward, California Institute of Technology.

Professor Ward discussed criteria for the integration of series term by term, and for the inversion of order of integration in iterated integrals in which both ranges of integration are infinite. The criteria involved no reference to uniform convergence, but to absolute convergence instead. The intent was to furnish easily applicable tests for use in such problems as occur frequently in the analysis employed in mathematical physics.

4. *An application of the Hartmann-Cornu formula to the determination of radial velocities*, by Professor E. M. Justin, University of California at Los Angeles.

The speaker gave a brief description of the telescope, spectrograph, and measuring engine, and employed slides for purposes of illustration. He then discussed the reduction formula by means of which wave lengths, and finally stellar velocities, are obtained from the measured lines on the sectral flats. A few remarks were made about stellar velocities in general along the line of sight.

5. *Some applications of solid analytic geometry to aircraft problems*, by J. J. Apalategui, Douglas Aircraft Company, introduced by L. J. Adams.

Mr. Apalategui remarked that solid analytic geometry can well be termed "mass production mathematics" as contrasted with the more common trigonometric and descriptive geometry methods of solving for the dimensions which define aircraft parts. Mass production of airplanes calls for correct detail parts which go together without rework in the final assembly of the plane. Some applications of solid analytic geometry to this problem were itemized as follows: the equations of basic wing and empennage lines used to determine the shapes of both normal and canted structural parts; determination of many true and projected angles which define the angularity of structural units; calculation of distances between points, lines, and planes; and the equations for the rotation of axes, which are used in the positioning of parts in any convenient subassembly position in the detail breakdown of an airplane. The speaker distributed in mimeographed form a detailed description of the methods used, complete with diagrams and tables.

6. *The engineer and his mathematics*, by Dr. Adam Zaborski, Lockheed Aircraft Corporation, introduced by Professor Clifford Bell.

The speaker observed that the engineer is interested in mathematics as a tool rather than as an abstract science. He stated that in order to use mathematics

effectively, the engineer should possess three qualifications, namely: he should know how to express physical laws in mathematical form; he should know how to perform mathematical operations according to set rules; he should be able to deduce the physical interpretation of mathematical results. Mr. Zaborski considered that the first and last of these qualifications are the most important, and that they are also the most neglected in our engineering schools. He stressed the need for instructors who can lead engineering students to appreciate the usefulness of certain phases of advanced mathematics such as the calculus of variations, the calculus of finite differences, operational calculus, integral equations, matrices, and projective geometry.

7. *The analytic viewpoint applied to certain aircraft problems*, by R. A. Liming, North American Aviation, introduced by Professor P. H. Daus.

Mr. Liming emphasized the necessity for functional organization within the factory to secure dimensional coordination in all departments which are concerned in any manner with the dimensions of the airplane. He explained how a master dimension group, using analytic methods, can obtain this desired result. The methods used to obtain this coordination in the building of the North American Mustang were discussed. He made a plea for students better trained to think analytically in geometric situations.

P. H. DAUS, *Secretary*

CALENDAR OF FUTURE MEETINGS

Twenty-Seventh Summer Meeting, Wellesley, Mass., August 12–14, 1944.
Twenty-Eighth Annual Meeting, Chicago, Ill., November 24–26, 1944.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN
ILLINOIS
INDIANA, Indianapolis, November 10, 1944
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA
NORTHERN CALIFORNIA, San Francisco, January 27, 1945
OHIO
OKLAHOMA
PHILADELPHIA, Philadelphia, November, 1944
ROCKY MOUNTAIN
SOUTHEASTERN
SOUTHERN CALIFORNIA, Los Angeles, March 10, 1945
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE
WISCONSIN

The Chauvenet Prize

In the year 1925, the MATHEMATICAL ASSOCIATION OF AMERICA established a prize of one hundred dollars for the best expository paper published in English during successive periods of five years by a member of the Association. Through two subsequent gifts the prize is now awarded every three years. The last award was made in December 1941 to Professor Saunders Mac Lane for his two papers in the *American Mathematical Monthly:* "Modular fields," volume 47, 1940, pp. 259-274, and "Some recent advances in algebra," volume 46, 1939, pp. 3-19.

As determined more recently by the Trustees, the prize is to be fifty dollars and is to be awarded for a noteworthy expository paper such as will come within the range of profitable reading of members of the Association. The purpose of the prize is to stimulate expository contributions in mathematical journals on the part of the younger American scholars. The award does not apply to books, although the CARUS MONOGRAPHS are expository in character and on this score might be included; they carry their own reward.

It is believed that clear expositions of mathematical subjects are greatly needed in this country and that the CHAUVENET PRIZE will tend to stimulate such production.

Note that the prize is to be awarded only to a *member* of the ASSOCIATION—one more of the many good reasons for membership.

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AUGUST-SEPTEMBER 1944

The AMERICAN MATHEMATICAL MONTHLY

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HILBERT'S AXIOMS OF PLANE ORDER

C. R. WYLIE, JR., Ohio State University

1. Introduction. Beyond the bare facts of the courses they will be called upon to teach, there are probably few things which can contribute more to the training of teachers of secondary school geometry than participation in a critical examination of the definitions and axioms of Euclid, and a comparison of them with a carefully developed modern equivalent, for instance the axiomatic system of David Hilbert.*

Of the various sets of axioms included in Hilbert's system, the axiom of parallels is in some ways the most interesting, opening up as it does the spectacular fields of non-euclidean geometry through its denial. However in another sense a careful scrutiny of the axioms of order affords a more profitable investment of time for those who have no lasting interest in any geometry but euclidean.

In the first place, the very existence of these axioms, devoted to such an obvious and familiar notion as *betweenness* serves to emphasize the fundamental modern attitude that geometry is an abstract science concerned with undefined elements whose properties are to be inferred from a set of consistent but otherwise essentially arbitrary assumptions, and not from any pseudo-isomorphism between the geometry elements and objects in the universe of experience. Secondly, it is in connection with the axioms of order that the logical structure of Euclid is weakest. In fact most of the fallacies of elementary geometry, such as the well-known "proofs" that all angles are right angles, and that all triangles are isosceles, spring immediately from the lack of clear-cut order relations.

The present note is essentially an expository account of the independence of these axioms, which may be of interest to those engaged in the training of secondary school teachers, and to those interested in the foundations of geometry, for some of the results obtained here are new, so far as the author is aware.

2. Hilbert's axioms. We shall consider exclusively the order axioms of Hilbert as they appear in the Carus Monograph,* The Foundations of Geometry. (*loc. cit.*) They are

1. If A, B, C, are points of a straight line, and B lies between A and C, then B lies between C and A.
2. If A and C are two distinct points of a straight line, then there exists at least one point B lying between A and C, and at least one point D, so situated that C lies between A and D.
3. Of any three points situated on a straight line, there is always one and only one which is between the other two.
4. Any four points A, B, C, D of a straight line can always be so arranged [*i.e.*, named] that B shall lie between A and C and also between A and D, and furthermore that C shall lie between A and D, and also between B and D.

* The Foundations of Geometry, David Hilbert; La Salle, Ill., 1938.

If we now define the segment AB to be the set of all points which are between A and B, we can add to the above axioms which define the notion of ***betweenness*** for points on a single line, the plane order axiom of Pasch

5. Let A, B, C be three points not lying in the same straight line and let a be a straight line lying in the plane of A, B, C and not passing through any of the points A, B, C. Then if the straight line a passes through a point of the segment AB it will also pass through either a point of the segment BC or a point of the segment AC.

3. Independence of the linear axioms. Considering the linear axioms 1–4 by themselves it is easy to establish their independence by concrete representations. For instance the following example shows that axiom 1 is independent of axioms 2–4.

Let the system consist of all the points on a line, and define "between A and B" to mean ***actually between*** if B is to the right of A, and ***between but not the mid-point of*** if B is to the left of A.

Similarly axioms 2, 3, 4 are shown to be independent by the following examples respectively. The system of points with coordinates $0, \pm 1, \pm 2, \cdots$ on a line with "between" defined to mean ***actually between***. The system of all points on a line with "between" defined to mean ***not between*** [!]. The system of all points on a line with "between A and B" defined to mean ***to the right of both A and B***.

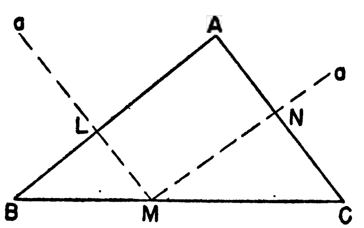


FIG. 1

4. Precise meaning of axiom 5. When axiom 5 is added to the four linear axioms certain dependencies are introduced. Before proceeding to a discussion of these it is desirable, however, to clarify a possible ambiguity in the meaning of this axiom. As stated, it is not entirely clear whether its alternatives are exclusive or not. That they are is an easy matter to demonstrate, and we now show that by appealing only to axiom 3 and the weak form of axiom 5, we may conclude that the line a which by hypothesis passes through a point of the segment AB can pass through a point in *only one* of the remaining segments BC or AC. In fact let the line a in Fig. 1, not passing through any of the points A, B, C, pass through points L, M, N of the segments AB, BC, and AC respectively. By axiom 3 one of the points L, M, N, say M, lies between the other two.* Now consider the three non-collinear points L, N, A. The straight line BC passes

* Henceforth we shall indicate such a "betweenness" relationship by writing simply (LMN).

through the point M of the segment LN. Therefore it must pass through a point in *at least* one of the segments NA or LA. But this is impossible, for by axiom 3 since N lies between A and C, C cannot simultaneously be between N and A, and for the same reason B cannot lie between L and A. This contradiction of the weak form of axiom 5 establishes the result.

5. Dependence of axiom 1. We now prove that axiom 1 is a consequence of axioms 2, 3, and 5. Let (AXB) be true on a line l, Fig. 2. Choose C not on l and Y on the line AC so that (ACY), (axiom 2). Now consider the line XY in relation to the three non-collinear points A, B, C. It passes through the point X, of the segment AB. It must therefore pass through a point of the segment BC

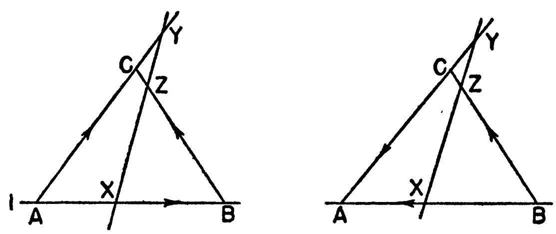


FIG. 2

or a point of the segment AC, by axiom 5. By axiom 3 it cannot pass through a point of AC, therefore it must pass through a point, say Z, of the segment BC. Now consider this same line in relation to the three points B, C, A. It surely passes through a point, namely Z, of the segment BC. It cannot pass through a point of CA, hence it must pass through a point of the segment BA. This is possible if and only if X is between B and A.

6. Dependence of axiom 4. Axiom 4 is also a consequence of axioms 2, 3, and 5, as was first pointed out by E. H. Moore* in 1902. His proof though entirely straightforward was somewhat lengthy, and a shorter proof may be of interest here. This seems doubly appropriate since in subsequent editions of Hilbert's *Foundations* this result is still not noted.†

We base our proof upon the following series of observations.

LEMMA: *Of four points, A, B, C, D, situated on a line l, no one can be "between" only once.*

We assume the contrary and suppose that of the four points A, B, C, D on l, Fig. 3, the point B lies between A and C and between no other pair. Choose E not on l, and on CE choose F so that (CEF), axiom 2. Now consider the line BF in relation to the three points A, C, E. It passes through the point B of the

* E. H. Moore, The projective axioms of geometry, Trans. Amer. Math. Soc., 1902.

† It is, of course, to be found in various places outside of Moore's paper. For instance J. W. Young makes explicit mention of this dependence in his book, Fundamental Concepts of Algebra and Geometry.

segment AC. Since by axiom 3 it cannot pass through a point of the segment CE, it must pass through a point, say G, of AE. Now consider this same line in relation to the three points A, E, D. It passes through the point G of the segment AE. It cannot pass through a point of the segment AD, since by hypothesis B is only between A and C on l. Therefore it must pass through a point of ED, say H. Now finally consider this same line in relation to the points E, D, C. It passes through a point of ED as we have just seen. However it cannot pass through a point of DC by hypothesis, and by axiom 3 it cannot pass through a point of EC. This contradicts axiom 5 and establishes our result.

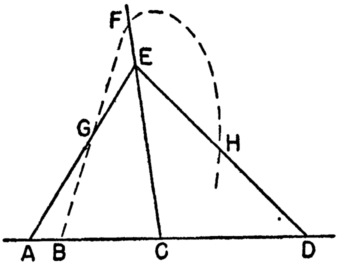


FIG. 3

As a first corollary of this lemma we conclude that of our points on a line, no one can be "between" three times. For there are only four combinations of three points each, existing in the set of points A, B, C, D, and by axiom 3 only one point in each triad can be "between." Hence there are only four possible "betweeness" relations insofar as the central term is concerned. If then, one point is the middle point in three of these, some other point must be "between" only once, which contradicts the lemma.

As a second corollary we observe that of four points on a line, two are necessarily "between" exactly twice, and two are "between" not at all.

The proof of axiom 4 as a consequence of axioms 2, 3, 5, and 1 (now known to be a theorem independent of axiom 4) follows at once from the second corollary. Of the four points A, B, C, D on a line l let the points which are never "between" be called A and D, the others B and C. Then of the respective sets of relations

$$(ABC) \quad (ABD) \quad (CBD)$$

and

$$(ACB) \quad (ACD) \quad (BCD)$$

two and only two of each set must be true. If first, (ABC) is true, then (ACB) is necessarily false by axiom 3. Hence (ACD) and (BCD) are true, and since the last is true, (CBD) is false, leaving (ABD) true. The points are then named according to axiom 4. If on the contrary (ABC) is false, then inevitably (ABD), (CBD), (ACB), and (ACD) are true, and if the names B and C are interchanged, the ordering of the axiom is again accomplished.

7. Independence of axioms 2, 3, and 5. The three axioms that remain may now be shown to be independent by the following concrete representations.

For the system consisting of three non-collinear points and the three lines through them, with "between" defined to mean *actually between*, axiom 2 is false and axioms 3 and 5 are (vacuously) true.

In the ordinary euclidean plane with "between" defined to mean *collinear with* axioms 2 and 5 are true but axiom 3 is false.

In the ordinary euclidean plane with "between" defined to mean *above both* on all vertical lines, and *to the right of both* on all other lines, axioms 2 and 3 are valid but axiom 5 is false.

8. A new form of axiom 5. We have now completed our exposition of the dependence and independence of the axioms of betweenness in the specific form in which Hilbert presented them. Numerous other sets of axioms have been devised to define this same notion. In general these differ substantially from Hilbert's system, and we do not propose to consider them here. There is however one other system almost identical with that of Hilbert which we shall touch on in conclusion.

Suppose we keep the axioms as quoted in paragraph I intact except that in axiom 5 we change the last phrase to read " . . . it will also pass through either a point of the segment BC *or a point of the segment* CA." The nature of this new set of axioms must be regarded as a new problem, entirely independent of those observations which led us to our conclusions concerning the dependence of Hilbert's own set. It may well be that axioms dependent in the first formulation will now be independent or vice-versa. And the proofs of such relations as are true in both systems may differ widely in their character.

9. Dependence of the new axioms. We shall not explore this new system in detail. The final results are curiously enough the same here as before: axioms 1 and 4 are consequences of axioms 2, 3, and 5. The proof of this follows roughly the outline used above. For preliminary clarification we can prove a strong form of axiom 5 precisely as we did in treating Hilbert's version of the axioms.

When we attempt to establish the dependence of axiom 1, however, we must devise an entirely new form of proof. In fact it must be based upon the lemma of paragraph 6 and its corollaries, instead of being independent of them. We omit the new proof of this lemma which follows *mutatis mutandis* from the earlier proof, and give only the new proof of the dependence of axiom 1.

Suppose that on a line l, (AXC) is true while (CXA) is false. Choose B not on l, Fig. 4, and choose Y so that (BYA), axiom 2. Now consider the line XY in relation to the points B, A, C. It passes through the point Y of the segment BA, and the point X of the segment AC. Therefore by the strong form of axiom 5* it cannot pass through a point of CB. Now choose D so that (AYD) and consider the same line in relation to the points A, D, C. It passes through a point

* Note that in all our discussion of Hilbert's axioms we did not need to use the strong form of axiom 5, and only called attention to it in order to secure a clearer understanding of the system.

of AD, namely Y, does not pass through a point of CA, since by hypothesis (CXA) is false, hence must pass through a point, say Z, of DC. Finally consider this line in relation to the points D, C, B. It passes through the point Z of DC, does not pass through a point of CB, as pointed out above, and therefore must pass through a point of BD. This is possible if and only if Y is between B and D. But then in the range A, Y, B, D, the point Y is between three times which is impossible by the first corollary. This contradiction establishes the result.

The proof that axiom 4 is dependent is literally identical in each system and we pass over it without mention.

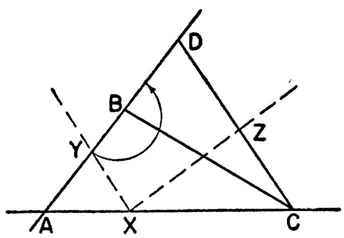
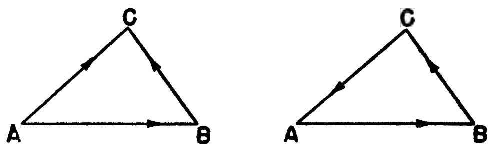


FIG. 4

10. Conclusion. We have considered as a separate problem the modified system that we have just discussed, not only as a simple experiment designed to make clear how a very slight change in an axiom may completely alter the character of a proof, but also to illumine a careless treatment of these axioms that is sometimes encountered. Axiom 5 which introduced the dependencies we have been discussing is sometimes loosely stated "A straight line which meets one side of a triangle and lies in the plane of the triangle meets at least one other side of the triangle."

Until a triangle is defined, this statement is meaningless. If the word triangle is taken to mean the configuration shown in Fig. 5a, associated as it were with vector addition, the axiom is simply an ill-stated version of Hilbert's axiom 5.



(a) (b)

FIG. 5

If, equally naturally, the word triangle is taken to imply the cyclic configuration shown in Fig. 5b we have the second version of axiom 5. If finally the word triangle is defined in the sense of elementary high school geometry, we pemit either of the above definitions, and commit the paralogism of incorporating axiom 1 in the definition of a term employed in another axiom. Small wonder that we can then prove it dependent!

MOTION WITH RESPECT TO MOVING AXES

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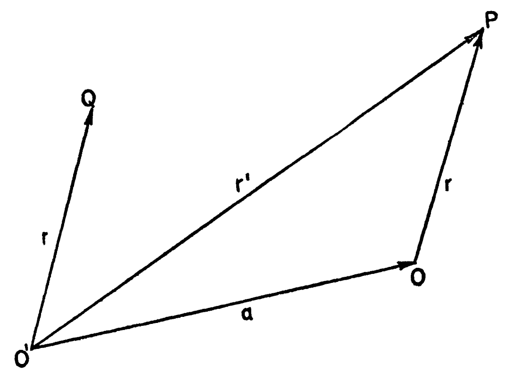
A paper by the author under a similar title has appeared previously.* The development there was in coordinate notation, and in the present paper the same results are obtained using vector notation for the discussion. The vector treatment also throws some additional sidelights on the problem.

1. Newtonian functions. For the sake of clarity we make a few explanations. In Newtonian mechanics when rates of change of quantities are referred to they are to be taken with respect to fixed or inertial axes. This is implied by the fact that Newton's laws of motion apply fundamentally only to such axes. It is often convenient, however, to use moving axes, and this necessitates extra care. Accordingly we introduce the idea of a Newtonian function, by which we shall mean a function which involves rates of change only with respect to fixed or inertial axes. Since for vector notation, however, we need only reference points and not axes, we can replace *axes* by *points* in the above definition when vector notation is to be used. But when interpreting results in component form we must take components with respect to fixed axes.



To illustrate what we mean, let O' be a fixed point and O a moving point whose position with respect to O' is $\mathbf{a}$; let P be a point whose position with respect to O is $\mathbf{r}$, and with respect to O' is $\mathbf{r}'$, so that

$$\mathbf{r}' = \mathbf{r} + \mathbf{a}; \tag{1}$$

also, for what follows, let O' be a former position of O.

Now let ϕ be a vector function of position with respect to any chosen reference point. Then $\phi(\mathbf{r}')$ is the function associated with P and with respect to O'; $\phi(\mathbf{r})$ is the function associated with P and with respect to O, or the function associated with Q and with respect to O'.

* J. W. Campbell, Motion re moving axes, Trans. Roy. Soc. of Can., Ser. III, vol. XXXVI, pp. 1–6.

But $\phi(\mathbf{r}')$ may depend on $\dot{\mathbf{r}}'$, $\ddot{\mathbf{r}}'$, $\cdots$ as well as on $\mathbf{r}'$ and t, and so we write

$$\phi(\mathbf{r}') = \phi(\mathbf{r}', \dot{\mathbf{r}}', \ddot{\mathbf{r}}' \cdots ; t).$$

In that case, if ϕ is a Newtonian function,

$$\phi(\mathbf{r}) = \phi(\mathbf{r}, \dot{\mathbf{r}}', \ddot{\mathbf{r}}', \cdots ; t).$$

For any rates of change occurring must be with respect to the fixed point with which the moving point instantaneously coincides, and the rates of change with respect to all fixed points are equal.

As an illustration consider the particular case of angular momentum. If a mass m were located at P its angular momentum about O' would be $\dot{\mathbf{r}}' \times (m\dot{\mathbf{r}}')$ and its angular momentum about O would be $\mathbf{r} \times (m\dot{\mathbf{r}}')$, not $\mathbf{r} \times (m\dot{\mathbf{r}})$. The latter would not be a Newtonian function.

Inasmuch, then, as the other arguments do not change with change of reference point, we shall for simplicity retain the position argument only and so we write $\phi(\mathbf{r}')$ and $\phi(\mathbf{r})$.

2. Derivatives. It follows from (1) that

$$\phi(\mathbf{r}') = \phi(\mathbf{r} + \mathbf{a}). \tag{2}$$

Now the Taylor's expansion

$$\begin{aligned} \phi(x + h, y + k, z + l) &= \phi(x, y, z) \\ &+ \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} + l \frac{\partial}{\partial z}\right)\phi(x, y, z) \\ &+ \frac{1}{2!}\left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} + l \frac{\partial}{\partial z}\right)^2 \phi(x, y, z) \\ &+ \cdots \end{aligned} \tag{3}$$

can be written in vector notation as

$$\phi(\mathbf{r} + \mathbf{a}) = \phi(\mathbf{r}) + (\mathbf{a} \cdot \nabla)\phi(\mathbf{r}) + \frac{1}{2!}(\mathbf{a} \cdot \nabla)^2\phi(\mathbf{r}) + \cdots .* \tag{4}$$

Hence equation (2) can be written

$$\phi(\mathbf{r}') = \phi(\mathbf{r}) + (\mathbf{a} \cdot \nabla)\phi(\mathbf{r}) + \frac{1}{2!}(\mathbf{a} \cdot \nabla)^2\phi(\mathbf{r}) + \cdots . \tag{5}$$

Differentiating with respect to t we have

$$\begin{aligned} \frac{d}{dt}\phi(\mathbf{r}') = \frac{d}{dt}\phi(\mathbf{r}) &+ (\dot{\mathbf{a}} \cdot \nabla + \mathbf{a} \cdot \dot{\nabla})\phi(\mathbf{r}) \\ &+ \text{terms involving } \mathbf{a} \text{ as a ``factor.''} \end{aligned}$$

* See J. G. Coffin, Vector Analysis, p. 131.

Proceeding to the limit as $\mathbf{a}\to 0$, we have

$$\frac{d}{dt}\phi(\mathbf{r}') = \frac{d}{dt}\phi(\mathbf{r}) + (\dot{\mathbf{a}}\cdot\nabla)\phi(\mathbf{r}). \tag{6}$$

Now since $\phi(\mathbf{r}')$ is the function associated with P and with respect to O', the left-hand side of (6) is the rate of change of ϕ with respect to the fixed point with which O instantaneously coincides. It is therefore the derivative in the Newtonian sense and so may be called the Newtonian derivative.

On designating the Newtonian derivative by a subscript N we have

$$\left(\frac{d}{dt}\right)_N \phi(\mathbf{r}) = \frac{d}{dt}\phi(\mathbf{r}) + (\dot{\mathbf{a}}\cdot\nabla)\phi(\mathbf{r}).$$

In more compact notation, if $\phi(\mathbf{r})$ is denoted by $\mathbf{V}$ and if $\mathbf{u}$ is the velocity of O with respect to a fixed point, then

$$\dot{\mathbf{V}}_N = \dot{\mathbf{V}} + (\mathbf{u}\cdot\nabla)\mathbf{V}. \tag{7}$$

To see the meaning of $\dot{\mathbf{V}}$, recall that $\phi(\mathbf{r})$ can be interpreted as the function associated with Q and with respect to O'. Therefore $\dot{\mathbf{V}}$ is the rate of change of $\mathbf{V}$ *as if O were fixed*, and the expression for $\dot{\mathbf{V}}$ is known from classical theory. Thus, if $\mathbf{V}=(V_1, V_2, V_3)$ and the axes are turning with the angular velocity $\boldsymbol{\omega}$, then

$$\dot{\mathbf{V}} = \dot{\mathbf{V}}_f + \boldsymbol{\omega}\times\mathbf{V}, \tag{8}$$

where

$$\dot{\mathbf{V}}_f = \dot{V}_1\mathbf{i} + \dot{V}_2\mathbf{j} + \dot{V}_3\mathbf{k}. \tag{9}$$

Therefore

$$\dot{\mathbf{V}}_N = \dot{\mathbf{V}}_f + \boldsymbol{\omega}\times\mathbf{V} + (\mathbf{u}\cdot\nabla)\mathbf{V}. \tag{10}$$

The first term on the right hand side of (10) gives the rate of change of $\mathbf{V}$ as if the axes were fixed, the second term gives the rate of change due to turning of axes, and the third term the rate of change due to translation.

Written in component form this equation gives

$$\begin{aligned}
\dot{V}_{1N} &= \dot{V}_1 - \omega_3 V_2 + \omega_2 V_3 + u_1\frac{\partial V_1}{\partial x} + u_2\frac{\partial V_1}{\partial y} + u_3\frac{\partial V_1}{\partial z},\\
\dot{V}_{2N} &= \dot{V}_2 - \omega_1 V_3 + \omega_3 V_1 + u_1\frac{\partial V_2}{\partial x} + u_2\frac{\partial V_2}{\partial y} + u_3\frac{\partial V_2}{\partial z},\\
\dot{V}_{3N} &= \dot{V}_3 - \omega_2 V_1 + \omega_1 V_2 + u_1\frac{\partial V_3}{\partial x} + u_2\frac{\partial V_3}{\partial y} + u_3\frac{\partial V_3}{\partial z}.
\end{aligned} \tag{11}$$

These results agree with those of the previous paper.

3. Remarks. It might be thought that the right hand side of (7) would introduce terms in $\dot{\mathbf{r}}$, but such is not the case. We can conclude this from the fact that the Newtonian derivative is a Newtonian function, but we can also see it analytically.

For (7) can be written in the form

$$\dot{\mathbf{V}}_N = \frac{D'V_1}{D't}\mathbf{i} + \frac{D'V_2}{D't}\mathbf{j} + \frac{D'V_3}{D't}\mathbf{k} + \boldsymbol{\omega} \times \mathbf{V} \tag{12}$$

when $D'/D't$ is an operation such that

$$\begin{aligned} \frac{D'V_1}{D't} &= \frac{dV_1}{dt} + u_1 \frac{\partial V_1}{\partial x} + u_2 \frac{\partial V_1}{\partial y} + u_3 \frac{\partial V_1}{\partial z} \\ &= (u_1 + \dot{x}) \frac{\partial V_1}{\partial x} + (u_2 + \dot{y}) \frac{\partial V_1}{\partial y} + (u_3 + \dot{z}) \frac{\partial V_1}{\partial z} \\ &\qquad + \text{terms not involving } x, y, z. \end{aligned} \tag{13}$$

That is

$$\frac{D'V_1}{D't}\mathbf{i} + \frac{D'V_2}{D't}\mathbf{j} + \frac{D'V_3}{D't}\mathbf{k} = (\dot{\mathbf{r}}'\cdot\nabla)\mathbf{V} + \text{terms in } t, \mathbf{r}, \dot{\mathbf{r}}', \ddot{\mathbf{r}}', \cdots. \tag{14}$$

Therefore by (12) and (13) $\dot{\mathbf{V}}_N$ contains no terms in $\dot{\mathbf{r}}$.

In particular, if $\mathbf{V} = \mathbf{r}$ we obtain from (10) and (12)

$$\dot{\mathbf{V}}_N = \dot{\mathbf{r}}_f' + \boldsymbol{\omega} \times \mathbf{r} = \dot{\mathbf{r}}'.$$

It is worth noting that it is only in the cases where $\mathbf{V}$ does not depend explicitly on the derivatives $\dot{\mathbf{r}}', \ddot{\mathbf{r}}', \cdots$ that (13) can be written in the form

$$\frac{D'V_1}{D't} = \frac{\partial V_1}{\partial t} + (u_1 + \dot{x}) \frac{\partial V_1}{\partial x} + (u_2 + \dot{y}) \frac{\partial V_1}{\partial y} + (u_3 + \dot{z}) \frac{\partial V_1}{\partial z}\cdot$$

This is analogous to, but not identical with, the hydrodynamical equation

$$\frac{DV_1}{Dt} = \frac{\partial V_1}{\partial t} + u_1 \frac{\partial V_1}{\partial x} + u_2 \frac{\partial V_1}{\partial y} + u_3 \frac{\partial V_1}{\partial z}\cdot$$

It therefore follows that the general dynamical equations for axes moving in any way are entirely independent of the equations of hydrodynamics in spite of some apparent analogy of form.

4. Functions of several points. In the foregoing discussion ϕ has been regarded as a function of a single point $\mathbf{r}$. It might, however, depend on a set of points r_i $(i = 1, \cdots, n)$. In that case there will be a variation of ϕ associated with each one of these points when the origin moves. By a sequential application of the Taylor's series process it will be found that if

$$\mathbf{V} = \phi(\mathbf{r}_1, \cdots, \mathbf{r}_n),$$

then

$$\dot{\mathbf{V}}_N = \dot{\mathbf{V}}_f + \boldsymbol{\omega} \times \mathbf{V} + \left(\mathbf{u} \cdot \sum_{i=1}^{n} \boldsymbol{\nabla}_i\right)\mathbf{V}, \tag{15}$$

where $\boldsymbol{\nabla}_i$ is the operator associated with r_i.

A special case of importance may arise. In the expression $\phi(\mathbf{r}_1, \cdots, \mathbf{r}_n)$ the $\mathbf{r}$'s may occur in pairs and be subtracted. In that case if we put $\mathbf{r}_i' = \mathbf{r}_i + \mathbf{a}$, the $\mathbf{a}$'s will all cancel in ϕ and so we shall have

$$\phi(\mathbf{r}_1', \cdots, \mathbf{r}_n') = \phi(\mathbf{r}_1, \cdots, \mathbf{r}_n).$$

On differentiation we find that

$$\dot{\mathbf{V}}_N = \dot{\mathbf{V}}_f + \boldsymbol{\omega} \times \mathbf{V}, \tag{16}$$

and so the term $(\mathbf{u} \cdot \Sigma \boldsymbol{\nabla}_i)\mathbf{V}$ is zero. This might also have been inferred from the fact that when two $\mathbf{r}$'s occur as a pair in ϕ and are subtracted, the $\boldsymbol{\nabla}$'s corresponding to these $\mathbf{r}$'s will cancel.

Vector functions of position are, therefore, divided into two classes, *viz.*, those that are affected by a translation of the origin and those that are not. When they are not affected by a translation of the origin the Newtonian derivative is given by (16).

If a function is explicitly a function of $\mathbf{r}$'s with the $\mathbf{r}$'s subtracted in pairs then we can conclude that a translation of the origin has no effect on the function and so we can use (16). But sometimes only one $\mathbf{r}$ appears explicitly and yet it is known that a translation of the origin has no effect on the function. In such cases, also, the Newtonian derivative is given by (16); another $\mathbf{r}$ is implicitly present to make this the correct result.

An example is the velocity vector. The Newtonian derivative of $\mathbf{r}$ is

$$\mathbf{v} = \dot{\mathbf{r}}_N = \dot{\mathbf{r}}_f + \boldsymbol{\omega} \times \mathbf{r} + \mathbf{u}\phi,$$

and this is explicitly a function of $\mathbf{r}$. But the velocity is obviously independent of the position of the origin and so we can write

$$\dot{\mathbf{v}}_N = \dot{\mathbf{v}}_f + \boldsymbol{\omega} \times \mathbf{v}.$$

In such cases it is not necessary to determine the $\mathbf{r}$ which appears implicitly, but in the case of the velocity vector it is not hard to find. For if $\mathbf{r}_1$ is a point on the instantaneous axis of the frame then $\mathbf{u} = -(\dot{\mathbf{r}}_1)_f - \boldsymbol{\omega} \times \mathbf{r}_1$, and therefore

$$\mathbf{v} = (\dot{\mathbf{r}} - \dot{\mathbf{r}}_1)_f + \boldsymbol{\omega} \times (\mathbf{r} - \mathbf{r}_1).$$

THE GEOMETRIC METHOD IN MATHEMATICAL STATISTICS

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1. Introduction. R. A. Fisher was the first to make use of n-dimensional geometry in the derivation of certain distributions. A direct approach is always possible, and is even to be preferred, by the use of the Fourier transform, with or without the transformation theory of positive definite quadratic forms. Nevertheless, the geometric method offers great advantages in brevity, clarity, and insight. It is in no way inferior in rigor to any other method and finally, its applications extend to a greater number of the distributions used in small-sample theory than is realized.

2. Geometric preliminaries. In an euclidean space of $n \geqq 1$ dimensions with coordinates $(x_1, x_2, \cdots, x_n)$ for a generic point x, the distance d between two points x and x' is given by

$$d^2 = \sum (x_i - x_i')^2. \tag{1}$$

In all summations, unless otherwise specified, the range is over values 1 to n of the index. This definition of distance, combined with the usual methods in use for 3-dimensional euclidean geometry, suffices to derive most of the formulas we need.

The vector $x' - x$ has direction components $a_i = x_i' - x_i$, $i = 1, 2, \cdots, n$, so that the square of its length is $d^2 = \sum a_i^2$, and the direction cosines $\alpha_i = a_i/d$, with $\sum \alpha_i^2 = 1$. The angle θ between two directions is given by $\cos \theta = \sum \alpha_i \alpha_i' = \sum a_i a_i'/dd'$.

A hyperplane in n-space is represented by a linear equation

$$\sum a_i x_i = y. \tag{2}$$

The coefficients a_i are the direction components of the normal to the plane. The perpendicular distance from a point x to the plane is $p = (\sum a_i x_i - y)/\sqrt{\sum a_i^2}$, where the proper sign is to be taken in the square root so as to make this distance positive. When the direction cosines α_i are used in place of a_i in equation (2), y is itself the perpendicular distance from the origin to the plane. The foot of the perpendicular to the plane from the point x is $\bar{x}$, given by

$$\bar{x}_i = x_i - \lambda a_i, \quad \text{where} \quad \lambda = \left(\sum a_i x_i - y\right)/\sum a_i^2. \tag{3}$$

The hypersphere of radius r centered at the origin is

$$\sum x_i^2 = r^2 \quad \text{or} \quad \sum x_i^2 \leqq r^2, \tag{4}$$

of which the first represents the surface, and the second the volume of the sphere. A plane section of the n-sphere is always an $n-1$ sphere where we understand by the one-dimensional sphere, the line-interval $(-r, r)$, with "surface" $x = \pm r$, the two-dimensional sphere being the circle, and so on. The volume of

the n-sphere is $2\pi^{n/2}r^n/n\Gamma(n/2)$. It suffices for our purpose that this volume should be cr^n, which is equivalent to the statement that the volume element for purposes of integration is

$$dV_n = dx_1dx_2 \cdots dx_n = r^{n-1}f(\theta_1\theta_2 \cdots \theta_{n-1})drd\theta_1d\theta_2 \cdots d\theta_{n-1}. \tag{5}$$

The actual form of f for any given n may be derived by simple extension of the 3-dimensional spherical polar coordinate formula.

3. The normal distribution and euclidean n-space. The most important basic distribution [important because of the "central limit theorem" which shows that it is the limiting form of the distribution of an average from any fairly general type of population] is the normal distribution, where the elementary probability is given by $(1/\sigma\sqrt{2\pi}) \exp -(x-\mu)^2/2\sigma^2dx$. By change of origin and scale, we can always take the population mean μ as zero, and the population variance σ^2 as unity; this we shall call standard measure.

For several normally distributed standard independent variables $x_1, \cdots, x_n$, the elementary probabilities are compounded by multiplication to give

$$\begin{aligned} dP &= \frac{1}{\sqrt{2\pi}}\, e^{-x_1^2/2}dx_1 \, \frac{1}{\sqrt{2\pi}}\, e^{-x_2^2/2}dx_2 \cdots \frac{1}{\sqrt{2\pi}} \, {}^{-x_n^2/2}dx_n, \\ &= \frac{1}{(\sqrt{2\pi})^n}\, e^{-r_n^2/2}dV_n. \end{aligned} \tag{6}$$

The basis of the geometrical method as applied to mathematical statistics is that this probability density depends only upon r_n^2, *i.e.*, it is isotropic in the n-space. Moreover, this is a characteristic property of the normal distribution. For, if the elementary probability were $f(x)dx$, and if we should ask ourselves when a relation of type

$$f(x_1)f(x_2) \cdots f(x_n)dx_1dx_2 \cdots dx_n = \phi(r_n)\, dV_n, \tag{7}$$

would be valid, we should obtain the functional equation

$$f(x_1)f(x_2) \cdots f(x_n) = \phi(\sqrt{\textstyle\sum x_i^2}), \tag{8}$$

to hold identically in x. Setting $x_2 = \cdots = x_n = 0$, we get $\phi(x_1) = f(0)^{n-1}f(x_1)$. The functional equation is thus equivalent to $f(x)f(y) = cf(\sqrt{x^2+y^2})$, which is known to have no continuous solutions except $f(x) = ae^{bx^2}$. This sort of argument is followed, for example, in the deduction of Maxwell's law in the kinetic theory of gases. If, now, the total range for each x_i is $-\infty$, $+\infty$, we have $b<0$, say $b = -1/2\sigma^2$, to make the total probability unity, $a = 1/\sigma\sqrt{2\pi}$. If the range be restricted, other distributions, including the uniform distribution, are possible; a fact that is forgotten by all who use this derivation for theoretical physics.

Not only may we build up the normal distribution in n dimensions from n independent individual distributions, but the process may also be reversed so

that one or more dimensions can be cut off as necessary. *Orthogonality is the geometric equivalent of independence in statistics.* We utilize this after performing linear transformations to new sets of orthogonal axes; for our purpose, dealing only with normal distributions in standard measure, rotations alone suffice. The effective number of dimensions in a given problem are called degrees of freedom by R. A. Fisher, who uses the letter n to denote this number. We shall use n for the original number of dimensions, keeping in mind that a random sample of n from an infinite basic population represents such an n-dimensional distribution as in (6). The degrees of freedom not in use are to be integrated out. Contrary to dynamical usage, the effective number of degrees of freedom in a statistical problem represents that number of dimensions in which the coordinates are free under the statistical conditions imposed only in the limited sense that a definite probability necessarily attaches to each region of the sub-space.

4. The distribution of the mean and variance in samples from normal. The χ^2-distribution. Let $y=\sum a_ix_i$ be a linear combination of n standard normal variables $x_1, \cdots, x_n$. To derive the distribution of y, we rotate so that one axis lies along the normal to the family of hyperplanes $y=\sum a_ix_i$, or with respect to the new axes, $y=\text{const.}$, and the remaining $n-1$ lie in the plane $\sum a_ix_i=0$. Taking $r=y/\sqrt{\sum a_i^2}$ we may split up the distance from the origin to a point x as $r_n^2=r_1^2+r_{n-1}^2$. That is, r_1 is the distance from the origin to the foot of the perpendicular to the particular hyperplane of the family $\sum a_ix_i=y$ which passes through the point $x=(x_1, \cdots, x_n)$; r_{n-1} is the distance in the plane from the foot of the perpendicular to the point x. The elementary probability of (6) may, therefore, be expressed as

$$(9) \qquad dP = \frac{1}{(\sqrt{2\pi})^n} e^{-r_n^2/2} dV_n = \frac{1}{\sqrt{2\pi}} e^{-r_1^2/2} dr_1 \cdot \frac{1}{(\sqrt{2\pi})^{n-1}} e^{-r_{n-1}^2/2} dV_{n-1}.$$

As the r_1 component alone interests us here, we may eliminate the rest by integration over the whole of V_{n-1}. Therefore, r_1 is normally distributed in standard measure. Hence, $y=r_1\sqrt{\sum a_i^2}$ is normal with zero mean and variance $\sum a_i^2$. Replacing $x_i+\mu_i$ by x_i, to pass from standard measure to the general normal distribution, we obtain:

THEOREM 1. *If n independent variates $x_1, \cdots, x_n$ are normally distributed with means μ_i and variances σ_i^2, any linear combination thereof $\sum a_ix_i$ is also normally distributed with mean $\sum a_i\mu_i$ and variance $\sum a_i\sigma_i^2$.*

On the other hand, we might have integrated out r_1 to concentrate upon r_{n-1}, noting that the resulting distribution in V_{n-1} would be normal, the original n-variables being now restricted to lie in a hyperplane, *i.e.*, to obey one identical linear restriction of type $\sum a_ix_i=\text{const.}$ As r_{n-1} is to be measured from the foot of the perpendicular from the origin of n-space, formula (9) may be applied to give

THEOREM 2. *If upon n originally independent standard normal variables a linear restriction $\sum a_i x_i = b$ is imposed, we obtain a normal distribution in $n-1$ variables*: $dP = c \exp - r_{n-1}^2/2 \cdot dV_{n-1}$, *where*

$$r_{n-1}^2 = \sum (x_i - \bar{x}_i)^2, \qquad \bar{x}_i = ba_i / \sum a_i^2. \tag{10}$$

If we abandon standard measure, putting $x_i - \mu_i$ for x_i, we should have to replace b by $b + \sum a_i \mu_i$. Thus, $\bar{x}_i$ shifts correspondingly except in the special case where all these separate additions cancel out in each bracket $(x_i - \bar{x}_i)$. To this end it is necessary and sufficient that

$$\mu_i = \frac{a_i \sum a_i}{\sum a_i^2}. \tag{11}$$

Since in sampling problems $\mu_i = \mu$, $i = 1, 2, \cdots, n$, we have

THEOREM 3. *For random sampling from a normal population, the sole linear restriction independent of the population mean is of type* $(x_1 + \cdots + x_n)/n =$ *constant.*

That is, in general, the population mean being unknown, we may measure from the sample mean $m = \sum x_i / n$ with the loss of a single degree of freedom. Moreover there is no other linear sample function from which such measurement may be made independent of the population mean. The statistic m has the further advantage that it is normally distributed, according to Theorem 1, with expectation equal to the population mean, and variance σ^2/n. Thus it is *unbiassed*, and using results and terminology due to Fisher, since no other estimate of μ can have a smaller variance than σ^2/n, m is the most *efficient* such estimate; finally, as $n \to \infty$, the probability for $m \neq \mu$ tends to zero, so that the estimate is consistent.

For the χ^2 and other tests, we need the distribution of the sum of squares of n normal standard variables, our r_n^2. That is,

$$P(r_n^2 \leqq t) = \int_{\Sigma x_i^2 \leqq t} \cdots \int e^{-(x_1^2 + \cdots + x_n^2)/2} dx_1 \cdots dx_n. \tag{12}$$

This is evaluated at once in spherical polar coordinates, integrating over thin spherical shells centered at the origin. This gives

$$P(r_n^2 \leqq t) = \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)(\sqrt{2\pi})^n} \int_0^{\sqrt{t}} e^{-r^2/2} r^{n-1} dr. \tag{13}$$

Transforming by $u = r^2$, $r = \sqrt{u}$, $dr = du/2\sqrt{u}$, we get

$$P(r_n^2 \leqq t) = \frac{1}{2^{n/2}\Gamma\left(\frac{n}{2}\right)} \int_0^{t} e^{-u/2} u^{n/2-1} du. \tag{14}$$

Provided the population mean is zero, this χ^2 or incomplete gamma function distribution holds even for a change of scale, in particular for $\sum x_i^2/\sigma^2$. The expection of $u = r_n^2$ is easily calculated by multiplying under the sign of integration by u and integrating to infinity. This amounts to replacing the exponent n by $n+2$, so that the result must be $2\cdot 2^{n/2}\Gamma(n/2+1)/2^{n/2}\Gamma(n/2) = n$. Therefore, r_n^2/n furnishes, when the population mean is known, an unbiassed estimate of σ^2. For unknown population mean, we apply the findings of Theorem 3 to obtain

THEOREM 4. *For random sampling from the same normal population,* $\sum(x-m)^2$ *with* $m = \sum x_i/n$ *has the* χ^2 *distribution with* $n-1$ *degrees of freedom, its expectation being* $(n-1)$.

To estimate the population variance without bias, we must divide $\sum(x-m)^2$ by $n-1$, and not by n.

5. The Student-Fisher *t*-distribution. The next step is to derive statistics independent of the population variance. The first of these is "Student's" $t = m\sqrt{n}/s = \sqrt{n}(m/\sigma)/(s/\sigma)$, which has the requisite property. We know that m is normally distributed with standard deviation $\sigma/\sqrt{n}$, and that $s^2 = \sum(x-m)^2/(n-1)$ is the best estimate of σ^2 in samples of n. For large samples, a consistent t tends therefore to normality with unit variance; its chief importance, however, lies in its use for small samples.

First, in standard variables, $m\sqrt{n}$ is the distance from the origin to the plane $\sum x_i = mn$. Moreover, $r_{n-1} = s\sqrt{n-1}$ is the distance within the hyperplane from the foot of the perpendicular, as before. Hence, the ratio $m\sqrt{n}/r_{n-1} = t/\sqrt{n-1}$ is the cotangent of the angle made by the radius vector from the origin to a point x of n-space with the normal to a family of parallel hyperplanes. Taking this normal as one axis, the distance from the origin may be integrated out over thin conical shells of angle θ with this direction. In one dimension, two lines can only make the angle 0 or π. In two dimensions, the thin sector has the "volume" element for the "cone" $2rd\theta dr$. In three dimensions, we apply the theorems of Pappus to get the volume element $2\pi r^2 \sin\theta dr d\theta$ over the conical shell, by simple extension to n dimensions, $(n-1)(n-2)cr^{n-1}\sin^{n-2}\theta dr d\theta$. Integrating out the r, the elementary probability is seen to be $c\sin^{n-2}\theta d\theta$, where c is hereafter treated throughout as a generic constant to be determined at the last step by equating the total probability to unity. Putting $u = \cot\theta$, dP transforms to $c(1+u^2)^{-n-2/2}(1+u^2)^{-1}du$. Finally, $u = t/\sqrt{n-1}$, which gives

THEOREM 5. *The probability of* $m\sqrt{n(n-1)}/\sum[x_i - m]^2 \leq t$ *where* $m = \sum x_i/n$ *and the* x_i *are independent normal variables with zero population mean and an identical variance is*

$$P = \frac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{n-1}\,\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n-1}{2}\right)}\int_{-\infty}^{t}\left(1+\frac{t^2}{n-1}\right)^{-n/2}dt. \tag{15}$$

Suppose we have two independent random samples of n_1, n_2 members respectively from the same standard normal population. Since $m_1\sqrt{n_1}$, $m_2\sqrt{n_2}$ are distances, independent and normally distributed, Theorem 1 applies to prove that m_1-m_2 is also normal with variance $1/n_1+1/n_2$, hence $(m_1-m_2)/\sqrt{(1/n_1+1/n_2)}$ is a normally distributed variable in standard measure. From this, we get a cotangent provided we can divide by some r_k. This is best done by taking $r_k^2=r_{n_1-1}^2+r_{n_2-1}^2$, *i.e.*, by combining the distance in the remaining n_1+n_2-2 dimensions, orthogonal to both m_1 and m_2. This gives

THEOREM 6. *For two independent samples from the same normal population*

$$\frac{m_1 - m_2}{\sqrt{\dfrac{(n_1-1)s_1^2+(n_2-1)s_2^2}{n_1+n_2-2}}\sqrt{\dfrac{1}{n_1}+\dfrac{1}{n_2}}} = \frac{m_1-m_2}{\sqrt{(n_1-1)s_1^2+(n_2-1)s_1^2}}\sqrt{\frac{n_1n_2(n_1+n_2-2)}{n_1+n_2}}$$

has the t distribution with n_1+n_2-2 degrees of freedom. If the mean from the first and the variance from the second are used, $t=m_1\sqrt{n_1}/s_2$ with n_2-1 degrees of freedom.

In the first portion, standard measure is unnecessary in view of the fact that the expression for t is independent of both μ and σ. The second part follows immediately upon consideration of the fact that the degrees of freedom were associated only with the estimate of variance; for, $m\sqrt{n}$ represents just one fixed direction, the movement over the "surface" of the cone being associated with $r_{n-1}^2=(n-1)s^2$ which gives the degrees of freedom in t.

6. Distribution of the variance ratio. A most important property of r_n^2 is that it may be broken up into components of the same type. Each component divided by its degrees of freedom gives an estimate of σ^2. The sum of squares is the only analytic non-negative function of the coordinates that may thus be resolved into additive components without change of form. This simple algebraic fact is the basis, for example, of dynamical theorems like those of Bertrand and Kelvin on the energy of a system after a certain number of constraints. In statistics, the use made of this resolution into components is called analysis of variance, and consists of testing the above independent estimates of σ^2 for compatibility. From the preceding sections, it is clear that the ratio of any two such estimates is independent of both the population parameters μ and σ^2.

Assuming therefore a normal standard population without loss of generality, $s_1^2/s_2^2=(n_2-1)r_{n_1-1}^2/(n_1-1)r_{n_2-1}^2$. Now r_{n_1-1}/r_{n_2-1} is again the cotangent (or tangent) of an angle with this difference: that in the t distribution, the numerator was confined to one fixed direction by taking $n_1=1$ whereas it is now allowed to move freely through n_1-1 dimensions independently of the denominator. Ac-

cordingly, the n_2 dimensions of the denominator contribute $c'' \sin^{n_2-2}\theta d\theta$ as before, which must be multiplied by $c' \cos^{n_1-2}\theta d\theta$ for the swing of r_{n_1-1} (at right angles). The total elementary probability is, therefore, $c \sin^{n_2-2}\theta \cos^{n_1-2}\theta d\theta$. With $u=\cot\theta$, we get

THEOREM 7. *In two independent random samples of size n_1, n_2 from the same normal population, the distribution of $F=s_1^2/s_2^2$ is given by*

$$P(F \leqq t) = \frac{\Gamma\left(\dfrac{n_1+n_2-2}{2}\right)(n_1-1)^{(n_1-1)/2}(n_2-1)^{(n_2-1)/2}}{\Gamma\left(\dfrac{n_1-1}{2}\right)\Gamma\left(\dfrac{n_2-1}{2}\right)} \cdot \int_0^t \frac{F^{(n_1-3)/2}dF}{[(n_2-1)+(n_1-1)F]^{(n_1+n_2-2)/2}}. \tag{16}$$

The distribution of $z=\log\sqrt{F}$ has special advantages over that of F in the way of asymptotic formulae, and is therefore commonly used for accurate work. In practice, s_1^2, s_2^2 are so labelled as to give $F \geqq 1$, $z \geqq 0$.

7. Distribution of the coefficient of correlation in samples from an uncorrelated normal universe. The product-moment correlation (briefly the correlation) coefficient is estimated from a random sample of n pairs $(x_1, y_1) \cdots (x_n, y_n)$ by

$$r = \sum (x_i - \bar{x})(y_i - \bar{y})/\sqrt{\sum (x_i - \bar{x})^2 \sum (y_i - \bar{y})^2}, \qquad \bar{x} = \sum x_i/n, \quad \bar{y} = \sum y_i/n. \tag{17}$$

The basic assumption is that the coordinate pairs are sampled from an uncorrelated bivariate normal population, the elementary probability being $2\pi \exp -(x^2-y^2)/2$, using standard measure without loss of generality because r as defined above does not depend upon the population parameters.

In this case, we superpose the y-space upon the x-space by rotation, much as two pictures on transparent film may be superposed upon a screen. In this, the x_i and the y_i axis are to coincide for each i, which does not imply any relationship between the generic points x and y. Then, clearly, $r=\cos\theta$, θ being the angle between the two (independent) directions from the points $\bar{x}$, $\bar{y}$ to the points x, y respectively. The family of hyperplanes $\bar{x}$-const. coincides with the family $\bar{y}$=const. so that we can rotate both spaces simultaneously without destroying the original correspondence, and have only to investigate the distribution of $\cos\theta$ in the superposed $n-1$-space within the hyperplanes.

Here, the x-radius vector traverses the whole space without restriction, hence may be integrated out, the y-vector has only the restriction of making the angle θ with the first, so that its radial distance may be integrated out also over thin conical shells as in the two preceding distributions. This gives the total elementary probability $c \sin^{n-3}\theta d\theta$, the exponent being 2 less than the dimensions

$n-1$ of the free y-space (under the restriction $\bar{y}=\text{const.}$). This gives for the distribution of r,

$$P(r \leqq R) = \frac{\Gamma\left(\frac{n-1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{n-2}{2}\right)} \int_{-1}^{R} (1-r^2)^{(n-4)/2} dr. \tag{18}$$

Our method of derivation shows the intimate relation between t and F, which is that $t=\sqrt{F}$ essentially when $n_1=2$. Moreover, the relation between r and t is also immediately evident, given that $r=\cos\theta$ with $t=\sqrt{(n-1)}\cdot\cot\theta$ for a sample of n. Seeing that for t we have one more dimension at the start than for r when generating the hypercone of angle θ, it follows that $t=r\sqrt{(n-2)}/\sqrt{(1-r^2)}$ has the Student distribution with $n-2$ degrees of freedom.

MORE MODIFIED SERIES

J. W. BRADSHAW, University of Michigan

1. Introduction. In a recent article* the author pointed out that in certain cases the remainder of an infinite series, after summing a certain number of terms, can conveniently be expressed as an infinite continued fraction. We recall, with a slight change of notation, the two formulas there obtained as illustrations:

$$\sum_{n=1}^{\infty} (-1)^{n-1}\frac{1}{n} = \sum_{n=1}^{k} (-1)^{n-1}\frac{1}{n} + (-1)^k \Big/ \left\{2k+1+\mathop{K}_{s=1}^{\infty}\frac{s^2}{2k+1}\right\},$$

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \sum_{n=1}^{k}\frac{1}{n^2} + 2\Big/\left\{2k+1+\mathop{K}_{s=1}^{\infty}\frac{s^4}{(2s+1)(2k+1)}\right\}.$$

As further illustrations of results obtained by the method, we cite similar formulas for the series $\sum_{n=1}^{\infty}(-1)^{n-1}n^{-2}$ and $\sum_{n=1}^{\infty}n^{-3}$ and a formula, obtained from the former, for $\sum_{n=1}^{\infty}n^{-2}$. Since the derivation and proof of the formulas of the earlier article are there given in some detail, and since the proofs of these new formulas can be carried through on similar lines, the formulas are stated without proof.

Attention may be called to a marked contrast between the continued fractions of the earlier formulas and those here involved, in that while the denominators in the former are linear in k and the numerators are positive, here the denominators are quadratic in k and the numerators are negative. This has an important bearing on the estimate of error involved in using a convergent; in the former case consecutive convergents are on opposite sides of the limit, while now all convergents are on the same side.

* J. W. Bradshaw, Modified series, this MONTHLY, vol. 46, 1939, pp. 486–492.

In this paper we call attention to the rapidity of convergence of the continued fractions involved to show that the method furnishes a really powerful tool for numerical calculation, powerful enough to give a feasible means of checking Stieltjes' calculation of $\sum n^{-2}$ and $\sum n^{-3}$, which presumably was based on the Euler-Maclaurin Sum Formula.

2. The three formulas. The formulas which give the remainder in the case of the three series mentioned are

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^2}=\sum_{n=1}^{k}(-1)^{n-1}\frac{1}{n^2}+\frac{(-1)^k}{2}\bigg/\left\{k^2+k+1+\mathop{K}_{s=1}^{\infty}\frac{-s^4}{s^2+(s+1)^2+k^2+k}\right\},\tag{1}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^3}=\sum_{n=1}^{k}\frac{1}{n^3}+1\bigg/\left\{2k^2+2k+1+\mathop{K}_{s=1}^{\infty}\frac{-s^6}{s^3+(s+1)^3+(2s+1)(2k^2+2k)}\right\},\tag{2}$$

and then, inasmuch as $\sum_{n=1}^{\infty}n^{-2}=2\sum_{n=1}^{\infty}(-1)^{n-1}n^{-2}$ also

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=2\sum_{n=1}^{k}(-1)^{n-1}\frac{1}{n^2}+(-1)^k\bigg/\left\{k^2+k+1+\mathop{K}_{s=1}^{\infty}\frac{-s^4}{s^2+(s+1)^2+k^2+k}\right\}.\tag{3}$$

In these formulas k is an arbitrary positive integer, the number of terms of the series summed directly. It may be of interest to remark that, if by $\sum_{n=1}^{0}$ we understand that no terms of the series are taken, the formulas reduce for $k=0$ to Euler's well known theorem of the equivalence of a series and a continued fraction.

3. Check of two of Stieltjes' values. Stieltjes in his table* of values of $S_k=\sum_{n=1}^{\infty}n^{-k}$ gives

$$S_2=1.64493\ 40668\ 48226\ 43647\ 24151\ 66646\ 03,$$
$$S_3=1.20205\ 69031\ 59594\ 28539\ 97381\ 61511\ 46,$$

to 32 decimal places, though he claims accuracy to only 30 places. To check the first of these values we take $k=40$ in formula (3) and compute $2\sum_{n=1}^{40}(-1)^{n-1}n^{-2}$ to 35 decimal places

$$1.64432\ 46821\ 00847\ 49998\ 87572\ 60979\ 57757.$$

* T. J. Stieltjes, Table des valeurs des sommes $S_k=\sum_{n=1}^{\infty}n^{-k}$, Acta Mathematica, vol. 10, 1887, pp. 299–302. See also J. W. L. Glaisher, Tables of $1\pm 2^{-n}\pm 3^{-n}\pm 4^{-n}\pm$ *etc.* and $1+3^{-n}+5^{-n}+7^{-n}+$ *etc.* to 32 places of decimals, Quarterly Journal of Mathematics, vol. 45, 1914, pp. 141–158.

To this we add the reciprocal B_s/A_s of a convergent of the continued fraction

$$1641 - \frac{1\,|}{|\,1645} - \frac{16\,|}{|\,1653} - \frac{81\,|}{|\,1665} - \frac{256\,|}{|\,1681} - \frac{625\,|}{|\,1701}$$
$$- \frac{1296\,|}{|\,1725} - \frac{2401\,|}{|\,1753} \cdots .$$

Now it can be shown that the error committed by using for the infinite continued fraction a particular convergent A_s/B_s is less than $2(s+2)$ times the difference between the reciprocal of this convergent and the reciprocal of the next. Computing B_6/A_6 and B_7/A_7 to 35 places, we find that the difference is less than 3 in the 34th place, and hence B_6/A_6 yields a result correct to 32 places. The value obtained by this method agrees even in the 32d place with that given by Stieltjes. Though A_7 and B_7 consist of 26 and 23 figures, respectively, the value of B_7/A_7 to 35 decimal places can be obtained by means of a calculating machine without excessive labor.

For a similar calculation with formula (2) a smaller value of k will answer the purpose. We may for S_3 take $k=20$ and again use B_6/A_6; we obtain a result which is smaller than Stieltjes' value by one unit in the 32d place. Formula (2) furnishes an easy verification of Kummer's remark* that direct summation would require 10,000,000 terms of $\sum_{n=1}^{\infty} n^{-3}$ to give a result correct to 14 places. We have only to put the error, 5 in the 15th place, equal to B_0/A_0 to determine the required value of k.

4. Generalizations. As in the illustrations given in the earlier article, generalizations of these formulas are readily derived:

$$\begin{aligned}\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(nx+y)^2} &= \sum_{n=1}^{k} (-1)^{n-1} \frac{1}{(nx+y)^2} + \frac{(-1)^k}{2} \Big/ \Big\{ (kx+y)^2 \\ &\quad + x(kx+y) + x^2 + \mathop{K}_{s=1}^{\infty} \frac{-x^4 s^4}{(kx+y)^2 + x(kx+y) + x^2(2s^2+2s+1)} \Big\},\end{aligned} \tag{4}$$

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{(nx+y)^3} &= \sum_{n=1}^{k} \frac{1}{(nx+y)^3} + \frac{1}{x} \Big/ \Big\{ 2(kx+y)^2 + 2x(kx+y) + x^2 \\ &\quad + \mathop{K}_{s=1}^{\infty} \frac{-x^4 x^6}{(2s+1)[2(kx+y)^2 + 2x(kx+y) + x^2(s^2+s+1)]} \Big\},\end{aligned} \tag{5}$$

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{(nx+y)^2} &= 2\sum_{n=1}^{k} (-1)^{n-1} \frac{1}{(nx+y)^2} + (-1)^k \Big/ \Big\{ (kx+y)^2 \\ &\quad + x(kx+y) + x^2 + \mathop{K}_{s=1}^{\infty} \frac{-x^4 s^4}{(kx+y)^2 + x(kx+y) + x^2(2s^2+2s+1)} \Big\}.\end{aligned} \tag{6}$$

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* E. E. Kummer, Eine neue Methode die numerischen Summen langsam convergierender Reihen zu berechnen, Journal für Mathematik, vol. 16, 1837, pp. 206–214.

DISCUSSIONS AND NOTES

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The Department of Discussions and Notes is open to all forms of activity in college mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

ON THE TEACHING OF DIFFERENTIAL EQUATIONS

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The purpose of this note is the exposition of a well-known fact of physics which might be used with benefit in teaching differential equations, especially, in helping students to visualize them. In solving the equations we shall point out minor shortcomings, from the logical point of view, of the traditional exposition in textbooks, and indicate how they should be corrected.

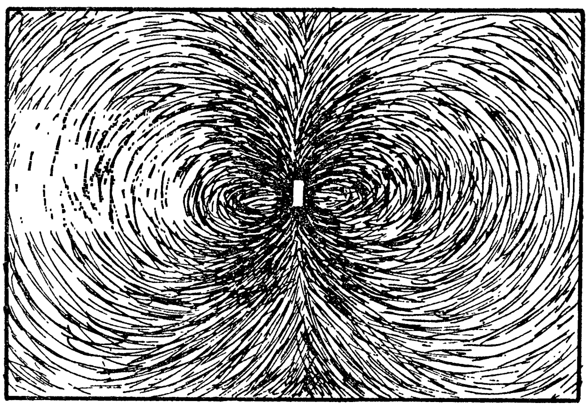

FIG. 1

The simplest introduction to the theory of differential equations is offered by nature. Cover a thin bar magnet with a horizontal piece of cardboard, and sprinkle fine iron splinters over the latter. Each splinter after coming to rest will assume a definite direction, depending upon its location. Before us is displayed the direction field associated with a first order differential equation. The splinters fit together along lines, called the lines of force of the magnet in the plane of the cardboard. These lines represent the solutions of the differential equation.

If the magnet covered by the horizontal cardboard is in a vertical position touching the cardboard at one pole, then each splinter will point toward the pole. If we choose this pole as the origin of a cartesian coördinate system, the splinter at the point (x, y) thus has the slope y/x. Hence the differential equation associated with the magnetic direction field reads $y'(x)=y(x)/x$ for $x\neq 0$. The splinters fit together along the straight lines through the origin. With the excep-

tion of the Y-axis,* these lines $y=Cx$ represent solutions $y(x)=Cx$ of the differential equation. The simplest correct way of showing that the equation has no other solution† seems to be by treating it as a Clairaut equation,‡ that is to say, by concluding that for each solution $y(x)$ of the equation and for each $x\neq 0$ we have

$$y''(x) = \left(\frac{y(x)}{x}\right)' = \frac{xy'(x) - y(x)}{x^2} = \frac{x\,\dfrac{y(x)}{x} - y(x)}{x^2} = 0$$

and consequently $y(x)=Cx+D$ where D obviously must equal zero.

If the magnet, NS, is horizontal (see Fig. 1), then in the plane of the cardboard we introduce 1) a cartesian coördinate system whose origin is at the center of the magnet and whose Y-axis passes through the poles N and S, and 2) a polar coördinate system whose axis coincides with OX. A splinter at the point $P=(x, y)=(r, \theta)$ is attracted by one pole, say, N and repulsed by the other (see Fig. 2). The angle ϕ between the resulting direction of the splinter and the radius vector OP satisfies the approximate equality§

$$\tan\phi = \tfrac{1}{2}\cot\theta \tag{1}$$

provided that the length $NS=2l$ of the magnet is small compared with $OP=r$. Since $\tan\theta=y/x$, we have $\tan\phi=x/2y$ and for the angle $\theta-\phi$ between the splinter and the positive X-axis

$$\tan(\theta-\phi) = \frac{\tan\theta - \tan\phi}{1+\tan\theta\tan\phi} = \frac{2}{3}\,\frac{y}{x} - \frac{1}{3}\,\frac{x}{y} \quad \text{for} \quad x\neq 0 \quad \text{and} \quad y\neq 0.$$

The differential equation associated with this magnetic direction field is thus‖

$$y'(x) = \frac{2}{3}\,\frac{y(x)}{x} - \frac{1}{3}\,\frac{x}{y(x)}\,. \tag{2}$$

* The Y-axis cannot be represented by any function $y(x)$. By choosing another cartesian coördinate system with the origin O we see that the former Y-axis represents a line of force.

† More precisely: no other solution *which is differentiable even for* $x=0$, though the differential equation is only defined for $x\neq 0$. Indeed, $y(x)=|x|$ is an example of a function which is not of the form Cx and yet is a solution of the differential equation where the latter is defined. The general solution in the latter sense is $y(x)=Cx+D|x|$.

‡ See the author's article, Differential Equations, in vol. 2 of Practical Mathematics, New York City, 1943.

The reasoning that $y'(x)/y(x)=1/x$ by integration implies $\log|y(x)|=\log|x|+c$ and hence $y(x)=Cx$, for some $C\neq 0$, is valid only for values of x for which $y(x)\neq 0$, since $y(x)$ appears in a denominator. All we can conclude in this way is that the solutions which for every $x\neq 0$ assume values $\neq 0$, are of the form Cx, $(C\neq 0)$. This reasoning must be supplemented by the slightly deeper remark that a solution which for one $x_0\neq 0$ assumes the value 0, is necessarily equal zero for each x. This function happens to be of the form Cx for $C=0$.

§ See, *e.g.*, S. G. Starling, Electricity and Magnetism, 1927, p. 17 sq.

‖ See, *e.g.*, Watson and Burbury, The Mathematical Theory of Electricity and Magnetism, vol. 1, 1885, p. 267.

Introducing the auxiliary function* $z(x)=y(x)/x$ we obtain

$$xz'(x)+z(x)=\frac{2}{3}z(x)-\frac{1}{3}\frac{1}{z(x)},$$

thus

$$z'(x)=-\frac{1}{3x}\left(z(x)+\frac{1}{z(x)}\right)\quad\text{and}\quad\frac{z(x)}{z^2(x)+1}\cdot z'(x)=-\frac{1}{3x}\cdot$$

By integration of these two equal functions of x, one of which is free of $z(x)$ while the other one is the product of $z'(x)$ and a function of $z(x)$, we find†

$$\tfrac{1}{2}\log\,(z^2(x)+1)=\log\,|x^{-1/3}|+\text{const.}\quad\text{and}\quad z^2(x)+1=Cx^{-2/3}$$

for some constant C which obviously must be positive. For the functions $y(x)$ in which we are interested we thus obtain

$$y^2(x)+x^2=Cx^{4/3}\quad\text{and}\quad(y^2(x)+x^2)^3=Cx^4\text{ for some }C>0,$$

hence the following explicit expressions

$$y(x)=+\sqrt{Cx^{4/3}-x^2}\quad\text{and}\quad y(x)=-\sqrt{Cx^{4/3}-x^2}$$

for some $C>0$ and $0\neq|x|\leqq C^{3/2}$.

The equations of these lines in our polar coördinate system are $r=C\cos^2\theta$. The equipotential lines of the magnet, *i.e.*, the orthogonal trajectories of the lines of force, satisfy the differential equation

$$y'(x)=-1\Big/\left(\frac{2}{3}\frac{y(x)}{x}-\frac{1}{3}\frac{x}{y(x)}\right)$$

which is homogeneous, too. The polar equations of the solutions are $r=C\sqrt{\sin\theta}$.

For the sake of completeness a short proof of the approximate equality (1) may be added (see Fig. 3). The only assumption is the general attraction law. According to it, the vectors PQ and PQ' representing the two forces acting on the point P are proportional to the reciprocal squares of the distances from P to N and S, respectively, that is to say,

$$PQ=\frac{k}{r^2+l^2-2rl\sin\theta}\quad\text{and}\quad PQ'=QR=\frac{k}{r^2+l^2+2rl\sin\theta}\cdot$$

If we set

* It seems to be about time to replace the terms "new variable" and "separation of variables" by expressions more adequate from the point of view of logic. (*Cf.* the author's article, Differential equations, mentioned in 3.)

† In order to avoid the inaccuracy criticized before, we may supplement the above reasoning by a remark taking into account that $z'(x)$ is undefined for values of x for which $z(x)=0$. From Rolle's theorem for functions which are differentiable in the interior of an interval and only continuous at its endpoints, we infer: No solution $z(x)$ can assume the value 0 at two different positive or two different negative values of x, since $z'(x)\neq 0$ for $x\neq 0$. It follows that also each solution $y(x)$ of (2) has at most one positive and one negative zero. The explicit expression of these solutions shows that each of them has exactly one positive and one negative zero, *viz.*, $\pm\sqrt{C^3}$.

$$\alpha = \measuredangle RPQ, \qquad \beta = \measuredangle QRP, \qquad \omega = \measuredangle PQR = \measuredangle NPS,$$

the tangent law yields

$$\tan\frac{\beta-\alpha}{2} : \tan\frac{\beta+\alpha}{2} \doteq (PQ - RQ):(PQ + RQ),$$

hence

$$\tan\frac{\beta-\alpha}{2} = \cot\frac{\omega}{2}\cdot\frac{PQ-RQ}{PQ+RQ}\,.$$

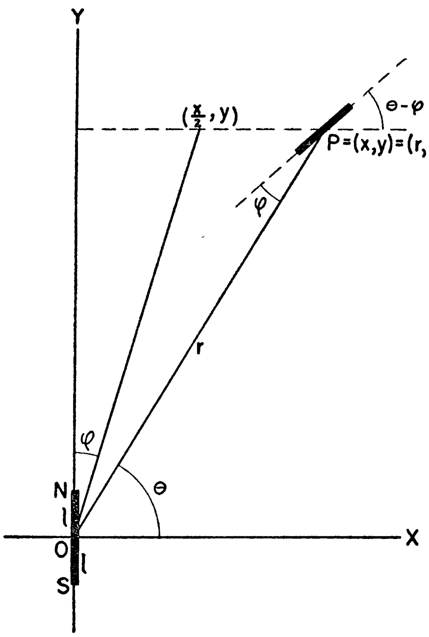

FIG. 2

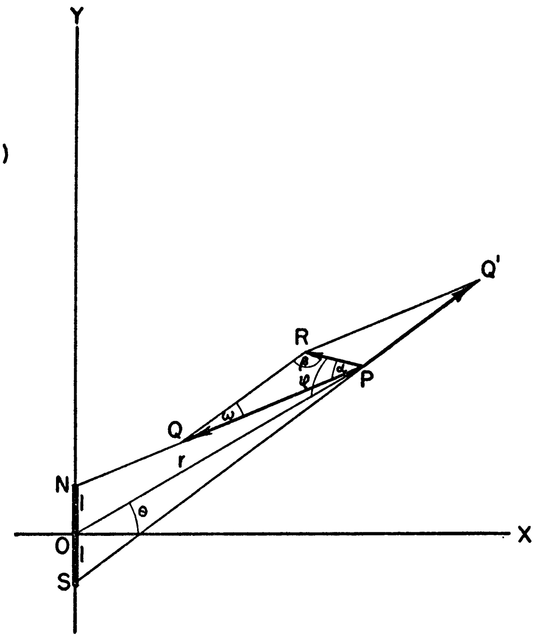

FIG. 3

If l is small compared with r, then ω is small compared with $90°$ and

$$\cot\frac{\omega}{2} \sim \csc\frac{\omega}{2} \sim \frac{r}{l\cos\theta} \quad \text{and} \quad \frac{PQ-RQ}{PQ+RQ} \sim \frac{4kl\sin\theta/r^3}{2k/r^2} = \frac{2l\sin\theta}{r}\,.$$

Hence

$$\tan\frac{\beta-\alpha}{2} \sim 2\tan\theta.$$

Moreover $\phi \sim \alpha \sim 90° - (\beta-\alpha)/2$. Thus

$$\tan\phi \sim \cot\frac{\beta-\alpha}{2} \sim \frac{1}{2}\cot\theta.$$

MORE GEOGRAPHICAL QUESTIONS

L. R. FORD, Illinois Institute of Technology

The geographical question propounded by Professor E. J. Moulton in the April MONTHLY suggests others.* If a man walks ten miles due south, then ten miles due east, then ten miles due north, then ten miles due west, he will ordinarily not be at the starting point. If his point of departure P is on the campus of Illinois Institute of Technology, he will find himself 114 feet west of P and with wet feet.

What is the locus of P in order that the man shall terminate his journey exactly at the starting point?

One obvious possible position of P is anywhere on the parallel of latitude 5 miles north of the equator. But there are many other possibilities. Let θ be the latitude of P counted positive to the north and negative to the south, and measured in radians; and let s (in this case 10 miles) be the length of each leg of his journey. Let R be the radius of the earth in miles.

The second part of his journey lies along a parallel of latitude θ', where

$$\theta' = \theta - s/R.$$

The radii of the two parallels involved are

$$r = R \cos \theta, \qquad r' = R \cos \theta'.$$

The angles subtended at the centers of these circles by an arc of length s are s/r and s/r'. The end of the journey will fall on P if these subtended angles are equal or if they differ by a multiple of 2π. We thus have

$$\frac{s}{R \cos \theta} - \frac{s}{R \cos \theta'} = 2n\pi,$$

or

$$\sec \theta - \sec \theta' = 2n\pi R/s,$$

where $n = 0, \pm 1, \pm 2, \cdots$. We shall show that this equation has exactly one solution for each value of n.

We shall show that

$$f(\theta) = \sec \theta - \sec \theta' = a$$

has exactly one solution in the interval to which θ is restricted,

$$-\frac{\pi}{2} + \frac{s}{R} < \theta < \frac{\pi}{2}.$$

As θ traces this interval from left to right $f(\theta)$ varies from $-\infty$ to $+\infty$. It thus

* Professor F. A. Foraker has called my attention to his interesting list of geographical questions in Education, vol. 38, 1917, pp. 157–159. Professor Ford's first question is essentially problem 13 of this list. M. J. W.

takes on any real value a. If a were taken on more than once, we should have

$$f'(\theta) = \sec\theta\tan\theta - \sec\theta'\tan\theta' = 0$$

somewhere in the interval, which we shall show to be impossible. From this

$$\sec^2\theta\tan^2\theta = \sec^2\theta'\tan^2\theta',$$

whence

$$(\tan^2\theta - \tan^2\theta')(1 + \tan^2\theta + \tan^2\theta') = 0.$$

This holds only if $\theta = \theta'$, which is untrue, or $\theta = -\theta'$. The latter is an extraneous root, since it gives $f'(\theta) \neq 0$. There is thus, exactly one solution of the equation.

For $n = 0$, we have the solution $\theta = s/2R = -\theta'$, which gives for P, the parallel of the latitude mentioned at the beginning. For $n = 1, 2, 3, \cdots$ we have a sequence of parallels converging to the north pole, on any one of which P may lie. For $n = -1, -2, -3, \cdots$ we have a sequence of parallels converging to the parallel of latitude s miles from the south pole. The north pole and this latter limiting circle are excluded from the locus.

We consider also the following problem.

Pedro's banana farm is bounded by two east-west fences exactly two miles long and two north-south fences exactly one mile long. José's farm is bounded by east-west fences exactly one mile long and by north-south fences exactly two miles long. Which has the larger farm?

It is clear that each farm is bisected by the equator and that the area of each is somewhat in excess of two square miles. Let us find the area A of a "rectangular" farm extending s miles on each side of the equator and bounded by east-west fences w miles long.

The latitudes of the latter fences are $\pm s/R$ radians. The height of the zone around the earth between these parallels is $2R\sin(s/R)$ and the area of the whole zone is

$$Z = 2\pi R\cdot 2R\sin(s/R).$$

The radius of the parallel of latitude is $r = R\cos(s/R)$ and its circumference is $2\pi R\cos(s/R)$. The required area is to Z as w is to this circumference, whence

$$A = \frac{wZ}{2\pi R\cos(s/R)} = 2Rw\tan\frac{s}{R}\cdot$$

The function $\tan(s/R)$ may be expanded in the usual power series for all s allowable by the problem; and the convergence is rapid for s/R small. We have

$$A = 2Rw\left(\frac{s}{R} + \frac{s^3}{3R^3} + \cdots\right) = 2ws + \frac{2ws^3}{3R^2} + \cdots.$$

For Pedro's farm we have $s = \frac{1}{2}$, $w = 2$, for José's, $s = 1$, $w = 1$. We have then respectively

$$A_p = 2 + \frac{1}{6R^2} + \cdots, \qquad A_j = 2 + \frac{2}{3R^2} + \cdots.$$

We see that José's farm is the larger by $1/(2R^2)$ square miles, which is almost 125 squares inches.

This is the area of the two pages of the MONTHLY that the reader has open before him, exclusive of the top margins. Pedro's farm exceeds two square miles by one-third of this area.

CLUBS AND ALLIED ACTIVITIES

EDITED BY J. S. FRAME

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to J. S. Frame, Michigan State College, East Lansing, Michigan.

ON TRISECTING AN ANGLE

BARBARA STEINBERG, Student at Queen's College

There is a proof, or so I'm told
By scientists respected,
That certain angles stubbornly
Refuse to be trisected
With compass, straight edge, pen and ink.—
And that's the only ticket,
For Euclid says that other tools
Simply are not cricket.
Yet every year some hardy souls
Will publish refutations
Of scientific treatises
And learned dissertations.
They send out methods by the score
And hope that by insistence
They will succeed in lowering
The scientists' resistance.
Some hint that mathematicians
(Ugly insinuation)
Claim no solution's possible
Out of sheer desperation.
You think you've found a method, eh,
Which cannot be refuted?—
I tell you, sir, I know it's right!
Aha, but is it Euclid?

CLUB REPORTS 1943-44

Harvard Mathematical Club

A distinctly increasing spirit of cooperation with neighboring institutions was in evidence in 1943–44. Members of the Mathematics Clubs and Faculties of Greater Boston colleges, including those of Boston University, M.I.T., Regis, and Tufts, were frequent and welcome visitors. And, breaking all traditions, students of Radcliffe College are now admitted to active membership on an equal basis with Harvard. During the year the following talks were presented:

Gödel's theorems, by Dr. Lynn H. Loomis

Symbolic probability, by Dr. Irving Kaplansky

Curves without tangents, by Dr. Ralph Boas, Jr.

Picturing fractions, by Peter Frank

Two hundred and fifty years of hydrodynamic theory, by Professor Garrett Birkhoff

1152 Varieties of algebra, by Professor R. M. Frye (Boston University)

Through the window, by Professor D. V. Widder

Mathematical economics, by Professor W. W. Leontief

Polyhedra, by R. C. Buck (first prize winner)

Philosophy for the scientist, by Professor Philip Frank

Mathematics in South America, by Professor M. H. Stone

The trisection of the angle, by Dr. Irvin Cohen

Tracking the quadratic to its native lair, by Professor Robert Bruce (Boston University)

Equations with too many solutions, by Mrs. Ralph Boas, Jr. (Tufts)

Foundations of mathematics, by P. R. Masani

Bernouillian numbers, by George Williams (second prize winner)

Some applications of Boolean algebra, by Robert Hoskins

Geometry of the triangle and two-dimensional relativity, by Professor G. D. Birkhoff

Recipients of the Robert Fletcher Rogers Prizes for the best student talks were Mr. Buck ($35) and Mr. Williams ($15). Officers for 1943–44 were: President, Robert Hoskins; Vice-Presidents, Daniel Gorenstein and Richard Arens; Secretary, Joseph Zilber; Treasurer, George Mostow; Faculty Advisers, Dr. George Mackey and Dr. Irving Kaplansky.

Rho Theta, Saint Louis University

The first of the monthly meetings of *Rho Theta* for the year 1943–44 was held in the fall. After the election of officers, plans for future meetings were discussed and committees appointed. It was decided to have guest speakers to talk on topics pertinent to mathematics, and seven lectures were arranged. Among these were:

Probability, by Professor Francis Regan

Seismology, by E. J. Walker, Instructor in Geophysics

The method of least squares, by Professor Regan

Simplifying production mathematics, by W. J. Biermann, Graduate Fellow in Chemistry

Curve-fitting, by Professor Regan

The application of the Clausius-Clapeyron equation to an earth problem, by Albert Frank, Instructor in Geology.

Round table discussions were held during the year to decide in which branch of the service the members would be able to use their mathematics more readily, and several problems were presented to the group by the members and solved at subsequent meetings. Two banquets were held, the latter being the final meeting, which was concluded by a discussion on the subject

Mathematics and the war, by Professor A. E. Ross of the Department of Mathematics.

Officers for the fall term were: President, Joseph Heithaus; Vice-President, Edwin Mertz; Secretary-Treasurer, Herbert Gebhardt. Officers for the spring term were: President, John Quinn; Vice-President, Herbert Gebhardt; Secretary-Treasurer, Helen Ann Jackson.

Square Circle Club, Woman's College of the University of North Carolina

Due to the accelerated program of education, the meetings of the *Square Circle Club* were decreased to two for each semester, in addition to the annual initiation meeting for the induction of new members.

Wartime mathematics was the theme for the year, and the program included the reading of letters from mathematics graduates who are now serving in the armed forces of our country or working in wartime industries.

Use of the slide rule was the subject selected by the Juniors, and slide rules were passed out to the club members so that they could practice various calculations.

Navigational stars was the subject of a dialogue given by two Seniors at the initiation meeting in February. The interest thus aroused in astronomy led the Sophomores to schedule for their meeting a lantern slide lecture entitled

The solar system, by Miss Cornelia Strong, the Faculty Adviser. This was followed by a naked-eye study of the constellations from the roof.

Radar and the range finder was the theme for the Freshmen program, which emphasized the importance of radar as a weapon in this war.

The major social event of the year was our formal initiation of the largest group of new members in the history of the club. After several of our meetings informal social hours were held during which mathematical games were enjoyed and refreshments were served. Officers for the year 1943–44 were: President, Janet Hubbard; First Vice-President, Lucy Taylor; Second Vice-President, Kathleen Wicker; Secretary-Treasurer, Sue McGee; Social Chairman, Elaine Atkin; Faculty Adviser, Miss Cornelia Strong. Officers elected for 1944–45 were: President, Katherine Simpson; First Vice-President, Sue McGee; Second Vice-President, Eleanor Holmstine; Secretary-Treasurer, Lois Russell; Social Chairman, Hilda Mattox.

RECENT PUBLICATIONS

EDITED BY VIRGIL SNYDER, Cornell University, and H. P. EVANS, University of Wisconsin

All books for review should be sent directly to the editors of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y., and not to any of the other editors or officers of the Association.

Mathematics for Navigators. By Delwyn Hyatt and B. M. Dodson. New York and London, McGraw-Hill Book Company, Inc., 1944. 7+106 pages. $1.25.

This book is intended primarily for those studying navigation "on their own." It is restricted to the mathematics of the subject with chapters on arithmetic, algebra, logarithms, plane geometry, plane trigonometry, spherical trigonometry, and oblique spherical triangles.

The first chapter deals only with arithmetical operations useful in navigation. Left-to-right addition and subtraction, and interpolation are used as unifying agents from the first and weld a number of isolated topics such as the sexagesimal systems, conversion of arc into time, and the finding of Greenwich hour angles from the *Nautical Almanac* into an interesting whole.

Approximate solution of spherical triangles by construction is outstanding and very useful, while the projection of the celestial triangle onto the plane of the meridian results in a very clear derivation of the formulas of Ageton's method, H. O. 211.

The compression of so many topics into such a little volume leads to occasional errors of omission and to inaccuracies of statement. These are perhaps most noticeable in the chapter on algebra, where a great many processes must be treated. Among them are the use of negative, fractional, and zero exponents after definition has been made for the positive integral case only; treatment of "cross-multiplying" in a proportion as though it would apply to any equation containing fractions; and the solution of linear equations in one unknown as though the method were generally applicable to equations of higher degree as well.

Two errors should perhaps be mentioned. On page 31, the following sentence might well be deleted: "In using base 10, the logarithm of 89, for example, means simply that 10 raised to the 'log of the 89th power' equals 89." And on page 60, in discussing the ambiguous case of the plane triangle with a, b, and A given, the authors write: "In actual practice there are two solutions when two sides and the angle opposite one of them are given," ignoring the fact that if a is not less than b, there is but one solution.

The orderly tabulation of solutions and the use of the haversine law are strong points of the book. No tables are included, it being assumed that the student will have ready access to Part II of *Bowditch's Useful Tables*, to which careful references are given.

R. C. HUFFER

Mathematics for Mariners. By C. E. Dimick and C. C. Hurd. New York, D. Van Nostrand Company, Inc., 1943. 7+182+69 pages. $2.75.

The authors state in the preface: "It is believed that the book contains the basic mathematics necessary for the undertaking of plane navigation, simple gunnery, seamanship, the problem of maneuvering at sea, and elementary marine engineering." They also say: "It is not to be expected that all the book up to and including Chapter X can be completed in a four or five weeks course."

They begin with arithmetic, since they have found that officer candidates and cadets who fail, do so "simply through slowness and inaccuracy in numerical calculations." The authors stress checks in all computations and use an elaborate system of casting out 9's in addition and subtraction as well as multiplication.

Exponents and logarithms are discussed in Chapter II and use of tables in Chapter III. Examples of interpolation, including double interpolation, are given with brief sections of tables such as are to be used for future marine work. While the Bowditch tables are not mentioned, it seems likely that the authors have in mind such tables which are widely used in navigation.

Chapter IV on Equations takes up linear, quadratic, and simultaneous linear equations with examples of problems from various phases of military use. The next chapter entitled "Geometry" gives some familiar definitions and theorems without proofs. Conversion of degrees to radians with appropriate definitions is carefully discussed, as is the use of the *mil,* which the army has asked to be introduced into the mathematical texts.

In three chapters on trigonometry, the authors give the student enough experience to enable him to solve any plane triangle. The trigonometric functions are defined by the right triangle method for angles less than 90°. Then follows a derivation of functions of angles of all sizes. It seems queer that they use an example from plane sailing involving departure and difference of latitude which is less familiar to most teachers than the standard definitions from a circle with rotating radius. The method used for determining the signs of the functions in the four quadrants makes use of what many teachers might consider an artificial method, but of course it gives the same results as the standard method.

Since celestial navigation, which is so important now, is based on spherical trigonometry, it is surprising that spherical trigonometry is omitted entirely from this text. Few people seem to be aware of a simple method of deriving at once the three sets of equations used in solving a spherical triangle. This method is due to Chauvenet and is given in less than four pages in a *Text-book of Field Astronomy* by G. C. Comstock (1908). All needed formulae, including those for right spherical triangles, can be derived for a class in two hours.

The text concluded with two chapters of vector diagrams for the solution of problems involving speed of ships and current or wind drift. The use of the maneuvering board is introduced and ten sample compass roses (one-half standard scale) are included at the end for practice use. This section of the book is

well-done and introduces simply and satisfactorily certain practical problems which are stressed in courses of deck officer training in the Navy and Coast Guard.

It is quite obvious that the mathematics given in this text is necessary to the student of marine problems. Some of it should have been thoroughly taught by the 10th grade. Some high schools have good courses in trigonometry which have been popular during the last two years. This book belongs to the level of such high schools.

A word should be added about the tables. It is gratifying to find five-place tables when the modern tendency is to four-place tables to save expense of printing. We wish the first figures of the mantissas of the logarithms had been printed. Also that the tables of tangents and cotangents had been printed alongside the sines and cosines, instead of making two separate tables. Tables of natural functions apart from their logarithms are acceptable and quite usual.

On the whole the authors have done well what they set out to do and have put into one book the mathematics necessary for mariners. The book will make a valuable handbook for those who have gotten the practical science of navigating and have neglected the theoretical.

C. M. HUFFER

Table of Reciprocals of the Integers from 100,000 through 200,009. 8+201 pages. \$4.00.

Table of Circular and Hyperbolic Tangents and Cotangents for Radian Arguments. 38+410 pages. \$5.00.

Table of the Bessel Functions $J_0(x)$ and $J_1(z)$ for Complex Arguments. 44+403 pages. \$5.00.

All of the above tables were prepared by the Mathematical Tables Project, Work Projects Administration of the Federal Works Agency, conducted under the sponsorship of the National Bureau of Standards, published by the Columbia University Press, New York, 1943.

The tabulated values are given (in order of mention) to 7 decimal places, 8 significant figures, 10 decimal places. The complex argument $\rho(\cos\phi+i\sin\phi)$ of the Bessel functions is tabulated for ρ from 0 to 10 in steps of .01 and for ϕ from 0° to 90° in steps of 5°. The prefaces contain valuable information on available literature in numerical tables, including lists of corrections and numerical errors carried over into quite recent publications. The present tables are the result of independent computations and re-computations. The cover size is 11″ by 8″, the photo-offset reproduction is immaculately distinct and clear.

M. A. SADOWSKY

NEW BOOKS RECEIVED

Aircraft Analytic Geometry Applied to Engineering, Lofting, and Tooling. By J. J. Apalategui and L. J. Adams. New York and London, McGraw-Hill Book Company, Inc., 1944. 17+285 pages. \$3.00.

Air Navigation Made Easy. By James Naidich. New York and London, McGraw-Hill Book Company, Inc., 1944. 9+124 pages. $1.75.

Basic Marine Navigation. By B. J. Bok and Frances Wright. Boston, Houghton Mifflin Company, 1944. 8+422 pages. $4.50. Kit of Practice Materials, $1.70.

Basic Mathematics for Engineers. By P. G. Andres, H. J. Miser, and Haim Reingold. New York, John Wiley and Sons, Inc.; London, Chapman and Hall, Ltd., 1944. 10+726 pages. $4.00.

Celestial Navigation. A Problem Manual. By Walter Hadel. New York and London, McGraw-Hill Book Company, Inc., 1944. 13+261 pages. $2.50.

Exact Values of the First 200 Factorials. By H. S. Uhler. New Haven, Uhler, 1944. 22 pages. $0.80.

Fourier Series. (Cambridge Tracts in Mathematics and Mathematical Physics, No. 38.) By G. H. Hardy and W. W. Rogosinski. New York, The Macmillan Company, 1944. 8+100 pages. $1.75.

General Mathematics in American Colleges. By K. E. Brown. New York, Teachers College, Columbia University, 1943. 4+167 pages. $2.35.

Marine and Air Navigation. By J. Q. Stewart and N. L. Pierce. Boston, Ginn and Company, 1944. 12+472 pages. $4.50.

Mathematical and Physical Principles of Engineering Analysis. By W. C. Johnson. New York and London, McGraw-Hill Book Company, Inc., 1944. 10+346 pages. $3.00.

Mathematics for Aircraft Engine Mechanics. By Harold Griffiths. New York and London, McGraw-Hill Book Company, Inc., 1944. 15+367 pages. $2.50.

Mathematics for Exterior Ballistics. By G. A. Bliss. New York, John Wiley and Sons, Inc.; London, Chapman and Hall, Ltd., 1944. 7+128 pages. $2.00.

Modern Operational Mathematics in Engineering. By R. V. Churchill. New York and London, McGraw-Hill Book Company, Inc., 1944. 10+306 pages. $3.50.

The Philosophy of Bertrand Russell. (The Library of Living Philosphers, vol. 5.) Edited by P. A. Schilpp. Evanston and Chicago, Northwestern University, 1944. 16+815 pages. $4.00.

Plane and Spherical Trigonometry. By H. P. Doole. New York, Thomas Y. Crowell Company, 1944. 8+183 pages. $1.75.

Sistemas de Ecuaciones Analiticas en Terminos Finitos, Diferenciales y en Derivadas Parciales. (Monografias publicadas por la Facultad de Ciencias Mathematicas, Fisico-Quimicas y Naturales, applicadas a la Industria, Universidad Nacional del Litoral, No. 1.) By Beppo Levi. Rosario, Argentina, 1944. 218 pages. $8.00.-m/n.

Spherical Trigonometry. By Aaron Freilich, H. H. Shanholt, and Joseph Seidlin. New York, Silver Burdett Company, 1943. 4+140 pages. $1.28.

Tables of Lagrangian Interpolation Coefficients. Prepared by the Mathematical Tables Project, conducted under the sponsorship of the National Bureau of Standards. New York, Columbia University Press, 1944. 36+392 pages. $5.00.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND H. S. M. COXETER

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to H. S. M. Coxeter, 24 Strathearn Boulevard, Toronto 10, Canada.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 631. *Proposed by J. A. Tierney, Norwich University, Northfield, Vt.*

Express the number 64, using two fours (and the operations of arithmetic).

E 632. *Proposed by V. Thébault, San Sebastián, Spain*

What is the smallest radix which admits a perfect cube of the form *abab*?

E 633. *Proposed by N. A. Court, University of Oklahoma*

Given a point M and four spheres (A), (B), (C), (D) whose centers form a tetrahedron, let MEE' be the transversal from M to the two opposite edges BC, DA, and let spheres (E) and (E') be constructed coaxal with the pairs of spheres (B), (C), and (D), (A). We have analogous spheres (F) and (F'), (G) and (G'). Show that the sphere (M) coaxal with (E), (E') is likewise coaxal with (F), (F') and with (G), (G').

E 634. *Proposed by H. S. Wall, Northwestern University*

Let a matrix be constructed according to the following rules:

$$\begin{aligned} &a_{1,1} = 1, \qquad a_{p,q} = 0 \quad \text{if} \quad p < q, && \\ &a_{p,1} = a_{p-1,2} && (p = 2, 3, \cdots), \\ &a_{p,q} = a_{p-1,q-1} + a_{p-1,q+1} && (p, q = 2, 3, \cdots). \end{aligned}$$

Show that the sum of the products of corresponding elements in any two rows is equal to an element in the first column, *viz.*,

$$\sum_{r=1}^{\infty} a_{p,r}a_{q,r} = a_{p+q-1,1} \qquad (p, q = 1, 2, \cdots).$$

E 635. *Proposed by R. A. Rosenbaum, U.S.N.R.*

Derive parametric equations for the involute of the involute . . . (n times) of a circle (with the same starting point for each process of unwinding).

Correction to E630[1944, 348]. *Proposed by R. A. Rosenbaum, U. S. N. R.*

For $s' = s + \pi d$ read $s' = s + 2\pi d$.

SOLUTIONS

The Reinforced Phalanx

E 596 [1943, 633]. *Proposed by W. C. Rufus, Observatory of the University of Michigan*

A square phalanx of less than five million soldiers was reinforced by ten equal squares and then formed a square of over ten million which could be divided into 400 equal squares. The reinforced army was less than three times the original. How many soldiers were in each of the 400 squares?

Solution by Colin Blyth, Queen's University. Let Y, X, and z be the respective sides of the original square, of a reinforcing square, and of one of the resulting 400 squares. Then

$$10X^2 + Y^2 = 400z^2.$$

We see that Y, and therefore also X, must contain the factor 10. Setting $X = 10x_1$, $Y = 10y_1$, we have $10x_1^2 + y_1^2 = 4z^2$. Thus y_1, and therefore also x_1, must be even. Setting $x_1 = 2x$, $y_1 = 2y$, we have

$$(1) \qquad 10x^2 + y^2 = z^2,$$

where $x = X/20$ and $y = Y/20$. Also, the given restrictions imply

$$35 < x < 50, \qquad 91 < y < 112, \qquad 158 < z < 194.$$

Now, the general solution of (1) (*cf.* H. N. Wright, *Theory of Numbers*, p. 96) is

$$x = 2kmn, \qquad y = k \cdot \left| rm^2 - n^2 \right|, \qquad z = k(rm^2 + n),$$

where k, m, n, r, s are positive integers, $rs = 10$, and

$$(m, n) = (r, s) = (r, n) = (s, m) = 1.$$

By testing the fifteen possible pairs m, n with $r = 10$, $s = 1$, and the eighteen possible pairs with $r = 5$, $s = 2$, we find that the only solution of (1) satisfying the given inequalities is

$$x = 36, \qquad y = 111, \qquad z = 159$$

(corresponding to $r = 5$, $s = 2$, $m = 3$, $n = 2$, $k = 3$). Thus the number of soldiers in each of the 400 squares is $z^2 = 25281$.

Also solved by E. M. Berry, W. E. Buker, H. D. Larsen, F. L. Miksa, E. P. Starke, Richard Woollett, and the proposer.

The Circular Baseball League

E 601 [1944, 46]. *Proposed by C. T. Tobin, St. Francis Xavier College, Antigonish, N. S.*

A baseball league is formed by seven teams belonging to towns located one mile apart around a circular railroad. In order that each team shall play each

other, three games are held every day for a week. No team plays in any town more than once, nor plays more than one game a day. In each town, there is no more than one game a day, and there are never games on two consecutive days. There are no home games. At night each team travels to the scene of the next day's game, by the shortest route along the circular railroad. (Trains run in both directions.) A team that is going to be idle on any day returns (or remains) home the preceding night. When a team has completed its schedule, it returns home that night. The first games are played on Sunday; so the travelling starts on Saturday night and is completed on the following Saturday night. The towns are 1, 2, 3, 4, 5, 6, 7, though not in that order. Teams 3, 5, 6 never travel more than 2 miles any night. Teams 3, 4, 7 all travel the same total number of miles during the schedule. Teams 4 and 6 play at town 5 on Friday, and on that day there is also a game played at town 1.

Solution by F. L. Miksa, Aurora, Illinois. The order of towns is

4, 5, 3, 2, 6, 1, 7.

For the schedule of games, see Fig. 1, which indicates that teams 2 and 5 play in town 4 on Sunday, teams 3 and 7 in town 5 on Monday, and so on. We see that team 1 travels $1+3+2+1+2+3+2=14$ miles altogether; team 2, 17 miles; team 5, 9 miles; team 6, 12 miles; while 3, 4, and 7 travel 13 miles each.

	4	5	3	2	6	1	7
Sun.	25			36		47	
Mon.		37			24		15
Tue.			27	14		56	
Wed.	67	12			35		
Thu.			16	57			34
Fri.		46			17	23	
Sat.	13		45				26

FIG. 1

	1	2	3	4	5	6	7
1	47			56	23		
2		45				37	16
3	26		57		14		
4		36		27		15	
5			12		67		34
6	35	17				24	
7			46	13			25

FIG. 2

Fig. 1 is a kind of modified Eulerian square: a distribution of the 21 pairs of the digits $1, \cdots, 7$ in suitable cells of a 7×7 square so that every row or column contains six of the seven digits. Each column is headed by the digit not contained in it (because there are no home games). Similarly, the digits missing from the respective rows are 1, 6, 3, 4, 2, 5, 7. Fig. 2 shows such a table in its standard form, with rows and columns permuted in such a way that the digit k is absent from the kth row and from the kth column. This suggests the new problem of enumerating such standard squares. To avoid too great multiplicity we may stipulate that the main diagonal (as in this instance) shall contain no empty cells.

A Corollary to E 552

E 604 [1944, 46]. *Proposed by Free Jamison, U. S. Navy Air Navigation School*

Is there any numerical solution of E 552 with $4<b<c<33800$?

Solution by E. P. Starke, Rutgers University. A negative answer to the question is indicated. Using the notation of Jamison's solution for E 552 [1943, 563], we have

$$c = mb, \qquad (a-1)^2 = nb, \qquad m^2b = n + a,$$

where n and b are perfect squares. With $n=s^2$, $b=y^2$, we have to consider $m^2y^2=s^2+sy+1$ or

$$x^2 - (4m^2+1)y^2 = -4, \tag{1}$$

where $x=2s+y$. Evidently x and y are both even or both odd.

In the latter case there is no solution of (1) with $m>1$. For, if $x=p$, $y=q$ is a solution of $x^2-Dy^2=\pm N$, $0<N<\sqrt{D}$, with p prime to q, then p/q is a convergent of the simple continued fraction expansion of $\sqrt{D}$. (See Wright, *Theory of Numbers*, New York, 1939, pp. 40–41.) But there is no convergent of

$$\sqrt{4m^2+1} = 2m + \frac{1}{4m+}\,\frac{1}{4m+}\,\frac{1}{4m+}\cdots \tag{2}$$

with both numerator and denominator odd.

Thus the x and y of (1) must be both even, say $x=2\xi$, $y=2\eta$, and we have to consider

$$\xi^2 - (4m^2+1)\eta^2 = -1. \tag{3}$$

The first three convergents of (2) are $2m$, $(8m^2+1)/4m$, $(32m^3+6m)/(16m^2+1)$. By the usual theory of the Pellian equation, every solution of

$$\xi^2 - (4m+1)\eta^2 = \pm 1$$

comes from a convergent ξ/η of (2). Since each period of the recurring continued fraction (2) contains a single quotient, every convergent provides a distinct solution, convergents of odd order giving the negative sign. Thus the two smallest solutions of (3) are

$$\xi = 2m, \quad \eta = 1; \qquad \xi = 32m^3 + 6m, \quad \eta = 16m^2 + 1.$$

As applied to (1), the first gives $b=4$; the second gives $b=4(16m^2+1)^2$, whence, if $m=c/b\geqq 2$, we have $b\geqq 16900$ and $c\geqq 33800$. Thus every solution lies outside the specified range.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known text-books or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4128. *Proposed by C. E. Springer, University of Oklahoma*

Consider the tangent planes to a sphere at three points A, B, C of a curve lying on the sphere. Let R be the limiting point of intersection of the planes as B and C move independently along the curve and approach coincidence with A. Each curve on the sphere through A has its corresponding R point. Prove that the curves through A, the locus of whose R points is a certain straight line lying in the tangent plane to the sphere at A, are the loxodromes through A.

4129. *Proposed by F. C. Gentry, Louisiana Polytechnic Institute*

Prove that the vertices of the original tetrahedron and those of either of the other tetrahedrons of a desmic system of tetrahedrons are the centers of the eight spheres which touch the faces of the third member of the system.

4130. *Proposed by J. R. Musselman, Western Reserve University*

Show that

$$\sum_{j=1}^{n} {}_nC_j \frac{(-1)^{j-1}}{j} = \sum_{j=1}^{n} \frac{1}{j}.$$

4131. *Proposed by V. Thébault, San Sebastián, Spain*

Let $ACBD$ be a skew quadrangle; planes perpendicular at A to AC, at C to CB, at B to BD, at D to DA form a tetrahedron $A_1C_1B_1D_1$ with the centroid G_1. Show that the powers of A with respect to the sphere (G_1C) on G_1C as diameter, of C with respect to (G_1B), of B with respect to (G_1D), of D with respect to (G_1A) are equal.

SOLUTIONS

Exponential Function as Quotient of Infinite Products

4072 [1943, 124]. *Proposed by Richard Bellman, Brooklyn College*

Show that

$$e^x = \frac{(1-x^2)^{1/2}(1-x^3)^{1/3}(1-x^5)^{1/5}\cdots}{(1-x)(1-x^6)^{1/6}(1-x^{10})^{1/10}\cdots}, \qquad |x| < 1,$$

where the exponents in the numerator are integers with an odd number of unrepeated prime factors; and those in the denominator have an even number of unrepeated prime factors.

Solution by R. C. Buck, Cambridge, Mass. Let us consider the function

$$f(x) = -\sum_{n=1}^{\infty} \frac{\mu(n) \log (1 - x^n)}{n}, \qquad |x| < 1$$

where $\mu(n)$ is the Möbius function. Then,

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \sum_{v=1}^{\infty} \frac{x^{vn}}{v} \\ &= \sum_{n=1}^{\infty} \sum_{v=1}^{\infty} \frac{\mu(n)}{nv} x^{nv}. \end{aligned} \qquad \text{for} \quad |x| < 1$$

The coefficient of x^m in this expansion is

$$\sum_{n|m} \frac{\mu(n)}{m} = \frac{1}{m} \sum_{n|m} \mu(n) = 0 \quad \text{unless} \quad m = 1.$$

Hence, $f(x) = x$, and

$$e^x = \sum_{n=1}^{\infty} (1 - x^n)^{-\mu(n)/n},$$

which gives the desired result.

Equilateral Triangles on Sides of a Triangle

4076 [1943, 204]. *Proposed by Harold E. Gove, Major, Washington, D. C.*

Given the triangle $A_1A_2A_3$ show how to construct the triangle $B_1B_2B_3$ so that the triangles $B_jB_iA_k$ will be equilateral and exterior to $B_1B_2B_3$.

Solution by Howard D. Grossman, New York City. We may view the plane on that side on which B_1, B_2, B_3 appear in clockwise order. Then in order that $B_jB_iA_k$ lie outside $B_1B_2B_3$, it is necessary that B_j, B_i, A_k be also in clockwise order, i, j, and k being any permutation of the numbers 1, 2, 3 in cyclical order. This is equivalent in the complex plane to $B_j + \omega^2 B_i + \omega A_k = 0$, which uniquely determine $B_i = \frac{1}{2}(A_i - \omega A_j - \omega^2 A_k)$, where $\omega = \cos 120° + i \sin 120°$. Geometrically this is equivalent to viewing the plane on that side on which A_1, A_2, A_3 appear in clockwise order, finding C_k on the same side of A_iA_j as A_k so that $A_iA_jC_k$ is equilateral and A_i, A_j, C_k are in clockwise order, and taking B_k as the midpoint of C_iC_j. This equivalence follows mechanically from $C_i + \omega^2 A_j + \omega A_k = 0$.

This however is not always sufficient, *e.g.*, an isosceles A-triangle of 120°, 30°, 30° or of $\theta (= \tan^{-1} 3\sqrt{3}$, about 79°), θ, $180° - 2\theta$ gives a null B-triangle and an A-triangle of $120+°$, $30-°$, $30-°$ or of $\theta+$, $\theta+$, $(180° - 2\theta)-$ gives a B-triangle overlapping one or more $B_jB_iA_k$-triangles. These cases have no solution.

Solved also by J. Rosenbaum.

Editorial Note. Rosenbaum gave the same method of locating the vertices B_i using geometrical considerations. He also gave remarks on the more difficult similar problem where the triangles $B_jB_iA_k$ are directly similar but not necessarily equilateral.

If $t_1, t_2, t_3, u_1, u_2, u_3$ are complex quantities for $A_1, A_2, A_3, B_1, B_2, B_3$, with the origin at any chosen point, $\lambda = e^{i\pi/3}$ then

$$u_1 = t_1 + \lambda t_2 + \lambda^{-1} t_3, \tag{1}$$

and u_2, u_3 are given by cyclic changes of subscripts. Since $\lambda + \lambda^{-1} = 1$ it is clear that the two triangles have the same centroid. If we set $t_3 = 0$, the equation (1) means that A_3A_2 is rotated through $+60°$ giving D_1 and the midpoint of A_1D_1 is B_1, similarly for D_2, B_2 and D_3, B_3, where D_1, D_2, D_3 are vertices of equilateral triangles constructed exteriorly on the sides of $A_1A_2A_3$. The line segments A_iD_i have equal lengths; for, if $A_1D_3A_3$ is rotated through $+60°$ about A_1, then A_3 falls on D_2 and D_3 on A_2. Also the points A_1, D_3, A_2, P are concyclic where P is the intersection of A_2D_2 and A_3D_3. Suppose that P lies inside $A_1A_2A_3$ then it easily follows that the three lines A_iD_i meet in a common point P such that the sides of $A_1A_2A_3$ subtend the angle of $120°$ at P which is called an isogonic center of the triangle. If P does not fall inside, the angles subtended at P are $60°$ or $120°$, see Johnson's *Modern Geometry*, p. 218. Denote by S_a and S_b the areas of $A_1A_2A_3$ and $B_1B_2B_3$ and let the origin of rectangular coordinates be A_1 with the x-axis along $A_1A_2 = a_3 > 0$, the order of rotation of A_1, A_2, A_3 being taken as positive. The coordinates $a_3, 0; X_3, Y_3 > 0$ of A_2, A_3 when used with the equation (1) and the expression for the area of $B_1B_2B_3$ give

$$(2\sqrt{3}\, X_3 + 10Y_3)a_3 - 2\sqrt{3}\,(X_3^2 + Y_3^2 + a_3^2) = 32S_b, \quad \text{or} \tag{2}$$

$$S_b = \frac{5}{8} S_a - \frac{\sqrt{3}}{32} \sum a_i^2. \tag{3}$$

Thus for given points A_1 and A_2 the locus of A_3 for which the B_1, B_2, B_3 are collinear is a circle which is easily constructed, and the line of collinearity goes through the centroid of $A_1A_2A_3$. The equation (2) says that the power of A_3 with respect to the circle, for which $S_b = 0$, is equal to $-16S_b/\sqrt{3}$. If A_3 is outside this circle then $S_b < 0$ and the order of rotation of B_1, B_2, B_3 is opposite to that of A_1, A_2, A_3 which has been taken as positive. If A_3 is inside the circle then $S_b > 0$ and the vertices of the two triangles have the same order of rotation. In this case there is a maximum S_b when A_3 is at the center $(a_3/2, 5a_3/2\sqrt{3})$, this maximum area is $\sqrt{3}a_3^2/12$ and the A triangle is isosceles.

We also find in a similar manner that

$$(A_1D_1)^2 = 2\sqrt{3}\, S_a + \sum a_i^2/2, \qquad \tfrac{3}{4}(A_1D_1)^2 = \sum a_i^2 - \sum b_i^2.$$

It is easily seen how to modify these results for the case where the equilateral triangles are drawn interiorly on the sides of $B_1B_2B_3$, and this gives the second isogonic center of the A triangle.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to B. W. Jones, White Hall, Cornell University, Ithaca, New York.

Michigan State College held an eight day school on Quality Control by Statistical Methods from June 9 to June 17, 1944, under the jurisdiction of the E.S.M.W.T. and the War Production Board.

Professor G. D. Birkhoff was awarded the honorary degree of Doctor of Science by Illinois Institute of Technology on June 18, 1944.

The following changes at Michigan State College are announced:

Assistant Professor J. M. Barbour has been promoted to an associate professorship.

Associate Professor W. D. Baten has been promoted to a professorship.

Dr. B. H. Bissinger has been granted a leave of absence to do mathematical work for the Army Air Forces.

Assistant Professor Paul Dressel has been appointed Chairman of the Board of Examiners and Director of Counselling.

Dr. W. L. Mitchell has been promoted to an assistant professorship and has been granted leave of absence to accept a position in the Aeronautical Research Department of the Bell Aircraft Corporation in Niagara Falls, N. Y.

Assistant Professor E. A. Nordhaus has been appointed major engineer in the Aerodynamics Unit of the Boeing Aircraft Corporation in Seattle, Washington.

Associate Professor J. E. Powell has been promoted to a professorship.

Assistant Professor C. P. Wells has been granted a leave of absence to do mathematical work for the Army Air Forces.

Assistant Professor E. T. Welmers has been granted a leave of absence to serve in the Flight Research Division of the Bell Aircraft Corporation in Niagara Falls, N. Y.

Assistant Professor J. W. Zimmer has been granted leave of absence to serve as senior development engineer with the Goodyear Aircraft Corporation in Akron, Ohio.

Assistant Professor G. E. Albert of Ohio State University has been granted a leave of absence to serve as research mathematician with the Lukas-Harold Corporation at Indianapolis, Ind.

Dr. P. H. Anderson of the War Production Board in Cleveland is now associated with the Bureau of Foreign and Domestic Commerce in Washington, D. C.

Associate Professor L. M. Blumenthal of the University of Missouri has been promoted to a professorship.

B. K. Brown of Colorado School of Mines has been granted a leave of absence to serve as a process instructor for the Tennessee Eastman Corporation.

Professor R. V. Churchill of the University of Michigan has been granted a leave of absence to serve as research specialist in the Army Air Force Instructors' School at Laredo, Texas.

Dr. Nancy Cole of Sweet Briar College has been appointed to an assistant professorship at Connecticut College.

Professor W. L. Duren, Jr. of Tulane University has been granted a leave of absence to serve as operations analyst for the United States Army.

Dr. J. E. Eaton of Queens College has been granted a leave of absence for work at the Radiation Laboratory at Massachusetts Institute of Technology.

Associate Professor W. W. Flexner of Cornell University has been granted a leave of absence to serve in the London office of the United Nations Relief and Rehabilitation Administration.

Professor C. W. Foard of Youngstown College has been appointed physicist at the Eastman Kodak Company in Rochester, New York.

E. L. Godfrey of Fenn College in Cleveland has been appointed to an assistant professorship at Northwest Missouri State Teachers College in Maryville, Missouri.

Dr. Theodore Hailperin of Cornell University has been appointed mathematician at Aberdeen Proving Ground.

Assistant Professor P. C. Hammer of Oregon State College has been granted leave of absence to serve with the Lockheed Aircraft Corporation of Burbank, California.

Dr. R. G. Helsel of Ohio State University has been granted leave of absence for war research at the Applied Physics Laboratory of the Johns Hopkins University.

Assistant Professor A. D. Hestenes of Carnegie Institute of Technology has been granted leave of absence to serve as research mathematician at the U. S. Naval Air Station in Patuxent River, Maryland.

Associate Professor M. R. Hestenes of the University of Chicago has been granted leave of absence to engage in war research.

Professor A. J. Hoare of the University of Wichita has retired.

Dr. Wilfred Kaplan of the University of Michigan has been granted leave of absence to serve as a research associate at Brown University, and has been promoted to an assistant professorship.

Dr. Irving Kaplansky of Harvard University has become a member of the applied mathematics group at Columbia University.

Dr. Mark Lotkin of Carleton College has been appointed to an assistant professorship at Wabash College, Crawfordsville, Indiana.

Assistant Professor Ingo Maddaus, Jr. of the University of Oregon is now serving in the Radiation Laboratory at Massachusetts Institute of Technology.

A. L. McCarty of San Francisco Junior College has retired.

Professor W. G. McGavock of Davidson College in North Carolina has been granted leave of absence to serve as visiting assistant professor at Duke University.

Assistant Professor Harriet F. Montague of the University of Buffalo has been promoted to an associate professorship.

Professor Richard Morris of Rutgers University has retired with the title of professor emeritus. Associate Professor E. P. Starke succeeds him as chairman of the mathematics department and has been promoted to a professorship.

Assistant Professor E. R. Ott of the University of Buffalo has been appointed executive engineer in the National Union Radio Corporation of Newark, New Jersey.

H. A. Palmer of the College of Idaho has been appointed Principal of the High School in Lund, Nevada.

Dr. A. M. Peiser of Cornell University has been appointed mathematician at Langley Field.

Professor W. G. Pollard of the University of Tennessee has been granted a leave of absence to serve at Columbia University with the N.D.R.C.

Assistant Professor H. A. Poppen of Illinois State Normal University has been appointed statistician with the firm of Stevenson, Jordan and Harrison of Chicago.

Dr. Louise Johnson Rosenbaum of Reed College has been promoted to an assistant professorship.

Associate Professor F. H. Steen of Allegheny College has been appointed Francis Asbury Arter Professor of Mathematics at that institution.

Assistant Professor N. E. Steenrod of the University of Michigan has been granted a leave of absence to work in the Office of Field Service in Washington, D. C.

Professor J. L. Synge of Ohio State University has been granted leave of

absence to serve as ballistics mathematician with the U. S. forces in the European theatre.

Professor T. Y. Thomas of the University of California at Los Angeles has been appointed professor of mathematics and chairman of the department at Indiana University. As chairman he succeeds Professor K. P. Williams.

Assistant Professor R. M. Thrall of the University of Michigan has been granted leave of absence to serve with the applied mathematics group at Columbia University.

Professor Agnes E. Wells of Indiana University has retired.

Dr. F. J. Weyl of Indiana University has been granted leave of absence to serve as mathematician in the Bureau of Ordnance of the Navy Department.

The following appointments to instructorships are announced:
Berea College, Kentucky: Mrs. Elizabeth C. Lukacs
College of the Holy Cross: Dr. V. O. McBrien
Emmanuel Missionary College (Berrien Springs, Michigan): C. L. Woods
Michigan State College: Elaine VanAken, Marian Michmerhuizen
The University of Notre Dame: Dr. Murray Mannos
The University of Pennsylvania: Dr. W. H. Gottschalk
The University of Rochester: W. N. Huff
The University of Virginia: W. R. Utz, Jr. (part-time)
Vassar College: Mrs. A. W. Goodman

Professor H. C. Hicks of Carnegie Institute of Technology died April 14, 1944.

Professor Emeritus David Eugene Smith of Columbia University died July 29, 1944. He was a charter member of the Mathematical Association.

Professor J. R. Wilton of the University of Adelaide, Australia, died during the week of April 20, 1944.

WAR INFORMATION

EDITED BY C. V. NEWSOM

Send news reports upon the utilization of mathematicians or mathematics in war activities to C. V. Newsom, Oberlin College, Oberlin, Ohio.

UNIVERSITY OF CALIFORNIA ESMWT PROGRAM FOR INDUSTRY

Many favorable comments have been received in regard to the ESMWT program being conducted by the University of California in the Southern California area. The following report concerning the program has been written, upon invitation, by Professor W. M. Whyburn of the University of California at Los Angeles.

The heavy concentration of aircraft manufacture in Southern California and of shipbuilding in the San Francisco Bay and Los Angeles—Long Beach harbor areas has led naturally to extensive demands on the University of California for courses given under the Engineering, Science, Management War Training program of the U. S. Office of Education. More than one hundred thousand workers in vital war plants have benefited by these courses, and actual class enrollments at all times total near the ten thousand mark. For the Southern California area the University maintains major training centers on its Los Angeles campus, and in downtown Los Angeles, Burbank, Inglewood, Long Beach, and San Diego. A large number of additional courses are given in war plants and in such neighboring cities as Downey, Glendale, Huntington Park, Riverside, Santa Barbara, Torrence, Upland, and Wilmington.

Mathematics courses constitute a substantial part of the program offered in this region, and approximately sixty such courses are in progress at all times. Courses in mathematics are organized and supervised by Professors Clifford Bell and W. M. Whyburn. These supervisors maintain close personal contact with engineering and educational personnel of aircraft and other companies, and spend considerable time inside the plants finding training needs as they show up in the various departments of operation. They organize the courses, select the instructors, and are responsible for maintaining high instructional standards in each class. The teachers used in the courses are, for the most part, selected from the engineering personnel of the war industries although members of the Department of Mathematics of the University participate heavily in the instructional work, especially in the more advanced courses. In all cases, an effort is made to secure instructors with actual experience in the teaching of college mathematics. Experience has shown that instructors with college teaching experience are more successful in teaching these specialized war training courses than are men whose sole contact with mathematics has been their study of the subject together with their use of it in their daily work at the war plant. Students in the courses show a decided preference for experienced college teachers.

The course most needed by aircraft workers is one which has been designated as Engineering Mathematics I. This course reviews arithmetic, plane and solid geometry, elementary algebra, and then gives a reasonably complete treatment of plane and spherical trigonometry. Approximately ten thousand local workers have taken the course, and it has done much to give these people the working knowledge of the elementary mathematics that is a daily necessity in their jobs. This course is followed in regular sequence by Engineering Mathematics II, III, and IV; these cover the essential substance of analytical geometry, differential and integral calculus, and applications of these subjects. Several classes in Differential Equations I and II have been taught, and recently there has been a heavy demand for courses in trigonometry. In addition to these somewhat regular courses, there are several specialized courses that have been given for specific groups of employees. There is ample evidence that such courses have done much to speed up the production of better airplanes. A course "Aircraft Analytical

Geometry" has been a pioneering project and through it, lofting and layout work in aircraft and shipbuilding plants have been considerably revamped along more efficient and accurate lines. Many sections of this course have been offered in the major aircraft plants and at the various training centers. Recently, courses in statistical methods and quality control have been in heavy demand, and three or four different courses have been designed to meet these needs. For radio and electrical workers, courses ranging up to and including Heaviside Operational Calculus and Mathematics of Wave Propagation are being offered. Other advanced courses for engineering personnel include Calculus of Variations, Tensor Analysis, Mathematics for Stress Analysis, and Mathematical Aspects of Aerodynamics and Hydrodynamics.

NOTES ON THE NAVY V-12 PROGRAM

1. The following is an excerpt from a letter, dated May 11, 1944, addressed by Captain A. S. Adams to presidents of institutions participating in the Navy V-12 Program. More recent information indicates that the total decrease in the number of students in the V-12 Program upon November 1, 1944, will be about 28%.

"(The Bureau of Naval Personnel) plans to continue in effect during the V-12 term beginning 1 July 1944 the previously established overall unit quotas at all V-12 institutions. · · · Plans for the term beginning 1 November 1944, include a substantial decrease in the total number of students in the V-12 Program. A detailed statement concerning changes contemplated at your Unit will be sent you within the next month, in order that you may have as much advance notice as possible to plan the necessary readjustments in your instructional staff and the facilities now under contract to the Navy."

2. "Effective with the class which entered the Navy V-12 Program on 1 November 1943, all apprentice seamen, V-12(a), will receive a minimum of three terms of college training in the V-12 Program, regardless of the amount of college work completed prior to reporting for active duty at a Navy V-12 Unit. (The V-12(a) group which reported at Navy V-12 Units on 1 November 1943 will therefore complete college training on 1 November 1944, instead of on 1 July 1944, as originally scheduled.)

"A reduction in the future output of the pilot training program for the calendar years 1945 and 1946 has necessitated a rearrangement of the training schedule to meet the reduced output requirements; at the same time an opportunity is thus afforded to raise the standards of performance and improve the quality of pilot output. The rearrangement of the pilot training schedule is further affected by the closing of the Navy CAA-War Training Service and Naval Flight Preparatory Programs. Students withdrawn from the V-12 Program for flight training, on and after 1 November 1944, will enter the flight training program at Pre-Flight Schools, which constitute at present the third stage of the flight training program.

"Prior to April 1942, a minimum of two years of college credits was required for acceptance for flight training as aviation cadet, and the majority of aviation cadets were college graduates. By April 1942, because of the greatly accelerated pilot training program, it was found necessary to lower the educational qualification for aviation cadet to graduation from high school in order to meet the then scheduled pilot output. However, as the scheduled pilot output has now been reduced, it is considered desirable to bring the educational backgrounds for prospective pilots to substantially the same minimum level as was formerly required. It is considered in the best interests of the Naval Service that preparation to assume the responsibilities of a naval aviator be as thorough as practicable within the exigencies of the war. The additional semester of V-12(a) training is intended to enhance the military and educational background of prospective aviation cadets to the end that the duties they will be expected to perform as naval aviators may be carried out more effectively."—*Navy V-12 Bulletin No. 213* (*Subject C*).

3. It was announced in the MONTHLY, March, 1944, that enlisted men gathered at assembly points before assignment to V-12 institutions would be given some review work in mathematics. This Pre-V-12 Program was not inaugurated until May. During April, an orientation program for Pre-V-12 instructors was held. Complete details of this important project will appear in an early issue of the MONTHLY.

DEFERMENT OF MATHEMATICIANS AGES 26 THROUGH 37

A man of age 26 through 37 may ("shall" in the case of men ages 30 through 37) be retained or replaced in Selective Service Class II-A if he is found to be "necessary to and regularly engaged in" an activity in support of the national health, safety, or interest, or in Class II-B if he is found to be "necessary to and regularly engaged in" an activity in war production. (The term "necessary" is omitted in the case of men 30 through 37.)

In view of the fact that all Activity and Occupation Bulletins have been rescinded, the question has been raised whether professional mathematicians are now regarded as essential within the meaning of the previous paragraph. The following sections bearing upon the question have been taken from the "List of Essential Activities" published May 12, 1944, by Selective Service.

Educational Services.—Public and private industrial and agricultural vocational training; elementary, secondary, and preparatory schools; junior colleges, colleges, universities, and professional schools, educational and scientific research agencies; United States Maritime Service Training Program; Civil Aeronautics Administration Civilian Pilot Training Program; armed forces contract flying, ground, and factory aviation schools; and the production of technical and vocational training films.

Technical, Scientific, and Management Services.—The supplying of technical, scientific and management services to establishments engaged in war production; publication of technical and scientific books and journals.

CODE FOR STUDENTS ON FORM 42-A SPECIAL

The *Dictionary of Occupational Titles* used by the United States Employment Service does not provide a code for students to be used in filling out the line reading, "United States Employment Service Dictionary Code (Occupation)," on DSS Form 42-A (Special) of the Selective Service System. For this purpose, the following code for students of science has recently been prepared by the National Roster and approved by the National Headquarters of the Selective Service System.

Physical Sciences

0-40.34 Geology
0-40.37 Meteorology
0-40.38 Chemistry
0-40.43 Mathematics
0-40.44 Physics
0-40.46 Geophysics

Engineering

0-40.47 Civil
0-40.48 Agricultural
0-40.50 Ceramic
0-40.51 Marine and Naval Architecture
0-40.52 Sanitary and Public Health
0-40.54 Industrial and Administrative
0-40.55 Chemical
0-40.56 Electrical
0-40.57 Mechanical
0-40.60 Metallurgical
0-40.61 Aeronautical
0-40.62 Mining
0-40.63 Petroleum
0-40.64 Radio
0-40.65 General & Other

THE AST RESERVE PROGRAM

A total of 166,761 men took the "Army-Navy Qualifying Examination" on March 15, 1944. Of this number, approximately 63,000 were seventeen-year-olds within the age range specified for the Army and Navy College Reserve Programs. Only an estimated 21,000 of this latter number, however, expressed Army preference, and, on the basis of experience, it can be anticipated that another 20% will not pass the physical examination. As a result, the AST Reserve Program seems destined to be smaller than the Army and institutions under contract to the Army had anticipated.

Several devices are being employed by the Army to increase the number of students registered in the AST Reserve Program. For instance, a passing mark was established for the latest qualifying examination which was lower than that originally required for the ASTP. Also, a large number of men who passed the examination and who expressed Navy preference have been given an opportunity to volunteer for the ASTRP without reexamination; this was made possible by the fact that the Navy decided to draw a very small quota from this source for the July term. Another potential source of recruits for the program is the large number of seventeen-year-olds who previously volunteered for the Air Force; since the Army Air Force College Program was completely eliminated by July 1, such men are being urged to enter the ASTRP.

Even though every encouragement has been given to the development of the

Reserve Program, it appears that the total number of men registered will only be a small fraction of the total withdrawn from the colleges with the elimination of the AAF Program and the reduction of the ASTP. Consequently, many institutions previously under contract to the Army cannot be utilized in connection with the ASTRP. Institutions selected to continue have been determined on the basis of a plan of priority developed by the Army. The order of assignment of ASTRP Units to institutions who have had AST or AAF Programs is (1) military colleges, (2) ROTC colleges and universities, and (3) men's colleges.

The ASTRP will be observed with considerable interest by educators. The men are still civilians, and can withdraw from the curriculum at their own discretion. Moreover, the Army has made no attempt to classify the registrants on the basis of special abilities, so institutions must accept responsibility for educational counseling. The program of study has not been rigidly prescribed, and some courses are being offered which are below the usual level of college work. If the Program is successful, it is probable that it will be continued over a considerable period of time, and perhaps expanded upon a broad basis.

ACTIVITIES OF THE COMMITTEE ON WAR TRAINING PROGRAMS

WILLIAM L. HART, University of Minnesota

During the summer of 1943, the War Policy Committee of the A.M.S. and the M.A.A. appointed a Subcommittee on War Training Programs. The Subcommittee now consists of Professors C. R. Adams (Brown); Ralph Beatley (Harvard); H. M. Bacon (Stanford); B. H. Brown (Dartmouth); H. J. Ettlinger (Texas); W. L. Hart (Minnesota), Chairman; C. V. Newsom (Oberlin); W. M. Whyburn (California). This Subcommittee decided to record its activities in the form of a series of reports, each dealing with the mathematical part of some specified war program. By the time the Subcommittee commenced its work, the curricula of the service schools maintained by the Armed Forces were thoroughly established and, in many cases, the flow of trainees was already beginning to decrease rapidly. Hence such curricula were eliminated from consideration by the Subcommittee. It decided to restrict its attention to war programs of the Armed Forces involving mathematical instruction of approximately collegiate grade carried on under the control of college departments of mathematics.

The following programs have been considered and the Subcommittee has presented reports on them to the War Policy Committee.

1. Mathematics in the College Training Program (Aircrew) of the Training Command of the Army Air Forces (report drafted by Professor Hart, November, 1943).*

2. Mathematics in the Army Specialized Training Program (report drafted by Professor Bacon, February, 1944).*

* The chairman of any collegiate department of mathematics can obtain a mimeographed copy of the report by writing to the office of Professor J. R. Kline, American Mathematical Society, University of Pennsylvania.

3. Mathematics in the College Training Program, Navy V-12, of the Training Division, Bureau of Naval Personnel (report drafted by Professor Newsom, March, 1944).*

When the Subcommittee commenced active operations, the Pre-meteorological Programs of the A.A.F. were in their final stages. Hence, in lieu of a report, the Subcommittee made arrangements with the Academic Office of the Programs to provide the War Policy Committee with copies of the very complete descriptive booklet† on the Programs which was issued by the University Meteorological Committee.

The War Policy Committee, through the office of Professor Kline, sent mimeographed copies of Reports 2 and 3 and the information booklet on the Pre-meteorological Programs to the officers of the A.M.S. and M.A.A. and to a selected list of administrators of departments of mathematics. Copies of Reports 2 and 3 also were sent to appropriate offices of the Armed Forces.

At the present time, the Subcommittee is giving consideration to the mathematical part of the program of the United States Armed Forces Institute. As a result of activity of the Subcommittee, Dr. Ralph W. Tyler, the Director of the Examinations Staff for U.S.A.F.I., cordially invited mathematicians to appoint a committee to give advice on mathematical problems in connection with examinations to be sponsored by the U.S.A.F.I. In accordance with this invitation, in June, 1944, Professor M. H. Stone, Chairman of the War Policy Committee, appointed a Subcommittee on Examinations to advise the staff of U. S.A.F.I. This Subcommittee has commenced active operations; its membership is as follows: Professors Ralph Beatley (Harvard); L. L. Dines (Carnegie Institute of Technology); W. L. Hart (Minnesota); C. C. MacDuffee (Wisconsin); W. T. Reid (Chicago), Chairman.

The Subcommittee on War Programs, through its chairman, will welcome at any time comments concerning mathematics in the various war programs.

REPORT ON MATHEMATICS IN THE V-12 PROGRAM

This report, to which reference was made in the previous article by Professor W. L. Hart, has been approved by the War Policy Committee. The report was written on the basis of the experience of the participating colleges up to March 1, 1944.

1. *General description of the Navy V-12 Program.* The Navy College Training Program (V-12) was inaugurated upon July 1, 1943. Its purpose, as phrased by the Navy Department, is to produce Naval officers chosen from high school seniors, high school graduates, and college students who appear to have the potentialities for ultimate selection as officers after carefully prescribed college

* Published as the next article in this MONTHLY.

† The Academic Office of the Programs is now closed. It sent copies of the booklet and amplified outlines of the pre-meteorological courses to the offices of admissions of colleges throughout the United States. Hence, the booklet and outlines should be accessible to every interested mathematician. At present, a few copies of the booklet remain in the hands of Professor Kline.

training. This college training shall be carried on while the men are on active duty, in uniform, receiving pay, and under general military discipline.

The Navy has insisted that all colleges and universities holding Navy V-12 contracts for undergraduate instruction offer three sixteen-week terms in each calendar year. In fact, there must be sixteen full weeks of instruction in each term, exclusive of vacations and holidays.

The several types of officer candidates to receive training in the V-12 Program are as follows: aviation candidates, deck candidates, supply corps candidates, pre-medical and pre-dental corps candidates, general engineering candidates, engineering specialist candidates, pre-chaplain corps candidates, and aerology specialist candidates. Prescribed curricula, which vary in length from two to eight terms, have been outlined for these various programs of training; for information upon the curricula, reference may be made to Navy V-12 Bulletin No. 101. All curricula contain at least two terms of elementary mathematical analysis. Moreover, technical programs involve two terms of calculus, and, in some cases, there is the equivalent of a term of differential equations.

Topical outlines for the individual courses composing the curricula have also been provided; they were drawn up with the assistance of academic consultants. The original outlines for the courses in mathematics were published in this MONTHLY, December, 1943. Since that date, no essential change has been made in their content except for the courses in navigation and nautical astronomy.

The curricula and courses, as prescribed, are for students who enter the V-12 Program as freshmen, without previous college work. Students entering the Program with advanced standing, based on previous college work, are not required to follow the prescribed curricula. Such transfer students are urged, however, to take as many of the prescribed subjects as possible, and certain minimum requirements for V-1 and V-7 transfers must be met. As a result of the liberal demands made of transfers, few institutions have had actual experience as yet with many prescribed V-12 courses above the freshman level.

2. *Selection and qualifications of the V-12 trainees.*

(a) *Transfers from other Navy College Programs.* Upon July 1, 1943, approximately 80% of the V-12 trainees assigned to institutions were students already in college; these students were previously enlisted in class V-1 or V-7, or held probationary commissions in the U. S. Naval Reserve or were enlisted in the Marine Corps or Coast Guard Reserves. Trainees in this general classification were a heterogeneous lot with considerable range of ability, since no rigorous academic criterion was a part of the basis for their selection.

(b) *Transfers from V-5.* V-5 candidates are selected by Naval Aviation Cadet Selection Boards. When temporarily transferred to V-12, as an incident in the adjustment of aviation training quotas, such trainees are allowed to take only the first two terms. After the completion of two terms, the qualifications of this group of men are subject to review, and, if found still qualified, the men are reclassified as V-5, and they then proceed to aviation training. Many V-12 trainees in this category appear to be misfits in the Program inasmuch as Naval

Aviation Cadet Selection Boards are admitting students for training who have had very little previous work in mathematics and science.*

(c) *Freshmen who qualifiy through the Army-Navy tests.* High school seniors in their last semester and high school graduates, within the age classification of seventeen to twenty-two years, may become candidates for the Army A-12 or for the Navy V-12 by successfully passing one of the "Army-Navy tests." Students taking the tests may choose the branch of service which they prefer. Approximately 300,000 young men took the first test given April 2, 1943, and about 78,000 took the second one on November 9, 1943; other tests will be given periodically in the future. Persons familiar with the first test generally believe that it was an adequate device for the elimination of young men of low intelligence. Unfortunately, there was insufficient emphasis upon mathematics, and many V-12 trainees, who successfully passed the first tests, are now having great difficulty with the mathematics in the Program. The second test was weighted more heavily in mathematics, and students in this category, who entered the V-12 Program in March, should be better prepared.

(d) *Men who have previously been on active duty.* The Navy has determined upon a policy of selecting some men on active duty for the V-12 Program. Present indications are that the quota in this category is to be considerably increased. Until recently, the only academic qualification required of such men was high school graduation; now, each man selected must have had a minimum of two years of high school mathematics, and must have made a high score upon a general classification test. In addition, it is understood that steps are now under way to give this group of trainees some refresher training in basic academic subjects before their enrollment in V-12.

Some V-12 trainees first selected in this category had no previous training in algebra or geometry, and a large number had less than two years of secondary mathematics. For such men, the courses in mathematics and science in the V-12 Program are obviously too advanced.

3. *Academic load of trainees.* All students in the V-12 Program are expected to be enrolled for a minimum of seventeen hours of academic work each term. Curricula 101 and 201, followed by most trainees during the first two terms, involve two laboratory subjects, namely, physics and engineering drawing, so the total number of contact hours per week in class and laboratory is twenty-three. This latter total increases to thirty, or more, in the last terms of some of the technical curricula, but the total of contact hours plus study hours should remain approximately the same.

The Navy has assumed that each lecture-recitation period would require approximately two additional hours of study and that the laboratory periods

* Since writing this report, Navy V-12 Bulletin No. 213 (Subject C) has appeared. This Bulletin specifies that Naval aviation students classified in V-12 will now receive a minimum of three terms of college work. Such a step may tend to aggravate present problems resulting from the deficiencies in mathematics and science to be found in this group of men, especially if they are to be classified as Deck Candidates.

would not require outside preparation. Thus, on the average, it is expected that a student will regularly devote about fifty-five hours per week to academic work.

4. *Class work of trainees.* In addition to the academic load, each V-12 trainee is required to devote about ten hours a week to physical training. Moreover, in spite of the utmost cooperation on the part of Naval administrative personnel, the term "military discipline" implies an additional demand upon the trainee's time. As a consequence, the total Program is definitely heavy, and many students have had great difficulty adjusting themselves to a situation which requires much more concentrated effort than other experiences to which they have previously been accustomed. When questioned, instructors generally have admitted that lessons assigned for preparation outside of the classroom must be somewhat shorter than usual; otherwise, the work is not completed. On the whole, however, it appears that students *with proper background* are able to handle the Program in a satisfactory manner, and there is no reason for any extensive modification of its content. The attitude and interest of trainees in the class room is generally reported to be good.

5. *Separation of trainees from Program.* Navy V-12 Bulletin No. 101 states, "Cases warranting separation from the program for scholastic reasons will be referred to the commanding officer for action." The experience of institutions in this connection has been very good. Upon the recommendation of scholastic officers of an institution, the Naval officer in command has generally taken immediate steps to remove the student in question from the Program. The Navy has used wisdom in awarding extensive authority to commanding officers in anticipation of the necessity for separating obvious misfits from the V-12 Program; this is in decided contrast to the policy in some of the other government-sponsored educational programs.

6. *Academic features of the Program.* The Navy has insisted that the V-12 Program is essentially a college program, and the usual standards of the contracting institution are to be maintained. No instance is known of any interference on the part of Navy authorities with the prerogatives of civilian teaching personnel. It is specified in Navy Bulletins that an institution will set examinations according to its own practices; the instructor in each course will select the text; and so on.

As already indicated, the Navy does specify the minimum content of courses through the medium of topical outlines supplied to contracting institutions. Repeated warnings have been issued that any fundamental deviation from prescribed patterns represents a serious threat to the program as a whole in view of the fact that there must be some transfer of students from one institution to another, especially at the end of the first two terms. It is possible, however, to rearrange the sequence of course content.

In addition to examinations given under institutional auspices, the Navy gives "a qualifying examination of its own to all first college year students during their second term, and may give other examinations at prescribed intervals for purposes of selection." The scores made on these tests are used to supplement

information otherwise available before assigning trainees to designated curricula at the end of the second term.

The first of these qualifying tests was given during the latter part of November, soon after the start of the second term. Instructors generally did not have an opportunity to look at the examination. Two reputable mathematicians, who did inspect the test after it was given, write, "(We) both felt that this examination was quite suitable and satisfactory for the purposes for which it was presumably intended." The part devoted to mathematics seems to have been constructed to provide a rough estimate of a trainee's knowledge of trigonometry.

The percentile scores of individual trainees, calculated upon a national basis, have been provided the various participating institutions for the one national examination so far given. Up to the present time, however, the Navy has issued no publicity in regard to the relative standings of the various college V-12 detachments. Moreover, to the best of our knowledge, such information has not been made available confidentially to academic authorities of the colleges or to commanding officers of the V-12 detachments. The Subcommittee does not know to what use, if any, the Bureau of Naval Personnel will put the available statistics about the examination scores, outside of employing them as a partial basis for assigning trainees to later duties.

7. *Problems relative to trainees who have inadequate preparation in mathematics.* The work in mathematics for the first two terms of the V-12 Program has been organized in two optional sequences of courses, namely, M1–M2, and M3–M4. Courses M1 and M2 are somewhat more elementary than M3 and M4; and the outlines of M3 and M4 include topics not suggested for M1 and M2. By Navy instruction, the M1–M2 sequence "is designed for students who enter with two or less units of mathematics or who, regardless of the number of units of previous preparation, are, in the judgment of the college authorities, not adequately prepared to undertake Mathematical Analysis 3, 4."

The situation pertaining to the beginning courses in mathematics has been considerably complicated by the admission of a large number of students into the Program who could not even cope with M1 and M2. This fact was acknowledged in Navy Bulletin No. 66, issued during the summer of 1943. Quotations from this bulletin follow: "It appears that some students have entered the V-12 Program whose preparation in mathematics is inadequate even for Mathematics 1. Colleges and universities are hereby authorized to establish the necessary refresher or makeup classes for such students, and to make necessary readjustments in the content and hours of subsequent classes in mathematics during the first two terms. It is required, however, that such students complete the substantial equivalent of Mathematics 2 by the end of their second term." "It is not contemplated that any V-12 student shall be permitted to remain in college more than the prescribed number of terms because of deficiency in high school preparation. He must, with necessary refresher classes and other adjustments in the curriculum—make up his deficiencies and complete all prescribed courses by the end of his second term in college, or be subject to separation from the program."

To ask a student to make up serious deficiencies in mathematics during the same short period of time that he must complete the content of an extremely heavy program of prescribed courses represents virtually an impossible request. Available data indicate that instructors have been able to salvage only a small fraction of such students, even after a tremendous amount of "sweat and tears" on the part of both students and faculty. Some trainees who were previously on active duty have been especially resentful of the situation; for instance, one said, "I was placed in the V-12 Program almost against my wishes; the assignment given me in the Program is impossible for me to handle, so now I shall have a permanent black mark on my record."

In the case of students who are finally certified at the end of two terms as having successfully passed the equivalent of M1 and M2, it is problematical whether many of them should continue to the calculus, at least, on the level of M5 and M6 as presently organized. Also, some students completing courses M3 and M4 have shown so little mathematical aptitude that it is doubtful that they should be required to study a standard course in calculus. It is apparent, therefore, that reconsideration should be given to the mathematics requirement for the third and fourth terms in certain technical curricula. The need for possible revision is especially urgent in the curricula for Deck Candidates and in the N R.O.T.C. general curricula. It may be assumed that there is no need for such revision in the engineering curricula, because it has been specified in the Navy Bulletins that "all engineering candidates shall be expected to be qualified for and to complete satisfactorily Mathematical Analysis III and IV!"

If the Navy expects to continue the present policy of requiring the calculus during the third and fourth terms for *all* trainees who are Deck Candidates or are students in the N.R.O.T.C. (general), and who have completed M1–M2 or M3–M4 as now organized, it would be desirable to consider the creation of a new course involving the calculus for those trainees who have demonstrated limited understanding or ability in mathematics. This provision might involve giving mathematical instruction at two levels for the first four instead of just the first two terms in some institutions.

8. *Comments on the organization of mathematics courses.* The outlines provided for the various courses in mathematics have now been subjected to the "test of experience." Although it is evident that the courses were organized by persons having considerable understanding of the entire situation, defects have since been observed by competent teachers and administrators, and the course outlines are being criticized on various grounds.

Probably most mathematicians prefer a program which is not too rigid, but this does not imply that the course outlines necessarily should be extremely brief and indefinite. In the hands of a conscientious teacher, an outline which clearly specifies minimum content may be a more flexible instrument than a briefer but indefinite description of the material to be covered. Moreover, in a Program where there is considerable shifting of students between terms from one institution to another, all outlines should be sufficiently specific in regard to

minimum content that it is possible for instructors to make plans for such continuation courses as M4 and M6. Experience has revealed that the course outlines can also be improved with respect to choice of material. In this regard, there appears to be considerable agreement among mathematicians relative to possible deletions and additions. It is undoubtedly true that mathematical consultants familar with V-12 and the experience of mathematics teachers participating in the Program could effectively improve the outlines of the mathematics courses.

Since M1 and M2 are for students with limited background, it is essential that every opportunity be given such students to become familiar with basic algebraic concepts. Instructors comment that they do not have ample time for necessary drill. It appears that the elimination of some topics from the outlines could be accomplished without destroying their effectiveness; then more time could be devoted to a thorough study of elementary notions. A student with only two units of high school mathematics is still very immature, mathematically. Consideration might be given to the elimination of the introduction to spherical trigonometry from the M2 outline, since it is presumed that trainees in this course will continue to non-technical curricula or to the curricula for deck candidates; in the latter case, a term course in spherical trigonometry is provided. Also it appears that the treatment of such topics as simultaneous quadratics, theory of equations, and complex numbers, could be quite brief.

It is noteworthy, also, that some important topics have been omitted from the M1–M2 outlines. In particular, it is unfortunate that provision has not been made for a treatment of the binomial theorem and the geometric progression. Moreover, some study of inequalities would be useful. If students completing M2 are to continue to the calculus, an early introduction to tangents and normals would be desirable.

Some minor adjustments might well be made in the M3–M4 outlines. It is the opinion of some mathematicians, for instance, that the time spent on permutations, combinations, and probability might be devoted to a consideration of inequalities and partial fractions.

Limited experience with courses M5–M6 indicates that a satisfactory distribution of the topics in the calculus has not been worked out between the two terms. It is virtually impossible to cover the work of M5 in a four-hour course with any semblance of thoroughness, whereas there is more than enough time to do rather satisfactory work in M6. Some of the difficulty may arise from the vagueness present in the outlines, inasmuch as it is not clear how detailed the treatment of the indefinite integral should be in M5.

It is somewhat surprising that approximate methods are ignored in connection with integration. This appears to be the most critical omission for a group of students who are seriously concerned with applications.

A very awkward situation exists in connection with courses M5 and M6 in view of the fact that the work of each term has been outlined upon the assumption that four class hours a week are to be devoted to the work. Instead of the

4–4 distribution, however, some curricula follow a 5–3 plan, and others have only the equivalent of a 4–3 distribution. In other words, some institutions are teaching the calculus simultaneously upon a 4–4, a 5–3, and a 4–3 plan. As a result, unnecessary complications are arising in the keeping of records, in the economical utilization of staff, and in the organization of teaching programs. It appears that only slight adjustments are necessary in the various curricula to permit the introduction of a uniform teaching program for courses M5–M6.

The outline of M7 sketches in general terms a typical first course in differential equations, based on elementary calculus. The great flexibility present in the outline appears to be consistent with the apparent purpose of the course.

Courses M8 and M9 in elementary navigation and nautical astronomy are listed in the mathematics section of Navy Bulletins, and presumably are handled in most institutions by mathematics departments.

The descriptions of these two courses provided to instructors previous to July 1, 1943, indicated that the Navy desired a technical presentation of navigation and nautical astronomy. Many institutions made serious efforts to obtain instruments as well as the elaborate sets of tables mentioned in the outlines. In fact, without some equipment, the courses as outlined could not be presented.

A few weeks after the start of the first term, Navy Bulletin No. 48 appeared. The following quotation is from that Bulletin:

"Navigation and Nautical Astronomy I and II are to be basic courses and are not designed to give any technical skill in the science of navigation. Practical navigation training is to be reserved for Naval Reserve Officer Training Corps and Midshipmen Schools. The purpose of Navigation I is to give the basic mathematics with emphasis on the elements of spherical trigonometry and elementary vector mathematics necessary for the student to comprehend the navigation courses he will be given later under Navy instruction. Problems may be drawn from nautical situations to keep the work alive but the emphasis shall be upon mathematical principles only. Any elementary text in Spherical Trigonometry would be an adequate guide to what is desired. The purpose of Navigation II is to give the student the astronomical background necessary to an understanding of the principles of celestial navigation and inherent in the practice of celestial navigation. In essence this course shall be a course in elementary nautical astronomy. Any elementary text in nautical astronomy would be an adequate guide to what is desired. Since Navigation and Nautical Astronomy I or II are not courses in practical navigation, equipment as charts, navigational instruments, and other devices for the study of practical navigation will not be necessary."

The outlines of M8 and M9 provided in Navy V-12 Bulletin No. 101, issued November 1, 1943, express the same point of view as that set out in Bulletin No. 48. It is unfortunate that a more thorough analysis of the needs and purposes of these two courses could not have been made before the inauguration of the Program. In many institutions, a change of text books was necessary before the start of the second term (November 1, 1943), and other adjustments had to be made.

Mathematicians generally are not familiar with the methods employed by the Navy in the field of celestial navigation. Hence, it would be desirable to have expanded outlines for the material which the Navy wishes taught in M8 and M9.

Summary of Recommendations

A. It is recommended that the Training Division, Bureau of Naval Personnel, obtain expert mathematical assistance to improve the course outlines for mathematics in the V-12 Program in the light of a careful analysis of the opinions of teachers in the participating colleges. This recommendation is not to be construed as a suggestion that the outlines should be revised in such a manner as to eliminate their present flexibility, which is a commendable feature.

B. In the case of trainees who come to the Program from active duty in the Navy, it is recommended that the Division give these men suitable refresher training, whenever a considerable interval of time has elapsed since their last academic studies or whenever mathematical tests show this to be desirable, before they commence any part of the regular curricula of the Program.

C. The Training Division is advised that problems of a serious nature are resulting from the plan followed at present in certain curricula of attempting to bring together in M5 two groups of students with distinctly different mathematical backgrounds. The members of one group enter the V-12 Program with adequate mathematical preparation, and complete courses M3 and M4 previous to registration in M5. Students in the other group enter the Program with deficiencies in mathematical preparation, and take M1 and M2 before M5. It seems unrealistic to expect the average member of the latter group to be able to carry successfully the work of M5 if it is taught at a level appropriate for most of the students who have had M3 and M4. In general, then, it would seem to be necessary either to increase the total time spent on mathematics by the members of the poorer group or to diminish their ultimate goal. Hence, we recommend:

C_1. That renewed consideration be given to the curriculum during the first two terms for men who must take M1–M2, and to the placement of these men in curricula for the third and fourth terms.

C_2. That curricula involving four terms of mathematics be re-examined in the light of the fact that many students who have passed M1–M2 (or M3–M4) do not give evidence of mathematical maturity sufficient to profit by the study of an extensive course in the calculus. It is suggested that especial attention be given to the curricula for Deck Candidates and for the General R.O.T.C.

D. If at any time the Training Division should decide to set comprehensive national examinations in specific courses in mathematics in the Program, it is recommended that the Division should consult competent mathematicians in the preparation of the tests. Such men should possess wide experience as teachers of mathematics at the undergraduate level and should have keen interest in examination techniques.

E. It is urged that consideration be given to the slight reorganization required of some technical curricula to permit the teaching of M5 and M6 upon a uniform time schedule for all curricula.

THE MATHEMATICAL ASSOCIATION OF AMERICA

NEW MEMBERS

The following thirty-four persons and one institution have been elected to membership on applications duly certified:

To Institutional Membership

McGill University, Montreal, P. Q., Canada

To Individual Membership

L. E. BABCOCK, M.S. (Illinois) Asso. Prof., Newberry Coll., Newberry, S. C.

R. A. BRADLEY. Student, Queen's Univ., Kingston, Ont., Can.

A. T. BRAUER, Ph.D. (Berlin) Asso. Prof., Univ. of North Carolina, Chapel Hill, N. C.

N. E. DODSON, A.B. (Berea Coll.) Instr., Newberry Coll., Newberry, S. C.

BROTHER DOMINIC, A.B. (St. Mary's Coll., Calif.) Instr., St. Mary's Coll., St. Mary's College, Calif.

W. S. ERICKSON, Ph.D. (Wisconsin) Prof., State Teachers Coll., Minot, N. D.

ELVY L. FREDRICKSON, A.B. (Willamette) Grad. Asst., Univ. of Colorado, Boulder, Colo.

R. A. GOOD, A.M. (Wisconsin) Acting Instr., Univ. of Wisconsin, Madison, Wis.

W. H. GOTTSCHALK, Ph.D. (Virginia) Asst. Prof., Univ. of Richmond, Richmond, Va.

LAURA Z. GREENE, M.S. (Chicago) Instr., Washburn Univ., Topeka, Kans.

J. R. HAMMOND, A.M. (Harvard) Asst. Prof., U. S. Naval Acad., Annapolis, Md.

M. A. HANHAUSER, A.B. (St. Bonaventure) Instr., St. Bonaventure Coll., St. Bonaventure, N. Y.

MRS. AUGHTUM S. HOWARD, Ph.D. (Kentucky) Asso. Prof., Kentucky Wesleyan Coll., Winchester, Ky.

M. A. HYMAN. Student, Univ. of Maryland, College Park, Md.

SAMUEL KARLIN, B.S. (Ill. Inst. of Tech.) Grad. Asst., Illinois Inst. of Tech., Chicago, Ill.

JULIUS KLEWANSKY, A.B. (Brooklyn) Recently Instr., Univ. of New Hampshire, Durham, N. H.

R. P. LALONDE. Inspector, R.C.A. Victor Co., Montreal, P. Q., Can.

ELLA E. LAUSMAN, A.M. (Michigan) Instr., Western Michigan Coll., Kalamazoo, Mich.

L. C. LAY, A.M. (Southern California) Design Engr., Lockheed Aircraft Corp., Burbank, Calif.

MRS. MARGARET B. LEHMAN, A.M. (U.C.L.A.) Instr., Univ. of California at Los Angeles, Los Angeles, Calif.

EMANUEL LEVINE, A.M. (Columbia), Ed.M. (Rutgers) Teacher, Gen. Sci., High School, Freeport, N. Y.

MARK LOTKIN, Ph.D. (Kiel, Germany) Instr., Carleton Coll., Northfield, Minn.

M. H. MARTIN, Ph.D. (Johns Hopkins) Prof., Chm. of Dept., Univ. of Maryland, College Park, Md.

K. A. MCMURTRIE, B.S. (Brown) Cpl., Officer candidate, U. S. Army Signal Corps.

R. S. NICHOLLS, B.Sc. (London) Chemist, Miles Laboratories, Inc., Elkhart, Ind.

W. J. ROBINSON, Ph.D. (Ohio State) Asst. Prof., Allegheny Coll., Meadville, Pa.

L. D. RODABAUGH, Ph.D. (Ohio State) Visiting Lecturer, Oberlin Coll., Oberlin, Ohio

BROTHER ROLAND, B.S. (Laval) Prof., Secretary, Superior School of Commerce, Quebec, P. Q., Can.

HANS SAMELSON, Dr.sc.math. (Fed. Inst. of Tech., Zurich) Asst. Prof., Syracuse Univ., Syracuse, N. Y.

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MARY M. TEMPLETON, A.M. (North Carolina) Prof., Lander Coll., Greenwood, S. C.

J. H. WHITE, A.B. (Princeton) Recently Instr., Lafayette Coll., Easton, Pa.

W. A. WILLIS, A.B. (Johns Hopkins) Mech. Engr., Bendix Aviation Corp., Towson, Md.

THE SPRING MEETING OF THE IOWA SECTION

The thirty-second meeting of the Iowa Section of the Mathematical Association of America was held at the Montrose Hotel in Cedar Rapids, Iowa, on the afternoon of Saturday, April 15, 1944, in conjunction with the meeting of the Iowa Academy of Science. Professor N. B. Conkwright, Chairman of the Section, presided.

The attendance was about thirty, including the following twenty members of the Association: E. W. Anderson, J. W. Beach, J. H. Butchart, E. W. Chittenden, L. M. Coffin, N. B. Conkwright, W. M. Davis, Cornelius Gouwens, Gertrude A. Herr, J. F. Heyda, Dora E. Kearney, A. T. Lonseth, R. B. McClenon, F. M. McGaw, E. N. Oberg, E. R. Smith, L. W. Swanson, Henry Van Engen, L. E. Ward, Roscoe Woods.

In the forenoon members of the Section attended the meeting of the Iowa Academy of Science, at which Professor E. R. Smith delivered a paper entitled *Mathematical continuations corresponding to physical impossibilities* as his Academy presidential address.

At the business meeting the following officers were elected for the next year: Chairman, J. H. Butchart, Grinnell College; Vice-Chairman, E. N. Oberg, University of Iowa; Secretary, Cornelius Gouwens, Iowa State College.

The following papers were presented:

1. *The mathematics situation in the high schools before Pearl Harbor*, by Professor Henry Van Engen, Iowa State Teachers College.

2. *The mechanism of gravitational force*, by Professor E. E. Watson, Iowa State Teachers College, introduced by the Secretary.

This paper will be published in the *Proceedings of the Iowa Academy of Science* for 1944.

3. *A design for sampling Iowa families*, by Mrs. Bernice Brown, Iowa State College, introduced by the Secretary.

This speaker gave a description of the sampling design devised for the Iowa canning and gardening survey. The paper included an explanation of method of selecting the sample, and a partial evaluation of the technique followed. This paper will be published in the *Proceedings of the Iowa Academy of Science* for 1944.

4. *A simple treatment of perturbations in linear algebraic systems*, by Dr. A. T. Lonseth, Iowa State College.

The speaker gave a brief derivation of the principal inequality (10, p. 335) in his paper on *Systems of linear equations with coefficients subject to error*, which appeared in the Annals of Mathematical Statistics, vol. 13, 1942, pp. 332–337.

5. *Repetitious numbers*, by Professor E. S. Allen, Iowa State College, introduced by the Secretary.

Professor Allen defined a repetitious number as one which, written decimally, has at some point every possible finite succession of digits. He pointed out that

every such number has then each such succession an infinite number of times. An example of a repetitious number was given, and methods for obtaining others were outlined. For instance, he explained a method for obtaining from a single one a set which can be put into a one-to-one correspondence with all real numbers.

6. *A mechanical calculator to solve the equation* $x=(d+z+c)y/(b+y)-c$, by Dr. Zabog V. Harvalik, Duluth State Teachers College, Duluth, Minnesota, introduced by the Secretary.

In this paper it was pointed out that the above equation can be written in the form $(x+c)/y=a-(x+c)/b$, in which form it can be interpreted geometrically by consideration of two similar triangles with a common vertex. The mechanical calculator constructed for the purpose of solving the equation was based upon this idea. The calculator was primarily designed to locate foreign bodies by a parallactic method employed in connection with X-rays. This paper will be published in the *Proceedings of the Iowa Academy of Science* for 1944.

CORNELIUS GOUWENS, *Secretary*

CALENDAR OF FUTURE MEETINGS

Twenty-Eighth Annual Meeting, Chicago, Ill., November 24–26, 1944.

The following is a list of the Sections of the Association with dates of future meeting so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN
ILLINOIS
INDIANA, Indianapolis, November 10, 1944
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA
NORTHERN CALIFORNIA, San Francisco, January 27, 1945
OHIO, Columbus, April 5, 1945
OKLAHOMA
PHILADELPHIA, Philadelphia, November, 1944
ROCKY MOUNTAIN
SOUTHEASTERN
SOUTHERN CALIFORNIA, Los Angeles, March 10, 1945
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE
WISCONSIN, Milwaukee, May, 1945

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COLLEGIATE MATHEMATICS

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OCTOBER 1944

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BIOGRAPHIES AND COLLECTED WORKS OF MATHEMATICIANS

T. J. HIGGINS, Illinois Institute of Technology

"No one should be recognized a master in any subject who does not know at least the outline of its history. Of course it would be foolish to expect him to have any deep historical knowledge, but he should know the main landmarks and the leading personalities—he should be acquainted with his scientific ancestors.

"This is almost a moral obligation. We might compare it to the obligation for any educated citizen to know the history of his country. The obligation is of the same kind and of the same order. . . . For a physicist not to be sufficiently familiar with Galileo and Newton is just as shocking as for an American not to know Washington and Lincoln."

These are the words of George Sarton,* historian of science. A decade ago they motivated the present writer to take up what has proved to be a most satisfying and profitable avocation: to seek out, to read, and to record:

I. All *book-length* biographies (individual and collected) in English of physicists, astronomers, mathematicians, chemists, metallurgists and engineers.

II. All of the more important writings in English on the historical development of electrophysics and electrical engineering and on the lives of noted electrophysicists and electrical engineers.

Literal accomplishment of this program is, of course, virtually impossible: but it can be carried to a high degree of completion. And now that the hunt for material has encompassed search of (i) complete files of the consequential serial publications in English devoted to physics or to electrical engineering; (ii) the stacks and the card catalogs of many of the important public, university and technical libraries located in the East and Middle West; (iii) the accumulated catalogs of the principal publishers of technical and scientific books; (iv) the lists of offerings, over a decade, of the larger dealers in used and rare technical and scientific works; (v) much relevant miscellaneous bibliographical material—book review journals, printed catalogs of private libraries, and kindred aids—it is believed that much the greater part of the more worthwhile material has been located and read. In such thought it seems desirable that this material now be made available to those similarly interested.

At present the file devoted to items of category I contains some 800 titles; that devoted to category II, some 1500 titles. The first part of the bibliography below comprises the items of I apposite to mathematicians. Elsewhere are to appear separate bibliographies on physicists and astronomers,† on chemists,‡ and on metallurgists and engineers;§ and the complete bibliography of some

* Sarton, G., The History of Science and the New Humanism, Harvard University Press, Cambridge, 1936, pp. 43–44.

† Higgins, T. J., Book-length biographies of physicists and astronomers, American Journal of Physics, vol. 12, 1944, p. 31–39; addendum, vol. 12, 1944, pp. 234–236.

‡ Higgins, T. J., Book-length biographies of chemists, School Science and Mathematics, vol. 44, 1944, pp. 650–665.

§ Higgins, T. J., Book-length biographies of engineers and metallurgists, Bulletin of Bibliography, forthcoming.

1500 titles on the historical development of electrophysics and electrical engineering and on the lives and work of the more noted of those who labored in these domains.

The writer will not attempt here to discuss what is to be gained through reading one, several, or many of the titles listed. His own views on the use of biographical material as a tool in teaching have been presented elsewhere.* Again, in several books†,‡ and in numerous papers published in Isis and in other journals, Dr. Sarton had discussed in detail the special values of the study of the history of science, technology and mathematics, and of the use of biographical material therein. Of unusual interest—with respect to the present paper—is his argument:‡

"The main reason for studying the history of mathematics, or the history of any science, is purely humanistic. Being men, we are interested in other men, and especially in such men as have helped us to fulfill our highest destiny. As soon as we realize the great part played by individual men in mathematical discoveries—for, however these may be determined, they cannot be brought about except by means of human brains—, we are anxious to know all their circumstances. . . . All these questions and many others are deeply interesting, especially for other mathematicians: if they are young, because of their dreams of the future and their hopes and doubts; if they are older, because of their memories of the past, and also, though in a different way because of their hopes and doubts."

Finally, we may note a fact remarked by many writers; their discussions may be taken as epitomized in the words of that noted British historian of engineering H. W. Dickinson,§ authoritative biographer of a half-dozen pioneer engineers of note:

"To the young person especially, but also to the old, the study of biography is valuable, as it broadens the mind, quickens the imagination, fires ambition and strengthens character. One thing the young man will find that will encourage him—and there is a period in everyone's life when encouragement is needed—and that is, that in one respect youth is at the helm, for practically all [mathematical!] inventions are made between the ages of twenty and forty. . . ."

The second part of the bibliography below comprises a list of the available "collected works" in English of those who worked in either (occasionally, in both) pure or applied mathematics. If biographical memoirs (a term we use as a collective for the various phrases: biographical sketch, personal recollections, obituary, *etc.*) are to be found in a "collected works," the authors thereof are

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* Higgins, T. J., The function of biography in engineering education, Journal of Engineering Education, vol. 32, 1941, pp. 82–92.

† Sarton, G., The Study of the History of Science, Harvard University Press, Cambridge, 1926, 75 pp.

‡ Sarton, G., The Study of the History of Mathematics, Harvard University Press, Cambridge, 1936, 113 pp.

§ Dickinson, H. W., The Value of History in Engineering Education, Rensselaer Polytechnic Institute Bulletin No. 55.

stated. Commonly, these men were intimates of the individuals about whom they wrote; for which reason it not unoften happens that these memoirs comprise the best existing account of the life and work of the individual in question.

This benefaction, however, is incidental to the major reason for setting forth the bibliography of "collected works" in English—which reason has been excellently stated by Dr. Sarton:

"The intelligent reading of the life of a mathematician would naturally lead to his works. Students may thus be induced to attack one of his treatises, or to examine a whole series of his papers. For this purpose my list will be useful, as it will tell them quickly whether 'collected works' are available or not. The existence of 'collected works' makes the study of the evolution of a man's thought much easier; without them one is obliged to refer to odd volumes of a number of periodicals, a process very tiresome at best, and for those out of reach of a large library, almost impossible.

. . .

"A student owning a good biography and the collected works of a mathematician can begin and continue at leisure a deep study of his thought and fate. The reading of monographs will give him a more intimate contact with the realities and creative processes of mathematics than almost any textbook."

And anticipating the purist who surely will decry the worth of such study—"The pursuit of deceased mathematics," to quote one belittler of this phase of the study of the history of mathematics—we may well conclude with the words of that eminent divine, Bishop William Stubbs, who, *re* the value of history in general, wrote:

"The roots of the present lie deep in the past and nothing in the past is dead to the man who would learn how the present comes to be what it is."

INDIVIDUAL BIOGRAPHIES*

John Arbuthnot, Mathematician and Satirist. By L. M. Beattie. Cambridge, Harvard University Press, 1935. 432 pp.

Archimedes. By T. L. Heath. London, Society for Promoting Christian Knowledge; New York, The Macmillan Company, 1920. 58 pp.

Passages from the Life of a Philosopher. By C. Babbage. London, Longman, Green, Longman, Roberts and Green, 1864. 496 pp.

Roger Boscovich, S. J., 1711–1787. By H. V. Gill. Dublin, M. H. Gill and Son, Ltd., 1941. 76 pp.

Yankee Stargazer; the Life of Nathaniel Bowditch. By R. E. Berry. London and New York, Whittlesey House, McGraw-Hill Book Company, 1941. 234 pp.

Memoir of Nathaniel Bowditch by His Son Nathaniel Ingersoll Bowditch. Third Edition. By N. I. Bowditch. Boston, C. C. Little and J. Brown, 1840. 172 pp.; Cambridge, University Press, 1884. 178 pp.

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* For the most part this section and that of COLLECTED BIOGRAPHIES encompass only those who worked wholly in, or made substantial contributions to, pure mathematics. For biographies of those who worked in applied mathematics (*i.e.*, mathematical physicists) see footnotes p. 433.

Eulogy on Nathaniel Bowditch, LL.D., President of the American Academy of Arts and Sciences; Including an Analysis of His Scientific Publications. By J. Pickering. Boston, C. C. Little and J. Brown, 1838. 101 pp.

Navigator; the Story of Nathaniel Bowditch. By A. Stanford. New York, William Morrow and Company, Inc., 1927. 308 pp.

An Eulogy on the Life and Character of Nathaniel Bowditch. By D. A. White. Salem, Mass., Printed at the Office of the Gazette, 1838. 72 pp.

A Discourse on the Life and Character of the Hon. Nathaniel Bowditch, LL.D., F.R.S. By A. Young. Boston, C. C. Little and J. Brown, 1838. 119 pp.

The Book of My Life (De Vita Propria Liber). By J. Cardan. Translated by J. Stoner. New York, E. P. Dutton and Company, 1930. 331 pp.

Jerome Cardan, The Life of Gerolamo Cardano of Milan, Physician. 2 Volumes. London, Chapman and Hall, Ltd., 1854.

Jerome Cardan, A Biographical Study. By W. G. Waters. London, Lawrence and Bullen, Ltd., 1898. 301 pp.

A Memoir of Zerah Colburn; Written by Himself. By Z. Colburn. Springfield, Mass., G. and C. Merriam, 1833. 204 pp.

Descartes: His Life and Times. By E. S. Haldane. New York, E. P. Dutton and Company, 1925. 398 pp.

Descartes. By J. P. Mahaffy. Edinburgh, W. Blackwood and Sons; Philadelphia, J. B. Lippincott and Company, 1881. 211 pp.

F. X. Edgeworth's Contributions to Mathematical Statistics. By A. L. Bowley. London, Royal Statistical Society, 1928. 139 pp. Contains no biographical details; an analysis of his mathematical work.

Carl Friedrich Gauss. By G. W. Dunnington. Baton Rouge, Louisiana State Univ. Press, 1937. 91 pp.

Willard Gibbs. By M. Rukeyser. New York, Doubleday, Doran and Company, Inc., 1942. 465 pp.

James Gregory. Tercentenary Memorial Volume. Containing His Correspondence with John Collins and His Hitherto Unpublished Mathematical Manuscripts, Together with Addresses and Essays Communicated to The Royal Society of Edinburgh, July 4, 1938. Edited by H. W. Turnbull for the Royal Society of Edinburgh. London, G. Bell and Sons, Ltd., 1939. 524 pp. This work contains an interesting chapter, Mathematicians of the Seventeenth Century, which encompasses brief sketches of the life and work of Collins, Wright, Briggs, Harriot, Oughtred, and Warner.

Life of Sir William Rowan Hamilton, Knt., LL.D., D.C.L., M.R.I.A., Andrews Professor of Astronomy in the University of Dublin, and Royal Astronomer of Ireland, etc.: Including Selections from His Poems, Correspondence and Miscellaneous Writings. By R. P. Graves. 3 Volumes. Dublin, Hodges, Figgis, and Company, 1882–89.

A Mathematician's Apology. By G. H. Hardy. Cambridge, University Press, 1940. 93 pp.

Thomas Hariot, the Mathematician, the Philosopher and the Scholar, Developed Chiefly from Dormant Materials. By H. Stevens. London, privately printed, 1900. 214 pp.

The Chequered Career of Ferdinand Rudolph Hassler, First Superintendent of the United States Coast Survey. By F. Cajori. Boston, The Christopher Publishing House, 1929. 245 pp.

Letters and Journals of W. Stanley Jevons. By W. S. Jevons. Edited by his wife. London, Macmillan and Company, 1886. 473 pp.

Sônya Kovalêvsky, a Biography by Anna Carlotta Leffler, Duchess of Cajanello—translated by A. de Furuhjelm and A. M. Clive Bayley; and *The Sisters Rajevsky, Being an Account of Her Life by Sonya Kovalevsky*—translated by A. M. Clive Bayley. London, T. Fisher Unwin, Ltd., 1895. 377 pp. Contains a memoir of the Duchess of Cajanello by L. Wolffsohn.

Sonia Kovalêvsky; Biography and Autobiography. I. Memoir, by *A. C. Leffler (Edgren) Duchessa di Cajanello; II. Reminiscences of Childhood Written by Herself*—translated by L. von Cossel. London, W. Scott, Ltd., 1895. 317 pp.

Sônya Kovalêvsky; Her Recollections of Childhood—translated by I. F. Hapgood; *with Biography by Anna Carlotta Leffler, Duchess of Cajanello*—translated by A. M. Clive Bayley. New York, Century Company, 1895. 318 pp. Contains a memoir of the Duchess of Cajanello by L. Wolffsohn.

Memoirs of John Napier of Merchiston, His Lineage, Life, and Times. By M. Napier. Edinburgh, W. Blackwood, 1834. 534 pp.

An Address on the Genius and Discoveries of Sir Isaac Newton. By G. Boole. Lincoln, Gazette Office, 1835. 23 pp.

The Life of Sir Isaac Newton. By D. Brewster. London, John Murray. 366 pp.; New York, J. and J. Harper, 1831, 1835; New York, Harper and Brothers, 1843. 321 pp. Revised and edited by W. T. Lynn. London, W. Tegg and Company, 1875. 346 pp. The 2 volume *Memoirs* (1855) is an expansion of this life.

Memoirs of the Life, Writing, and Discoveries of Sir Isaac Newton. By D. Brewster. 2 Volumes. Edinburgh, T. Constable and Company, 1855. Second Edition, Edinburgh, Edmonston and Company, 1860.

Sir Isaac Newton: A Brief Account of His Life and Work. By S. Brodetsky. London, Methuen and Company, Ltd., 1927. 165 pp.

Life of Sir Isaac Newton. By H. (Lord) Brougham. London, Library of Useful Knowledge, 1829. 40 pp.

Isaac Newton. By H. Crew. New York, Scripta Mathematica, 1943. 227 pp.

Essays on the Life and Work of Newton. By A. De Morgan. Edited with notes and appendices by P. E. B. Jourdain. Chicago, The Open Court Publishing Company, 1914. 198 pp.

Newton: His Friend: and His Niece. By A. De Morgan. Edited by S. E. De Morgan and A. C. Ranyard. London, E. Stock, 1885. 161 pp.

Newton: The Man. By R. De Villamil. London, G. D. Knox, 1931. 111 pp.

The Elogium of Sir Isaac Newton. By B. le Bovier de Fontenelle. London, J. Tonson, 1728. 32 pp.

The Life of Sir Isaac Newton; with an Account of His Writings. By B. le Bovier de Fontenelle. London, J. Woodman and D. Lyon, 1728. 26 pp.

The Life of Sir Isaac Newton, Containing an Account of His Numerous Inventions and Discoveries; and a Brief Sketch of Astronomy Previous to His Time, Compiled from Authentic Documents. By G. Grant. Dublin, J. M'Glashan, 1849. 311 pp.

Isaac Newton, 1642–1727. A Memorial Volume Edited for the Mathematical Association. By W. G. Greenstreet. London, G. Bell and Sons, Ltd., 1927. 181 pp.

Isaac Newton; a Biography. 1642–1727. By L. T. More. London and New York, Charles Scribner's Sons, 1934. 675 pp.

Sir Isaac Newton, A Biographical Sketch. By V. E. Pullin. London, E. Benn, 1927, 80 pp.

Newton and the Origin of Colours. By M. Roberts and E. R. Thomas. London, G. Bell and Sons, Ltd., 1934. 133 pp.

Matter and Gravity in Newton's Physical Philosophy; a Study in the Natural Philosophy of Newton's Time. By A. J. Snow. London, Oxford University Press, Humphrey Milford, 1926. 256 pp.

Memoirs of Sir Isaac Newton's Life by William Stukeley, 1752, Being Some Account of His Family and Chiefly of the Junior Part of His Life. By W. Stukeley. Edited by A. H. White. London, Taylor and Francis, Ltd., 1936. 86 pp.

Isaac Newton, 1642–1727. By J. W. N. Sullivan. New York, The Macmillan Company, 1938. 275 pp.

Correspondence of Sir Isaac Newton and Professor Cotes, Including Letters of Other Eminent Men, Now First Published from the Originals in the Library of Trinity College, Cambridge; Together with an Appendix Containing Other Unpublished Letters and Papers by Newton; with Notes, Synoptical View of the Philospher's Life, and a Variety of Details Illustrative of His History, by J. Edleston. London, J. W. Parker, 1850. 316 pp. For reference to other correspondence of Newton see *A Bibliography of the Works of Isaac Newton; Together with a List of Books Illustrating His Works, with Notes.* By G. J. Gray. Second Edition. Cambridge, Bowes and Bowes, 1907. 80 pp.

Sir Isaac Newton, 1727–1927, a Bicentenary Evaluation of His Work; a Series of Papers Prepared under the Auspices of the History of Science Society. . . . Edited by F. E. Brasch. Baltimore, The Williams and Wilkins Company, 1928. 351 pp.

William Oughtred, a Great Seventeenth-Century Teacher of Mathematics. By F. Cajori. Chicago, Open Court Publishing Company, 1916. 100 pp.

Pascal, the Life of Genius. By M. Bishop. New York, Reynal and Hitchcock, Inc., 1936. 398 pp.

Blaise Pascal. By J. Chevalier. Translated by L. A. Clare. London, Sheed and Ward; London and New York, Longmans, Green and Company, 1930. 336 pp. Essentially a study of Pascal's thought.

Pascal and the Port Royalists. By W. R. Clark. New York, C. Scribner's Sons, 1902, 235 pp.

The Physics of Pascal. By I. Leavenworth. New York, Publication of the Institute of French Studies, Inc., 1930. 164 pp.

Pascal; the Man and the Message. By R. H. Soltan. London, Blackie and Sons, Ltd., 1927, 216 pp.

Karl Pearson; an Appreciation of Some Aspects of His Life and Work. By E. S. Pearson. Cambridge, University Press, 1938. 170 pp.

Speeches Delivered at a Dinner Held in University College, London, in Honour of Professor Karl Pearson, 23 April 1934. Cambridge, privately printed, University Press, 1934.

Benjamin Pierce, 1809–1880: Biographical Sketch and Bibliography. By R. C. Archibald. Oberlin, The Mathematical Association of America; Chicago, The Open Court Publishing Company, 1925. 30 pp.

Benjamin Pierce . . . A Memorial Collection. Edited by M. King. Cambridge, Mass., 1881. 64 pp.

The Life-Romance of an Algebraist. By G. W. Pierce. Boston, J. G. Cupples, 1891. 167 pp. The writings of an eccentric.

Thomas Simpson and His Times. By F. M. Clarke. Baltimore and New York, Waverly Press, Inc., 1929. 215 pp.

Account of the Life and Writings of Robert Simson, M.D. By W. Trail. London, G. and W. Nicol, 1812. 190 pp.

James Stirling: a Sketch of His Life Along with His Scientific Correspondence. By C. Tweedie. Oxford, Clarendon Press, 1922. 213 pp.

The Mathematical Work of John Wallis, D.D., F.R.S. 1616–1703. By J. F. Scott. London, Taylor and Francis, Ltd., 1938. 240 pp.

The Life and Times of James Wilkins, Warden of Wadham College, Oxford. By P. A. Wright-Henderson. Edinburgh and London, W. Blackwood and Sons, 1910. 131 pp.

COLLECTED BIOGRAPHIES

Men of Mathematics. By E. T. Bell. New York, Simon and Schuster, 1937. 592 pp. Zeno, Archimedes, Pythagoras, Descartes, Fermat, Pascal, Newton, Leibniz, The Bernoullis, Euler, Lagrange, Laplace, Monge, Fourier, Poncelet, Gauss, Cauchy, Lobatchewsky, Abel, Jacobi, Hamilton, Galois, Sylvester, Cayley, Wierstrass, Kowalewski, Boole, Hermite, Kronecker, Riemann, Kummer, Poincaré, Cantor.

Memoirs of the Royal Academy of Sciences in Paris Epitomized, with the Lives of the Late Members of That Society and a Preface by Monsieur Fontenelle, Secretary and Author of the History of Said Academy. By B. le Bovier de Fontenelle. Translated by J. Chamberlayne. London, W. and J. Innys, 1721. 464 pp. Viviani, L'Hospital, James Bernoulli, Duhamel, Tschirnhaus, Carrée, Cassini among others.* There is a 1717 edition with a longer, somewhat different title.

* The phrase, "among others," signifies that these latter are not mathematicians.

Portraits of Famous Philosophers Who Were Also Mathematicians, with Biographical Accounts. By C. J. Keyser. New York, Scripta Mathematica, 1939. 52 pp. Pythagoras, Plato, Aristotle, Epicurus, R. Bacon, Descartes, Pascal, Spinoza, Leibniz, Berkeley, Kant, C. Peirce.

Lectures on Ten British Mathematicians of the Nineteenth Century. By A. Macfarlane. New York, John Wiley & Sons, Inc., 1916. 148 pp. Peacock, De Morgan, Hamilton, Boole, Cayley, Clifford, H. J. S. Smith, Sylvester, Kirkman, Todhunter.

Biographia Philosophica. Being an Account of the Lives, Writings, and Inventions of the Most Eminent Philosophers and Mathematicians Who Have Flourished from the Earliest Ages of the World to the Present Time. By B. Martin. London, W. Owen and B. Martin, 1764. 565 pp. Biographical sketches of some 150 men, including Greek and early English mathematicians not mentioned in more modern works.

Some Great Mathematicians of the Nineteenth Century: Their Lives and Their Works. By G. Prasad. 2 Volumes. Benares, The Benares Mathematical Society, 1933. Vol. 1. Gauss, Cauchy, Abel, Jacobi, Weierstrass, Riemann. Vol. 2. Cayley, Hermite, Kronecker, Brioschi, Cremona, Darboux, Cantor, Mittag-Leffler, Klein, Poincaré.

Correspondence of Scientific Men of the Seventeenth Century, Including Letters of Barrow, Flamsteed, Wallis, and Newton. Printed from the Originals in the Collection of the Right Honourable the Earl of Macclesfield. By S. P. Rigaud. 2 Volumes. Oxford, University Press, 1841. See also, *Contents of the Correspondence of Scientific Men of the Seventeenth Century.* Compiled by A. De Morgan. Oxford, University Press, 1862. 36 pp.

A Portfolio of Portraits of Eminent Mathematicians. Edited by D. E. Smith. Chicago, The Open Court Publishing Company, 1905. Descartes, Pythagoras, Archimedes, Fermat, Leonardo of Pisa, Euclid, Cardan, Leibnitz, Napier, Vieta, Newton, Thales.

Portraits of Eminent Mathematicians; with Brief Biographical Sketches. By D. E. Smith. New York, Scripta Mathematica, 1936. Archimedes, Copernicus, Vieta, Galileo, Napier, Descartes, Newton, Leibniz, Lagrange, Gauss, Lobachevsky, Sylvester.

Portraits of Eminent Mathematicians with Brief Biographical Sketches. Portfolio No. 2. By D. E. Smith. New York, Scripta Mathematica, 1938. Euclid, Cardan, Kepler, Fermat, Pascal, Euler, Laplace, Cauchy, Jacobi, Hamilton, Cayley, Chebishef, Poincaré.

The Great Mathematicians. By H. W. Turnbull. London, Methuen and Company, 1929. 128 pp. The Greeks, Ferro, Fontana, Cardan, Vieta, Napier, Descartes, Fermat, Desargues, Pascal, Brouncker, Wallis, Gregory, Newton, The Bernoullis, Euler, Maclaurin, Lagrange, Gauss, Hamilton.

Biographies of Men of Science. By T. Young. Volume 2 of his collected *Miscellaneous Works*, see below. Lagrange, Fermat, Lambert among others.

COLLECTED WORKS

Pure Mathematics

Collected Scientific Papers of John Aitken. Edited by C. G. Knott. Cambridge, University Press, 1923. 591 pp. Contains a memoir by C. G. Knott.

The Collected Logical Works of George Boole. 2 Volumes. Vol. 1 unpublished. Vol. 2. *The Laws of Thought.* London and Chicago, The Open Court Publishing Company, 1916; photo-reprint, 1940.

The Collected Mathematical Papers of Arthur Cayley. 14 Volumes. Cambridge, University Press, 1889–1899. Vol. 8 contains a memoir by A. R. Forsyth.

Lectures and Essays by the Late William Kingdon Clifford, F.R.S. Edited by L. Stephen and F. Pollock. London, The Macmillan Company. 2 Vols., 1879; 1 Vol., 1886; 2 Vols., 1901. Vol. 1 contains a memoir by F. Pollock; those in the 1886 and 1901 editions are identical and slightly longer than that in the 1879 edition.

The Mathematical Papers of William Kingdon Clifford. Edited by R. Tucker. London, Macmillan and Company, 1882, 658 pp. Contains memoirs by R. Tucker and H. J. S. Smith.

Mathematical Researches of Sir James Cockle, M.A., F.R.S., Chief Justice of Queensland, 1864–77. Reprints from various journals; the copy available to me had no data *re* publisher, year, *etc.*, nor have I been able to ascertain such.

The Mathematical and Other Writings of Robert Leslie Ellis. Edited by W. Walton. Cambridge, Deighton, Bell and Company; London, Bell and Daldy, 1863. 427 pp. Contains a memoir by H. Goodwin.

The Mathematical Writings of Duncan Farquharson Gregory. Edited by W. Walton. Cambridge, Deighton, Bell and Company, 1865. 267 pp. Contains a memoir by R. L. Ellis.

The Mathematical Papers of Sir William Rowan Hamilton. Vol. 1 edited by A. W. Conway and J. L. Synge; Vol. 2 edited by A. W. Conway and A. J. McConnell. Cambridge, University Press, 1931, 1940. Vol. 1 contains a memoir by C. Graves.

The Collected Works of George Abram Miller. Urbana, University of Illinois Press, 1935– . Vols. 1 and 2 have been issued to date.

The Mathematical and Physical Papers of B. O. Peirce, 1903–1913. Cambridge, Harvard University Press, 1926. 444 pp.

Collected Papers of Charles Sanders Peirce. 6 Volumes. Edited by C. Hartshorne and P. Weiss. Cambridge, Harvard University Press, 1931–1935.

Collected Papers of Srinivasa Ramanujan. Edited by G. H. Hardy, P. V. S. Aiyer and B. M. Wilson. Cambridge, University Press, 1927. 355 pp. Contains memoirs by R. V. Sesha Aiyar and R. B. Rao and by G. H. Hardy. See also *Ramanujan.* By G. H. Hardy, Cambridge, University Press, 1940. 236 pp. This text contains valuable comment on Ramanujan's work.

The Collected Mathematical Papers of Henry John Stephens Smith. Edited by

J. W. L. Glaisher. 2 Volumes. Oxford, Clarendon Press, 1894. Contains memoirs by C. H. Pearson, B. Jowett, Lord Bowen, J. L. Strachan-Davidson, and A. Robinson.

The Collected Mathematical Papers of James Joseph Sylvester. Edited by H. F. Baker. 4 Volumes. Cambridge, University Press, 1904–1912. Vol. 4 contains a memoir by H. F. Baker.

Students' [William Sealy Gosset] *Collected Papers.* Edited by E. S. Pearson and J. Wishart. Cambridge, University Press, 1942. 244 pp. Contains a memoir by L. McMullen.

Mathematical Astronomy

The Scientific Papers of John Couch Adams. Vol. 1 edited by W. G. Adams. Vol. 2, part 1 edited by R. A. Sampson, part 2 edited by W. G. Adams. Cambridge, University Press, 1896, 1900. Vol. 1 contains a memoir by J. W. L. Glaisher.

Scientific Papers of Sir George Howard Darwin. Edited by F. J. M. Stratton and J. Jackson, 5 Volumes. Cambridge, University Press, 1907–1916. Vol. 5 contains memoirs by F. Darwin and E. W. Brown.

The Scientific Papers of Sir William Herschel. Including Early Papers Hitherto Unpublished with a Biographical Introduction Compiled Chiefly from Unpublished Material by J. L. E. Dreyer. 2 Volumes. London, The Royal Society and the Royal Astronomical Society, 1912.

The Collected Mathematical Works of George William Hill. 4 Volumes. Washington, Carnegie Institution of Washington, 1905–1907. Vol. 1 contains a memoir in French by H. Poincaré.

Scientific Papers of William Parsons, Third Earl of Rosse, 1800–1867. Republished by Sir C. Parsons. Bradford and London, P. Lund, Humphries and Company, Ltd., 1926. 221 pp.

Mathematical Physics

The Scientific Papers of the Late Thomas Andrews. London, Macmillan and Company, 1889. 514 pp. Contains memoirs by P. G. Tait and A. C. Brown.

The Works of Archimedes. Edited in modern notation by T. L. Heath. Cambridge, University Press, 1897. 326 pp. Chapter I of the "Introduction" comprises a memoir by T. L. Heath.

The Method of Archimedes Recently Discovered by Heiberg; a Supplement to the Works of Archimedes, 1897. Edited by T. L. Heath. Cambridge, University Press, 1912. 51 pp.

Collected Physical Papers of Sir Jagadis Chunder Bose. London, Longmans, Green and Co., 1927. 404 pp.

The Philosophical Works of the Hon. Robert Boyle Esq.; Abridged, Methodized, and Disposed under the General Heads of Physics, Statics, Pneumatics, Natural History, Chemistry and Medicine. Second edition, 3 Volumes. London, Peter Shaw, 1738.

The Works of the Honourable Robert Boyle. 5 Volumes. London, A. Millar, 1744. 6 Volumes. Vol. 1, London, J. F. Rivington; Vols. 2–6, London, W. Johnston, 1772. Vol. 1 of each edition contains "The Life of the Honourable Robert Boyle" by Thomas Birch.

The Collected Papers of George Ashley Campbell. Printed by Lancaster Press, Lancaster, Pa., for the American Telephone and Telegraph Company, New York, 1937. 586 pp. Contains memoirs by V. Bush and E. H. Colpitts.

The Electrical Researches of the Honourable Henry Cavendish, F.R.S., Written Between 1771 and 1781. Edited by J. C. Maxwell. Cambridge, University Press, 1879. 454 pp. Contains a memoir by J. C. Maxwell.

The Scientific Papers of the Hon. Henry Cavendish. Volume 1 edited by J. C. Maxwell, revised by J. Larmor; Volume 2 edited by T. E. Thorpe. Cambridge, University Press, 1921. Vol. 1 contains a memoir by J. C. Maxwell.

Collected Papers of Sir James Dewar. Edited by Lady Dewar and J. D. H. Dickson, H. M. Ross, and E. C. S. Dickson. 2 Volumes. Cambridge, University Press, 1927.

Scientific Memoirs, Being Experimental Contributions to a Knowledge of Radiant Energy. By J. W. Draper. London, Sampson Low, Marston, Searle and Rivington; Harper and Bros., 1878. 467 pp.

Experimental Researches in Chemistry and Physics. By M. Faraday. London, R. Taylor and W. Francis, 1859. 496 pp.

The Scientific Writings of the Late George Francis Fitzgerald. Edited by J. Larmor. Dublin, Hodges, Figgis and Company, Ltd.; London, Longmans, Green and Company, 1902. 576 pp. Contains memoirs by J. Larmor, O. J. Lodge, and F. T. Trouton.

A Commentary on the Scientific Writings of J. Willard Gibbs. Edited by F. G. Donnan and A. Haas. 2 Volumes. London, Oxford University Press, Humphrey Milford; New Haven, Yale University Press, 1936. Each of these two volumes deals with the corresponding portion of the *Collected Works.*

The Collected Works of J. Willard Gibbs: Memorial Edition. Edited by W. R. Longley and R. G. Van Name. 2 Volumes. New York, Longmans, Green and Company, 1928. Vol. 1 contains a memoir by H. A. Bumstead.

The Scientific Papers of J. Willard Gibbs. Edited by H. A. Bumstead and R. G. Van Name. 2 Volumes. London and New York, Longmans, Green and Company, 1906. Vol. 1 contains a memoir by H. A. Bumstead.

Mathematical Papers of the Late George Green. Edited by N. M. Ferrers. London, Macmillan and Company, facsimile reprint, 1871; Paris, A. Hermann, 1903. 336 pp.

Electrical Papers. By O. Heaviside. 2 Volumes. London and New York, Macmillan and Company, 1892.

Electromagnetic Theory. By O. Heaviside. 3 Volumes. London, "The Electrician" Printing and Publishing Company, Ltd., 1893–1912.

Scientific Writings of Joseph Henry. 2 Volumes. Washington, Smithsonian Institution, 1886.

The Collected Works of H. Hertz. Edited by P. Lenard.

Vol. 1. *Miscellaneous Papers.* Translated by D. E. Jones and G. A. Schott. London and New York, Macmillan and Co., 1896. 340 pp.

Vol. 2. *Electric Waves; Being Researches on the Propagation of Electric Action with Finite Velocity Through Space.* Translated by D. E. Jones. London and New York, Macmillan and Co., 1893. 278 pp.

Vol. 3. *The Principles of Mechanics Presented in a New Form.* Translated by D. E. Jones and J. T. Walley. London and New York, Macmillan and Co., 1899. 276 pp.

Scientific Papers of Bertram Hopkinson. Edited by J. A. Ewing and J. Larmor. Cambridge, University Press, 1921. 480 pp. Contains memoirs by J. A. Ewing and A. V. Hill.

Original Papers by the Late John Hopkinson. Edited by B. Hopkinson. 2 Volumes. Cambridge, University Press, 1901. Contains a memoir by B. Hopkinson.

The Scientific Papers of James Prescott Joule. Published by the Physical Society of London. 2 Volumes. London, Taylor and Francis, Ltd., 1884–1887.

Baltimore Lectures [of Lord Kelvin] *on Molecular Dynamics and the Wave Theory of Light . . . Followed by Twelve Appendices on Allied Subjects.* Cambridge, University Press, 1904. This volume contains papers referred to only by title in the *Mathematical and Physical Papers.*

Mathematical and Physical Papers by William Thomson, 1st Baron Kelvin. 6 Volumes. Vols. 4–6 edited by J. Larmor. Cambridge, University Press, 1882–1911.

Popular Lectures and Addresses. By Lord Kelvin. 3 Volumes. London and New York, Macmillan and Co., 1891–1894.

Reprint of Papers on Electrostatics and Magnetism. By W. Thomson (Lord Kelvin). London, Macmillan and Company, 1872. 592 pp. Second edition, 1884. 596 pp.

Mathematical and Physical Papers of Sir Joseph Larmor. 2 Volumes. Cambridge, University Press, 1929.

Collected Works of J. MacCullagh. Edited by J. H. Jellett and S. Haughton. Dublin, Hodges, Figgis and Company, 1880. 381 pp.

Collected Scientific Papers of Samuel Bruce McLaren. Edited by H. R. Hasse, J. Larmor and others. Cambridge, University Press, 1925. 112 pp. Contains memoirs by H. Walker and J. W. Nicholson.

The Scientific Papers of James Clerk Maxwell. Edited by W. D. Niven. 2 Volumes. Cambridge, University Press, 1890; photo-reprint, Paris, 1927. Contains a memoir by W. D. Niven.

The Works of John Playfair, Esq. Edited by J. G. Playfair, 4 Volumes. Edinburgh, A. Constable and Company, 1822. Contains a memoir by J. G. Playfair; also biographies of Matthew Stewart, James Hutton and John Robinson.

Collected Scientific Papers of John Henry Poynting. Edited by G. A. Shakespear and G. Barlow. Cambridge, University Press, 1920. 768 pp. Contains memoirs by O. Lodge, J. Larmor, J. J. Thomson, and G. A. Shakespear.

Miscellaneous Scientific Papers by W. J. Macquorn Rankine. Edited by W. J. Millar. London, C. Griffin and Company, 1881. 567 pp. Contains a memoir by P. G. Tait.

Collected Scientific Papers of John William Strutt, Baron Rayleigh. 6 Volumes. Cambridge, University Press, 1899–1920.

Papers on Mechanical and Physical Subjects by Osborne Reynolds. 3 Volumes. Cambridge, University Press, 1900–1903.

The Physical Papers of Henry Augustus Rowland. Baltimore, Johns Hopkins Press, 1902. 704 pp. Contains a memoir by T. C. Mendenhall.

The Complete Works of Count Rumford. 4 Volumes. Boston, American Academy of Arts and Sciences, 1870–1875.

Collected Papers on Acoustics of W. C. Sabine. Cambridge, Harvard University Press, 1922.

Scientific Papers of Shizuwo Sano. Tokyo, Iwanami Shoten, 1926. 340 pp.

Scientific and Technical Papers of Werner von Siemens. Translated from the 2nd German edition. 2 Volumes. London, John Murray, 1892–1895.

The Scientific Papers of C. William Siemens, Kt., F.R.S., D.C.L., LL.D., Civil Eng. Edited by E. F. Bamber. 3 Volumes. London, John Murray, 1889.

Mathematical and Physical Papers by Sir George Gabriel Stokes. 5 Volumes. Cambridge, University Press, 1880–1905. Vol. 5 contains a memoir by Lord Rayleigh.

Scientific Papers by Peter Guthrie Tait. 2 Volumes. Cambridge, University Press, 1898–1900.

Collected Papers in Physics and Engineering by James Thomson. Edited by J. Larmor and J. Thomson. Cambridge, University Press, 1912. 484 pp. Contains a memoir by J. T. Bottomley.

The Scientific Papers of Sir Charles Wheatstone. London, Physical Society of London, Taylor and Francis, Ltd., 1879. 380 pp.

Miscellaneous Works of the Late Thomas Young. 3 Volumes. Vols. 1 and 2 edited by G. Peacock, vol. 3 by J. Leitch. London, John Murray, 1855.

TANGENT TRIANGLES TO A BIQUADRATIC CURVE

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1. Sketching a biquadratic curve whose inflection points are real. Ask a sophomore calculus student how to sketch the graph of a biquadratic polynomial, and he might answer "Set the derivative equal to zero, locate the maximum and minimum points, and then sketch the curve." One difficulty with this procedure is, that unless the biquadratic has been especially fixed up to have a derivative with integral roots—or at the worst, rational roots—the student may find himself confronted with an irreducible cubic equation which he does not

know how to solve. Furthermore, the maximum- and minimum-point procedure fails to show a certain symmetry of the biquadratic curve which will be noted below.

Restricting our discussion to *biquadratic polynomial graphs*, $y=f(x)$, *having real inflection points*, we shall show how certain important tangent lines with their contact points may be located geometrically after finding both coordinates of each of the inflection points $(h\pm a,\ k+4b\pm ma)$ and the ordinate $f(h)=k+9b$ midway between these—five independent quantities involving at most quadratic irrationalities in terms of the five coefficients of the polynomial. Six important tangents form a characteristic configuration of two intersecting triangles, similar in the ratio one to two, which guide the curve as it bends away from its inflection points. Two other tangents give the directions of the curve further out. Sketching in the curve from these eight tangents is then a simple matter.

2. Affine reduction to normal form. The general biquadratic polynomial graph with real inflection points has an equation of the form

$$y = f(x) = c_0x^4 + 4c_1x^3 + 6c_2x^2 + 4c_3x + c_4, \qquad c_1^2 > c_0c_2. \tag{1}$$

This can be transformed into the particular biquadratic polynomial

$$Y = F(X) = X^4 - 6X^2 + 9, \tag{2}$$

by first shifting to a suitable new origin (h, k), then subtracting a linear term $m(x-h)$, and then changing the scale on the new axes by factors a and b respectively. Five constants h, k, m, a, b are involved, which are expressible in terms of the five coefficients $c_0, c_1, \cdots, c_4$ as follows:

$$\begin{aligned} &a = \sqrt{c_1^2 - c_0c_2}/c_0, \qquad b = c_0a^4, \\ &h = -c_1/c_0, \qquad k = f(h) - 9b, \qquad m = f'(h). \end{aligned} \tag{3}$$

The explicit equations of transformation are

$$\begin{aligned} aX &= x - h, & x &= h + aX, \\ bY &= (y - k) - m(x - h), & y &= k + bY + maX, \\ & & \frac{dy}{dx} &= \frac{b}{a}\frac{dY}{dX} + m. \end{aligned} \tag{4}$$

Important properties of this transformation (4) which is called an *affine transformation preserving vertical lines*, are the preservation of collinearity of points, concurrency or parallelism of lines, midpoints, centroids, tangents and contact points, and ratios of distances along parallel lines. A certain configuration of tangents to the simple biquadratic curve (2) leads to a corresponding configuration for the general biquadratic curve (1) which makes it easy to sketch.

3. Theorems on centroid points. We start by proving a theorem defining an important centroid point C associated with a biquadratic curve (Fig. p. 449).

THEOREM 1. *The centroid C of the contact points, on the three tangents of a given slope λ which can be drawn to a given biquadratic curve, is a fixed point independent of λ.*

Proof. Consider first the biquadratic (2) and let the three contact points on tangents of slope λ be (X_iY_i), $i=1, 2, 3$.

Then X_1, X_2, X_3 are roots of the equation $dY/dX-\lambda=0$, namely

$$4X^3 - 12X - \lambda \equiv 4(X - X_1)(X - X_2)(X - X_3) = 0. \tag{5}$$

Comparing coefficients of X^2 and X, respectively, we have

$$\begin{gathered} X_1 + X_2 + X_3 = 0, \\ 2(X_1X_2 + X_1X_3 + X_2X_3) = -6 = (X_1 + X_2 + X_3)^2 - (X_1^2 + X_2^2 + X_3^2). \end{gathered} \tag{6}$$

Hence the coordinates of the centroid C are $(0, 3)$, for all λ, since

$$\begin{aligned} \tfrac{1}{3}(X_1 + X_2 + X_3) &= 0, \quad Y_i = (X_i^4 - 3X_i^2) - 3X_i^2 + 9 = \lambda X_i/4 - 3X_i^2 + 9, \\ \tfrac{1}{3}(Y_1 + Y_2 + Y_3) &= (\lambda/12)(X_1 + X_2 + X_3) - (X_1^2 + X_2^2 + X_3^2) + 9 \\ &= 0 - 6 + 9 = 3. \end{aligned} \tag{7}$$

Since tangency, parallelism, and centroids are unchanged by the transformation (4), the transform of the point C, whose coordinates are $(h, k+3b)$, will be the centroid of contact points for parallel tangents of biquadratic (1).

There are just three slopes for which two of the three parallel tangents coincide. The doubly counting tangents are the two inflectional tangents d_1 and d_2, each having a doubly counting contact at one the inflection points A_1 and A_2; and double tangent d, having two distinct contact points B_1 and B_2. (We call the segment B_1B_2 the *base* and denote its midpoint by B.) The three doubly counting tangents form a triangle $D\ D_1D_2$. Parallel to each of the sides d_i of this triangle there is but one other (singly counting) tangent s_i, and the three tangents s, s_1, s_2 form a triangle $S\ S_1\ S_2$ similar to triangle $D\ D_1D_2$ but twice as large. The contact points M, C_1, C_2 are determined from the centroid theorem proved above. Since A_1 and A_2 are doubly counting as contact points, and B is midway between B_1 and B_2, the segments BM, A_1C_1, A_2C_2 are trisected by C.

The vertical line through the base center B we call the *axis* of the biquadratic curve, and the point M where it cuts the curve we call the *midpoint* of the curve. The axis passes through the centroid C and through the inflectional center A, midway between the inflection points. It also contains the vertices D and S of the two principal tangent triangles. An important property of the axis is based on the following theorem.

THEOREM II. *The centroid of the four points (real or imaginary) in which an arbitrary line cuts a biquadratic polynomial curve is a point on a vertical line called the axis, which is defined by setting the third derivative of the biquadratic polynomial equal to zero.*

Proof. The intersections x_1, x_2, x_3, x_4 of an arbitrary line $y=px+q$ with the biquadratic curve (1) are given by the equation

$$
\begin{aligned}
(8) \qquad & c_0x^4 + 4c_1x^3 + 6c_2x^2 + 4c_3x + c_4 - px - q \\
& \qquad = c_0(x - x_1)(x - x_2)(x - x_3)(x - x_4) = 0.
\end{aligned}
$$

Comparing coefficients of x^3, we have $4c_1=c_0(-x_1-x_2-x_3-x_4)$. Hence the average of the x_i is given by

$$(9) \qquad x = \tfrac{1}{4}(x_1 + x_2 + x_3 + x_4) = - c_1/c_0 = h,$$

which is easily seen to be the value of x for which the third derivative $24c_0x+24c_1$ is zero.

Let the arbitrary line be in particular one of the inflectional tangents d_i which intersects the curve three times at the inflection point A_i, and cuts the axis at D. Then its fourth intersection G_i is a point on d_i such that $\overline{DG_i}=3\overline{A_iD}$. The two points G_1 and G_2 so determined are useful in sketching the curve beyond the contact points C_1, C_2 already determined.

4. Coordinates of special points. Returning to the special biquadratic (2) we next compute by elementary calculus and analytic geometry the coordinates of the various points and the slopes of the various tangents we have discussed. We have

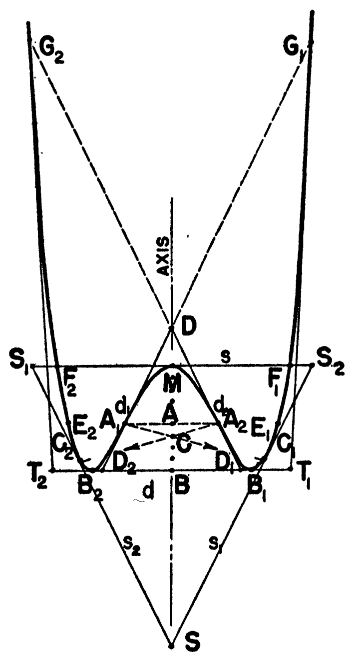
$$(10) \qquad Y' = 4X^3 - 12X, \qquad Y'' = 12X^2 - 12.$$

The inflection points A_1, A_2 are at $(\mp 1, 4)$ and the slopes of d_1 and d_2 are ± 8. Hence the inflectional center A is at (0, 4), and the vertex D is at (0, 12). The double tangent d is the x-axis, having slope 0, its contact points B_1, B_2 are at $(\pm\sqrt{3}, 0)$, and the vertices D_1, D_2 of the doubly tangent triangle are at $(\pm 1.5, 0)$. The midpoint M is at (0, 9), the base center B is at (0, 0) and the centroid C is at (0, 3), (as we proved). Since C trisects the segments $\overline{A_iC_i}$, the points C_1, C_2 are at $(\pm 2, 1)$. Since the slopes at these contact points are the inflectional slopes ± 8, the point S is at $(0, -15)$, while S_1, S_2 are at $(\mp 3, 9)$. The line A_1A_2 through the inflection points is parallel to the double tangent and intersects the biquadratic again in points E_1, E_2 at $(\pm\sqrt{5}, 4)$. The tangent line at the midpoint M is parallel to A_1A_2 and meets the biquadratic again in points F_1, F_2 at $(\pm\sqrt{6}, 9)$. The inflectional tangents intersect the biquadratic again in points G_1, G_2, at $(\pm 3, 36)$ respectively, and since the corresponding slopes are ± 72, respectively, the tangents g_1, g_2 intersect the base line at points T_1, T_2 with coordinates $(\pm 2.5, 0)$. Points and slopes are given in the following table:

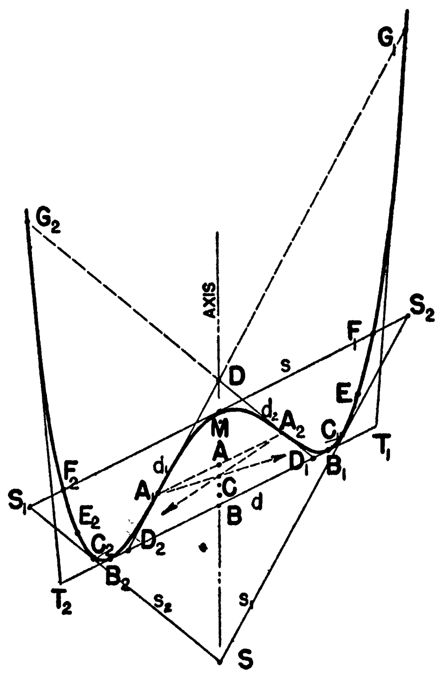
Auxiliary points:	A	B	C	D	D_1	D_2	S	S_1	S_2	T_1	T_2
X-coordinates:	0	0	0	0	1.5	-1.5	0	-3	3	2.5	-2.5
Y-coordinates:	4	0	3	12	0	0	-15	9	9	0	0

Points on Curve:	G_2	F_2	E_2	C_2	B_2	A_1	M	A_2	B_1	C_1	E_1	F_1	G_1
X-coordinates:	-3	$-\sqrt{6}$	$-\sqrt{5}$	-2	$-\sqrt{3}$	-1	0	1	$\sqrt{3}$	2	$\sqrt{5}$	$\sqrt{6}$	3
Y-coordinates:	36	9	4	1	0	4	9	4	0	1	4	9	36
Slope:	-72			-8	0	8	0	-8	0	8			72

5. Geometric construction of the tangent configuration. The corresponding points and slopes on the arbitrary biquadratic curve (1) are obtainable from these by means of the transformation (4), but it is easier to sketch the configuration without using the explicit coordinates of any points except the inflection points A_1, A_2 and the midpoint M, which are computed from the original function. Having located the points A_1, A_2, M, we proceed geometrically to draw the configuration as follows:



$Y = X^4 - 6X^2 + 9.$



The biquadratic polynomial graph.

Step. 1. Locate A, midway between A_1 and A_2.

Step. 2. Locate C on the vertical axis MA extended through A so that $\overline{AC} = \frac{1}{5}\overline{MA}$.

Step 3. Locate B beyond C on MA so that $\overline{AB} = 4\overline{AC}$.

Step 4. Locate D on AM extended through M so that $\overline{DM} = \overline{CB}$. (Check: $2\overline{DM} = \overline{MC}$.)

Step 5. Locate S on the axis MA beyond B so that $\overline{CS} = 2\overline{DC} = 2\overline{MB}$.

Step 6. Locate C_1 and C_2 by extending A_1C and A_2C each twice its own length through C.

Step 7. Draw tangents A_1D, A_2D, and parallel tangents C_1S and C_2S. (Check parallelism.)

Step 8. Complete the triangles DD_1D_2 and SS_1S_2 by drawing parallels to A_1A_2 through B and M. (Check: $\overline{S_1M} = \overline{MS_2} = \overline{D_2D_1}$.)

Step 9. Locate G_1 and G_2 by extending A_1D and A_2D each three times its length through D.

Step 10. Locate T_2 and T_1 on D_2D_1 (but not between D_2 and D_1), so that $\overline{T_2D_2}=\overline{D_1T_1}=\overline{A_1A}$; and draw the tangents T_1G_1 and T_2G_2.

Step 11. Estimate contact points B_2, B_1 on the base D_2D_1 at distances $\sqrt{3}\,\overline{A_1A}$ from the axis; and also the intersections E_2, E_1 at distances $\sqrt{5}\,\overline{A_1A}$ from axis on A_1A_2, and the intersections F_2, F_1 at distances $\sqrt{6}\,\overline{A_1A}$ from axis on S_1S_2.

Step 12. Sketch in the curve from A_1 to M to A_2 inside the triangle DD_1D_2, using tangents at these three points and making the curve concave towards A. Then use the other constructed contact points and tangents on the two arms of the curve and draw the curve to be concave in the opposite direction (upward if b is positive, downward if b is negative).

A final check on the drawing is based on a certain axial symmetry of a biquadratic curve when viewed obliquely along lines parallel to the double tangent. The following theorem defining this "symmetry" is left as an exercise to the reader.

THEOREM III. *A line parallel to the double tangent which intersects a biquadratic polynomial curve in real points, intersects it either in one pair or in two pairs of points, such that the points of each pair are equidistant from the axis of the curve.*

THREE-LINE LATIN RECTANGLES

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This note shows the relation of the reduced *problème des ménages* to the enumeration of three-line Latin rectangles. A simple solution of the former has recently been published by I. Kaplansky [1]; the latter has been studied by S. M. Jacob [2] and by S. M. Kerawala [3].

A Latin rectangle is an array in which each row is a permutation of elements 1 to n and each column has distinct elements.

The *problème des ménages* asks for the number of ways of seating n married couples, husbands alternating with wives, at a circular table, so that no husband sits next his wife. Fixing positions of wives (or husbands) the problem is reduced to the enumeration of permutations discordant with the two permutations:

$$\begin{matrix} 1 & 2 & 3\cdots n-1 & n \\ 2 & 3 & 4\cdots \quad n & 1 \end{matrix}$$

or, more generally, to the enumeration of permutations discordant with two permutations, one of which is the identity, the other a cycle of n.

This enumeration may be used for three-line Latin rectangles in the following way. With the identity as the first line (which introduces an $n!$ factor in the enumeration) the second line may be any permutation discordant with the identity. The number of these is known to be the subfactorial of n, or $\Delta^n 0!$,

where Δ is the finite difference operator. Then for each of these the number of permutations discordant with both lines is enumerated as a linear function of *problème des ménages* numbers.

Consider in cycle form, the second line permutations, since they leave no element unchanged, contain no cycles of degree one, that is, the cycle structure is a partition of n without unit parts; *e.g.*, for $n=4$, the permutations and their cycle structure are as follows:

Perm.	*Cycle Struct.*	*Perm.*	*Cycle Struct.*	*Perm.*	*Cycle Struct.*
2143	2^2	3142	4	4123	4
2341	4	3412	2^2	4312	4
2413	4	3421	4	4321	2^2

By a known formula [4], the number of permutations of cycle structure $2^{a_2}3^{a_3}\cdots n^{a_n}$, where $(a_2, a_3, \cdots, a_n)$ is some solution of $2a_2+3a_3+\cdots+na_n=n$, is:

$$C_a = \frac{n!}{2^{a_2}a_2!\cdots n^{a_n}a_n!}. \tag{1}$$

Kaplansky's solution of the *problème des ménages* may be written:*

$$u_n = \sum_{i=0}^{n}(-1)^i\frac{2n}{2n-i}\binom{2n-i}{i}E^{n-i}0!, \tag{2}$$

where E is the shift operator of finite differences: $Ef_n=f_{n+1}$, $E0!=1!$ $E^{n-i}0!=(n-i)!$. Thus $u_n\equiv U_n(E)0!$, with $U_n(E)$ a polynomial of degree n in E.

The number of permutations discordant with both the identity and a permutation of class $a=(a_2, a_3, \cdots, a_n)$ is (*cf.* Kaplansky [5]):

$$u_{[a]} = U_2^{a_2}U_3^{a_3}\cdots U_n^{a_n}0!. \tag{3}$$

In this symbolical expression, E is treated as an ordinary algebraical quantity until all operations are completed.

Then the number of three-line Latin rectangles is

$${}_3L_n = n!\,{}_3K_n = n!\sum C_a u_{[a]} = \sum\frac{n!^2 U_2^{a_2}\cdots U_n^{a_n}0!}{2^{a_2}a_2!\cdots n^{a_n}a_n!}, \tag{4}$$

where the summation is over all partitions of n without unit parts.

Now U_n has the recurrence:

$$U_n = (E-2)U_{n-1} - U_{n-2}, \tag{5}$$

which, with the substitution $e=E-2$, is the recurrence for Tchebycheff polynomials, and in fact U_n may be written:

* A more compact form is $u_n=2\cos(2ny)$ with $2\cos y=\sqrt{v}$ and $v^n=E^n0!=n!$, as given without proof by Touchard [6]; this is an immediate consequence of equation (6) below.

$$U_n \doteq 2 \cos (nx), \qquad \cos x = \tfrac{1}{2}e. \tag{6}$$

This agrees with (2) except for the initial values U_0 and U_1, where the function has no meaning in the enumeration sense. For the use of (6) which follows, these initial values are $U_0=2$, $U_1=e=E-2$.

Then,

$$\begin{aligned} U_iU_j0! &= [4 \cos ix \cos jx]0! \\ &= u_{i+j} + u_{i-j}. \end{aligned} \tag{7}$$

which is readily generalized to

$$U_{i_1}U_{i_2} \cdots U_{i_k}0! = \sum u_{i_1\pm i_2\pm \cdots i_k} \tag{8}$$

(*cf.* Touchard [6]), where the sum is over the 2^{k-1} possible assignments of $+$ and $-$ signs.

Hence (3) may be evaluated as a sum of values of u with proper subscripts. (Note that for interpretation of (7) or (8), $u_{-n} \equiv u_n$.)

This evaluation may be used in (4) without difficulty for the smaller values of n; *e.g.*, for $n=4$, the results may be tabulated as follows:

a	C_a	$u_{[a]}$	$C_au_{[a]}$
4	6	2	12
2^2	3	4	12

so that ${}_3K_4=24$, in agreement with Jacob and Kerawala.

Also the behavior for large values of n follows immediately on noting that, for all possible $a=(a_2, a_3, \cdots, a_n)$

$$u_{[a]} \sim u_n \sim n!e^{-2}.$$

Then ${}_3K_n$ behaves like the product of u_n and the subfactorial of n, which for n large is $n!e^{-1}$; that is,

$${}_3K_n \sim (n!)^2e^{-3}, \tag{9}$$

a result which Kerawala surmised but failed to prove, though his numerical evidence was practically conclusive (agreeing with e^{-3} to seven decimal places).

Bibliography

1. Irving Kaplansky, Solution of the "problème des ménages," Bull. A.M.S. 49, 10, pp. 784–5, Oct. 1943.
2. S. M. Jacob, The enumeration of the Latin Rectangle of depth three . . . , Proc. Lond. Math. Soc. 31, 5, pp. 329–354, Sept. 1930.
3. S. M. Kerawala, The enumeration of the Latin Rectangle of depth three by means of a difference equation, Bull. Calcutta Math. Soc. 33, 3, pp. 119–127. Sept. 1941.
4. F. D. Murnaghan, The Theory of Group Representations, Baltimore 1938, p. 9.
5. I. Kaplansky, On a generalization of the "Problème des recontres," this MONTHLY, 46, pp. 159–161, March 1939.
6. J. Touchard, Sur un problème de permutations, Comptes Rendus, 198, pp. 631–3, Feb. 1934.

MATHEMATICS FOR STUDENTS OF THE HUMANITIES

OYSTEIN ORE, Yale University

1. Mathematics and the liberal arts. Most colleges are looking forward already to the postwar period; new plans are being discussed and the old curricula are coming under close scrutiny. Undoubtedly the teaching of mathematics will be fundamentally affected in this process of revaluation and reorganization. In the college faculties we shall experience searching discussions in regard to the role which mathematics should naturally play in a liberal arts curriculum.

Historically mathematics always constituted an integral part of the liberal arts, in all the meanings which this term has had through the centuries. There is every indication that it will retain this position in the future. In the present scientific trend of our civilization and education the mathematical theories represent not only a body of important facts, but they constitute also a language without which many phases of scientific thought would be cumbersome to represent and difficult to understand. Furthermore, mathematics embodies the framework for the theory of logic and other philosophic constructions. Mathematical terminology and methods of expression gradually are becoming more prevalent. Even the daily press and publications which usually strive to present their subject matter in an entirely popular form do not shrink from using mathematical terms. For this reason there does not seem to be any trend away from the inclusion of mathematics in the liberal education; perhaps one may expect a gradually strengthened position. This situation makes it imperative that the position of mathematics in the educational scheme should be closely analyzed so that a well considered basis for its role in a liberal arts curriculum can be established.

2. Relation of students to mathematics. The students in a college usually fall into different groups in regard to their interests and their studies of mathematics.

One important group consists of those students who are to become engineers or intend to specialize in various physical sciences or to major in mathematics. Most of these men are aware of the fact that their studies will require mathematics of a rather advanced character and also that proficiency in mathematical manipulations and knowledge of technical procedures will be of great value to them. Most of the students in this group will take several courses in mathematics and some will extend their study of the subject throughout their whole period of attendance. Many of the courses given for these men are well defined through the prospective applications, and these students usually constitute the great majority in the advanced mathematics courses in the curriculum. There are, of course, at most institutions improvements which could be made in the instruction for these men. Certainly their courses must be revised from time to time according to needs and applications and according to the development of

new mathematical ideas and points of view. To me it seems desirable, for instance, if possible, to broaden the basis for these courses with more emphasis on some of the fundamental mathematical ideas. This would probably be advantageous both from the pedagogical and scientific points of view, and it would certainly reduce some of the now so common repetitions in the advanced calculus courses where the previous theories must be put on a more satisfactory logical basis. However, the possibility of more emphasis on fundamental ideas is counteracted somewhat from the beginning by the fact that the instruction of engineers and science students must to a certain extent be subordinate to the requirements of some of the fields of application. Thus it may be necessary to bring the students very quickly to a certain level of proficiency in handling the differential and integral calculus and the instruction often becomes one-sidedly and narrowly directed at attaining this goal at the earliest possible moment.

In spite of these remarks it appears that from the point of view of instruction this group of mathematics students does not represent as difficult a problem as other large groups which we shall discuss presently. Science and engineering students will usually broaden their mathematical views through their several courses. Furthermore, the students in this class have an easier task both through the clearer understanding of the necessity of studying the subject and through their natural interest and gifts for mathematics which enable them to supplement the instruction by independent studies. Therefore, I wish to make it quite clear, that the following considerations are not essentially concerned with this group of students.

In any liberal arts college there will be large groups of students whose relations to the field of mathematics are entirely different trom those of the science students and engineers. To illustrate this in specific figures, some statistics from Yale College extending over some years before the war may be quoted: Close to 60 per cent of the students take no mathematics course whatever, about 35 per cent take one course only, while a mere 5 per cent take more than one course. (In order that these figures shall not be misleading it may be stated that Yale College includes very few science students, since these men usually graduate from the Sheffield Scientific School.)

If one analyzes the group of students who take one course in mathematics, one finds that many take another science course—physics, chemistry, biology, geology, psychology, *etc.* Some intend to study medicine or major in economics. In all these fields a certain familiarity with the mathematical procedures and methods is useful so that a basic course in mathematics often is made a requirement. It may be recalled specifically, that a particular method which is common to many of these fields is statistical analysis, and its increased use makes it extremely desirable that its terminology and basic principles be known to the students. To this group of students taking one course in mathematics should be added a considerable number of students specializing in the humanities, who correctly feel that a liberal education is not complete without some knowledge of mathematics. Some of them like the subject well enough, but there are many

among them, who have but little gift for mathematics, and nevertheless are willing to go through its supposed ordeals for the cultural value and the general usefulness of the subject.

This brings us finally to the great number of students who prefer to stand on the sidelines as far as mathematics is concerned. In attempting to discover their reasons, one finds that this is not always because of an unwillingness to take mathematics. Many of them have a real desire to gain some mathematical knowledge and feel that it would be advantageous in their education, but they fear that they are not capable of fulfilling the requirements of the courses, so that they avoid difficulties and unhappiness by turning to other subjects.

3. Organization of a special course. When it is recognized that this situation exists one must ask the question: On which principles should a single mathematical college course be organized so that it serves as well as possible the educational interests of those students for whom it will most likely represent the end of their mathematical studies? At present many institutions make no particular provision for these men. They are simply offered one or two courses which are identical with those given to beginning science and engineering students. For many students this may be too difficult, but a much more important deficiency lies in the fact that the students after such a course are left with an impression confirming their previous school experiences that mathematics is mainly a technical tool which requires some special mental skill. They are presented with no broader view of mathematics and given little understanding of mathematics as a method and as an important branch of human thought and philosophy developed through centuries.

There are certain principles, it seems to me, which offer themselves naturally as guides in the organization of a terminal one-year course in mathematics.

First, as a general remark it is desirable to give the course a larger measure of broadness with greater emphasis on first principles than those courses now provide.

Secondly, as every teacher of mathematics knows, the ability to solve fairly complicated classroom problems is very short-lived. Most students after graduation retain only a vague memory of mathematics and little interest in the subject remains. In the choice of subject matter one should therefore attempt to include some information on problems which they may encounter later in everyday life and conversation.

A third basic aim of the course should be to present the development of mathematics as an important branch in the history of thought. For this purpose it is essential that the historical side of the subject should be touched upon in connection with all topics which are discussed. This, in my experience, contributes in no small measure to the interest of the students, and it serves the additional purpose of breaking the dry presentation of the average mathematics course.

Some of the classical problems which occasionally appear within a cultured

man's sphere of interest should not be omitted. For instance, a college graduate should not use the term "squaring of the circle" without having an idea of what this problem involves. Similarly, the teacher should at times step out of the beaten path of the subject to attempt to give the students a view of those peaks of scientific achievement which have already been reached, or some of the unsolved problems on which the interests of mathematicians have been focussed.

4. Content of the course. After these general considerations let us turn to more definite proposals for the content of a course of this nature. It would seem desirable to begin with a discussion of the basic concept, the number system from integers to real numbers and complex numbers. There is no reason why imaginary numbers should be unintelligible to a college student. The Greek dilemma in regard to the real numbers serves as an illustration of the decisive importance which a single idea may have. Trigonometry, with its welter of special formulas which even the teacher often has difficulty in remembering, should be reduced to a bare minimum, probably not more than the simplest facts for a right triangle with an indication of how other figures can be handled. A brief review of analytic geometry would probably be necessary with only one or two of the simplest curves discussed. A rapid excursion into space provides an opportunity to explain what the mathematician understands by higher dimensions so that the concept of fourth dimension can be stripped of the veil of mysticism with which it is usually draped in ordinary conversation.

Calculus must of course be introduced and its importance should be stressed. However, only the very simplest parts should be included in the actual course work. Differentiation and integration of simple polynomials are quite sufficient to illustrate the main ideas. Higher functions like exponentials, logarithms and trigonometric functions should not be touched upon. This is also a more straightforward acknowledgment of certain barriers than those pseudo-explanations which are now often used in this connection, stating conveniently at difficult points that "it can be shown." Simple maxima and minima problems should be treated since they usually impress and interest the students.

This brings us to some of the further topics which I should like to see included in the course and for which certainly some time will be available. Number theory is a subject of peculiar interest to many students. The Euclid algorithm is presumably known; it yields quite simply the solution of linear Diophantine equations and the unique factorization in primes. The infinity of primes, the distribution theorem for primes as well as the largest known primes, should certainly be mentioned in this connection. It adds to the interest of the students to point out that a great number of the puzzles which commonly appear in magazines can be solved by linear Diophantine equations. The sum of the divisors of a number is easily obtained, and it enables one to explain the numerological mysticisms in Plato in regard to perfect numbers and amicable numbers. Practically no students are acquainted with simple rules for checking multiplications and divisions by casting out nines and other rules. Perhaps this may all

appear as a somewhat long tribute to number theory, yet all the topics mentioned here can be presented in a few classroom hours.

An excursion into the theory of equations and its long history is quite fascinating. The only drawback is that the classroom work would be limited; only the solution of quadratic and cubic equations could be achieved. But one should explain that every equation has an many roots as its degree when the imaginary ones are counted. The historical milestones in the theory should be reported up to the result that not all equations can be solved by radical expressions. One should explain the fact that geometric constructions with compass and ruler lead to expressions involving square roots and thus one has gained an introduction to a brief discussion of the classical Greek problems of squaring the circle, trisecting the angle and doubling the cube. If one had gone so far in number theory as to determine the number of integers less than and relatively prime to a given one, even the results on the construction of regular polygons can be stated completely.

Another topic which is very inviting is the theory of probability. Here one has a field whose problems may occur in any one's experience and whose basic rules can be explained briefly and fully without the use of complicated mathematical apparatus. Some properties of combinations and permutations must be derived but this again seems advantageous since it leads to the kind of reasoning everyone runs into occasionally, be it in arranging dinner-guests or other objects. A variety of probability problems of interest to the students can be solved and the fact that they occur in connection with card games and gambling is in line with the history of the subject and should not cause any moral scruples.

There may be many other subjects which could be adapted to the course: those mentioned here should only be taken as specific examples; perhaps it may not be possible to find time even for all these. The detailed subject matter should depend to a considerable extent on the wishes and tastes of the instructor since the success of the course will depend largely upon his personal qualifications and enthusiasm. It requires a teacher who himself has broad interests in mathematics and an ability to transfer some of his own enthusiasm to the students. On the other hand, this freedom represents a challenge to the younger instructors which they, in my experience, will heartily welcome.

5. Concluding remarks. Of course there may be objections to the course as outlined, and it will probably be scorned as being "easy" by many of the old mathematical drill masters. If it is an easy course because it is entertaining and because a certain effort has been made to find topics which many of the students in this group can more readily absorb, then the charge is true. Too little effort is often being made in college courses to retain any of the quality of entertainment of mathematics which it undoubtedly had in many ancient cultures, particularly emphasized in Indian mathematics. The gentleman mathematicians of the seventeenth and eighteenth centuries who studied mathematical problems *con amore* in their spare time hardly exist any more and I feel this is to be de-

plored. However, if one means by an easy course, a course in which the students do little serious work, I think the charge will be without foundation. There are as many opportunities for serious problems and logical deliberation in the fields outlined above as in any others, and it should be insisted that those questions which are actually discussed from more than the historical side should actually be mastered. There is no intention that the course should be effortless on the part of the students.

And so a final remark. College instruction in this country, and for that matter in most other countries of the world, is faced with the problems of the scientific versus the humanistic education. Fortunately one must say, the question is not to be decided one way or the other. The present so common comparisons of values in the two fields, presumably to vindicate one field over the other, contributes nothing to the problem. Only a serious attempt to bring the two in contact will bring any positive results. Mathematics in the last century has experienced a brilliant growth in conjunction with the natural sciences. It should not be forgotten, however, that mathematics by its traditions and long history belongs to the liberal arts; it is certainly not to be regarded mainly as a technical tool of the sciences. There is an earnest desire among many humanistic students for information on the subject. They may not have the ability or time to partake in the regular, more technical courses. However, such a serious demand we shall have to satisfy even if it may mean a compromise in some of our traditional ways of instruction. This in the long run will undoubtedly be of as much benefit to mathematicians and the position of mathematics in our higher institutions of learning as to those for whom these concessions are made. And the instructors in mathematics can take pride if in this attempt some small bridge has been thrown over the chasm between the scientists and the humanists.

CLUBS AND ALLIED ACTIVITIES

EDITED BY J. S. FRAME

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to J. S. Frame, Michigan State College, East Lansing, Michigan.

CLUB REPORTS 1943–44

Pi Mu Epsilon, University of Nebraska

This year six meetings were held, including business meetings, the initiation banquet, and the annual spring picnic. The following talks were given:

Invariants, by Professor M. G. Gaba

The isograph, by Mr. H. W. Linscheid

Gnomonic projections, by Professor O. C. Collins.

The annual Freshman and Sophomore competitions were held May 5, and the Freshman prize was won by Mr. Fay Fuchser. At the annual banquet held on February 29, eleven men were initiated. Because of wartime conditions, it was decided to invite members of the Army Specialized Training Program stationed at this University, who showed exceptional ability in mathematics, to become members of *Pi Mu Epsilon*. At the spring picnic on May 12, the pledge was administered to ten civilian and fifteen army students.

The officers elected for next year are: Director, Charles Lantz; Vice-Director, Margaret Stewart; Secretary, Mary Lou Weaver; Vice-Secretary, Noboru Tosaya; Treasurer, Mary Ann Mattoon; Vice-Treasurer, Anton Kashas. The faculty adviser is Professor F. S. Harper.

Kappa Mu Epsilon, Upsala College

The New Jersey Alpha chapter of *Kappa Mu Epsilon* concluded a very successful year with its fifth annual banquet, on June 2, at which two new members were initiated: Miss Audrey Richter and Mrs. Mary Louise McKim. At the banquet an interesting talk was given entitled

Cryptography, by Dr. D. R. Davis of Montclair State Teachers College.

Other topics and speakers for the year were:

Boolean algebra, by Elizabeth Ebel

The group concept, by Joseph Prieto

Duplication of a cube, squaring of the circle, and trisection of an angle, by Zelda Meisel

Practical applications of mathematics to engineering problems, by Dr. Fithian of Newark College of Engineering

Finite differences, by Betty Rudebock

Vector analysis, by Dorothy Shaw.

Officers for the coming year are to be: President Thales, Betty Rudebock; Vice-President Apollonius, Joseph Prieto; Secretary Abel, Elizabeth Ebel; Treasurer Fibonacci, Mrs. Mary Louise McKim; Historian Gauss, Marjorie Wolfe; Faculty Sponsor and Corresponding Secretary Descartes, Dr. M. A. Nordgaard.

Mathematics-Physics Club, College of Saint Teresa

The club numbered forty-five members who devoted the bi-monthly meetings to the study of aeronautics, which included the identification of planes according to the Wetfurn system, aerodynamics, and the engine; as well as the arithmetic, algebra, geometry, and trigonometry involved. A guest speaker discussed the bomber of the future. Slides and films illustrated certain topics. A Valentine party and a mathematical picnic conducted as a Naval Air Base comprised the social meetings of the year. Officers were: President, Dorothy Hilty; Vice-President, Josephine Schmelzle; Secretary, Mary Theo Maze; Treasurer, Dolores Raleigh; Faculty Advisors, Sister Thomas à Kempis, Sister Roswitha, Sister Leontius.

Mathematics Club, Illinois Institute of Technology

During the winter semester, the Mathematics Club met to hear papers read by various faculty members and students, as follows:

Elasticity, by Wally Ito
Transcendental numbers, by Dr. Herbert Busemann
Projective geometry, by Harley Flanders
Number theory, by Allen Devinatz
Binary system, by Felix Rosenthal
Brachistochrone problem, by Samuel Karlin
Laplace transformation, by Dr. W. B. Caton
Series summation, by Harley Flanders
Perfect numbers, by Samuel Karlin
Transformations of lineal elements, by Dr. John DeCicco

During the spring semester, bimonthly meetings were held, at which the following papers were presented:

Euler's summation formula and Stirling's formula for $n!$, by Samuel Karlin
Fourier's series, by Sidney Baker
Buffon's needle problem, by Gerald Rosenberg
Ordered systems—Lattices, by Dr. L. R. Wilcox
Continued fractions, by Alan Grant
Probability—Tchebycheff's problem, by Harley Flanders
Evaluation of $\int_0^\infty e^{-t^2}dt$ *and related forms*, by Berny Miretzky
Pascal's theorem and areas in plane geometry, by Don Slager

Interest was encouraged on the part of members in a broad range of topics, particularly in subjects not on the regular curriculum.

Officers for the winter semester were: President, Samuel Karlin; Secretary, Harley Flanders; Publicity, Morris Shapiro; Faculty Sponsor, Dr. H. Busemann. Officers for the spring semester were: President, Sidney Baker; Secretary, Harley Flanders; Faculty Sponsor, Dr. John DeCicco.

Kappa Mu Epsilon, Albion College

The 1943–44 program of the Michigan Alpha Chapter of *Kappa Mu Epsilon* included one meeting each month from October to May, and opened with a talk entitled

The history and purpose of Kappa Mu Epsilon, by K. R. Ferguson.

This was followed by biographical sketches by the following initiates: George Peacock, R. D. Barrow, Augustus DeMorgan, F. Aburano, George Boole Mari-Anne Gordon, James Sylvester, Beryl Voelker.

The November meeting was preceded by a roll call, calling for a *Recent mathematical article and comments*. Two talks were presented:

Special devices for integrating unusual forms, by J. R. Barcroft
Mathematics in war, by W. F. Voglesong.

Topics presented at the two joint meetings with the *Physics Club* in December and January were

Early world mathematics, by F. Aburano
Vectors applied to physics, by E. J. Dinger
Asymptotes oblique to the axes, by Betty Hossfeld
Some unusual applications to physics, by W. F. Voglesong.

Following the initiation at the February meeting, each of the new members gave a brief talk, and then all joined in a social hour with refreshments. Both the March and April meetings opened with a roll call. A feature of the April meeting was that seniors had complete charge of the program. Topics presented in March and April were

March roll call: *Continental mathematicians*
Homothetic properties and their application, by R. D. Barrow
Mathematical fallacies, by Virginia Tripp
April roll call: *An unusual application of mathematics*
Mathematics applied to physics, by E. J. Dinger and K. R. Ferguson
Mathematics applied to chemistry, by Betty Hossfeld and W. F. Voglesong.

The annual program was concluded with a joint picnic of the Mathematics, Physics, and Chemistry Clubs on May 8. Officers for the year 1943–44 were: President, K. R. Ferguson; Vice-President, J. R. Barcroft; Secretary-Treasurer, Betty Hossfeld; Member of Program Committee, Virginia Tripp.

Kappa Mu Epsilon, Louisiana State University

A total of fifty-eight members have been initiated during the year, at initiations which were held each quarter because of present conditions. Papers presented during 1943–44 were as follows:

Games of chance, by Dr. K. L. Nielsen
Base e, by Dr. Frank A. Rickey
The accident of ten fingers, by Dr. C. J. Thorne
The theorem of Desargues, by Miss Gloria McCarthy
Mathematics in nature, by Mr. Milton Peacock
Some notes on relativity, by Dr. Houston T. Karnes.

Each year the chapter gives two awards. An award of fifteen dollars, to the graduating senior who has proved superior according to the rules set up in the local chapter by-laws, was granted this year to Mr. R. J. Hoskins. A second award, conferred upon the freshman who ranks highest in a special *K.M E.* examination, was granted to Mr. R. B. Stobaugh. The local chapter has obtained a conspicuous place in the Mathematics Library for the purpose of setting up a reading shelf of mathematical books of interest to students at large. Some twenty volumes have been placed on this shelf. The chapter also bought a one hundred dollar war bond in the Fourth Bond Drive.

Officers for 1944–45 are: President Gauss, Milton Peacock; Vice-President Poincare, Mike Fossier; Secretary Fermat, Gene Ventre; Treasurer Galois, Yvonne Broussard; Historian Cajori, Albert Saliba; Faculty Sponsor, Marelena White; Secretary Descartes, Dr. H. T. Karnes.

Pi Mu Epsilon, University of Kentucky

Due to wartime conditions, and the presence of an Army Specialized Training unit at the University of Kentucky this year, *Pi Mu Epsilon* held only one meeting each quarter. The programs follow:

Problems in mathematical definitions, by Dr. H. H. Downing

The need for a mathematics for the social and biological sciences, by Professor Edward Newbury

Problems in inversion, by the following mathematics students: Mary Ann Mache, Virginia Baskett, Virginia Mitchell

Highlights from the history of Pi Mu Epsilon at University of Kentucky, presented by Dean P. P. Boyd, after the April initiation and dinner.

A picnic meeting was held at Needmore Farm in May. During the year, *Pi Mu Epsilon* voted to spend one hundred dollars for the purchase of books for the mathematics library. Officers for 1944–45 are: Director, Dr. Sallie Pence; Vice-Director, Mary Ann Mache; Secretary-Treasurer, Virginia Baskett; Librarian, Dr. L. W. Cohen.

DISCUSSIONS AND NOTES

EDITED BY MARIE J. WEISS, Sophie Newcomb College, New Orleans 18, La.

The Department of Discussions and Notes is open to all forms of activity in college mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

THE METHOD OF UNDETERMINED COEFFICIENTS

TOMLINSON FORT, Lehigh University

The method of undetermined coefficients for determining a "Particular Integral" in the theory of the linear differential equation with constant coefficients is explained in many text books, usually without proof of the rules that are announced. Proofs have been given by A. B. Coble,* also by Reddick and Miller† and possibly others. However, the following seems simpler. The use of Taylor's formula is rigorous and compact and a treatment of special cases is not necessary. The author has not seen a previous discussion for the difference equation.

1. The differential equation. Let us be given the differential equation with constant coefficients

$$F(D)y = (D^n + C_1D^{n-1} + \cdots + C_n)y = ke^{ax}x^m, \tag{1}$$

* A. B. Coble, Concerning a method for finding a particular integral, this MONTHLY, vol. 26, 1919, p. 12.

† Reddick and Miller, Advanced Mathematics for Engineers, p. 50.

where m is zero or a positive integer and a is any number real or complex. First establish the formula

$$D^j(e^{ax}\mu(x)) = e^{ax}(a+D)^j\mu(x). \tag{2}$$

This is readily done by mathematical induction.

From (2) it follows that

$$F(D)e^{ax}\mu(x) = e^{ax}F(a+D)\mu(x). \tag{3}$$

Rewrite (3) by the use of Taylor's formula utilizing well known properties of D

$$F(D)e^{ax}\mu(x) = e^{ax}\left(F(a) + F'(a)D + \frac{F''(a)}{2!}D^2 + \cdots + \frac{F^{(n)}(a)}{n!}D^n\right)\mu(x). \tag{4}$$

Now let us suppose that a is an r-fold root of the auxiliary equation

$$F(\alpha) = 0. \tag{5}$$

It is permitted that r be zero, that is that a be not a root. Assume a particular solution of the type

$$Y = (A_1x^{m+r} + A_2x^{m+r-1} + \cdots + A_{m+1}x^r)e^{ax}.$$

Substitute this in (1) and use (4) noting that $F(a) = F'(a) = \cdots = F^{(r-1)}(a) = 0$ since a is an r-fold root. We have

$$\begin{aligned} e^{ax}\Bigg[& x^mA_1\frac{(m+r)\cdots(m+1)}{r!}F^{(r)}(a) \\ & + x^{m-1}\left(A_1B_{11} + A_2\frac{(m+r-1)\cdots m}{r!}F^{(r)}(a)\right) \\ & + x^{m-2}\left(A_1B_{21} + A_2B_{22} + A_3\frac{(m+r-2)\cdots(m-1)}{r!}F^{(r)}(a)\right) \\ & + \cdots\cdots\cdots\cdots\cdots\cdots \\ & + x^0\left(A_1B_{m1} + A_2B_{m2} + \cdots + A_{m+1}\frac{r!}{r!}F^r(a)\right)\Bigg] = ke^{ax}x^m. \end{aligned}$$

Here the B's depend upon terms arising from operating with $D^{r+1}, \cdots, D^n$. Equate coefficients. Inasmuch as $F^{(r)}(a) \neq 0$ as a is only an r-fold root of (5), and inasmuch as $(m+r-j)\cdots(m+1-j) \neq 0$ since $r \geqq 0$ and $j \leqq m$ we can determine successively $A_1, A_2, \cdots, A_{m+1}$. The coefficients thus determined make Y a solution.

In case the right hand member is a sum of functions of the type $ke^{ax}x^m$ treat each such function separately. The work, of course, can be arranged to suit the convenience of the computer.

Suppose the right hand member contains a term of the form $ke^{ax}x^m \cos \lambda x$. Replace it by the sum of two terms of the respective types $k\frac{1}{2}x^me^{ax}(\cos \lambda x + i \sin \lambda x)$

and $k\frac{1}{2}x^m e^{\alpha x}(\cos \lambda x - i \sin \lambda x)$ or in more compact notation $k\frac{1}{2}x^m e^{\alpha x}$ and $k\frac{1}{2}x^m e^{\bar{\alpha} x}$. Denote the resulting particular solutions by Y_1 and $\overline{Y}_1$. If the coefficients in (5) are real, then if α is an r-fold root so is $\bar{\alpha}$. If k is also real the A's and the corresponding $\overline{A}$'s are conjugates. Consequently Y_1 and $\overline{Y}_1$ are conjugate and $Y = Y_1 + \overline{Y}_1$ is real. It is not, however, necessary to the scheme that coefficients in (5) be real. If the coefficients in (5) are real and if k and a are real one can work wholly in the domain of reals if he so desires. Assume the trial form for Y in the final known form:

$$Y = (A_1 x^{m+r} + A_2 x^{m+r-1} + \cdots + A_{m+1} x^r)e^{ax} \cos \lambda x$$
$$+ (B_1 x^{m+r} + B_2 x^{m+r-1} + \cdots + B_{m+1} x^r)e^{ax} \sin \lambda x.$$

We know that such a particular solution exists and consequently that the A's and B's are determinable. If the right hand member contains a term of the form $ke^{ax}x^m \sin \lambda x$, modifications of our discussion are but slight and are left to the reader.

2. The difference equation. The work here is so similar to the above that it is only sketched. Suppose that we are given

$$F(E)y = K\beta^x x^{(m)},$$

where

$$E\mu(x) \equiv \mu(x+1) \equiv (1+\Delta)\mu(x).$$

The auxiliary equation is

$$F(\alpha) = 0.$$

Suppose that β is an r-fold root of this.

Assume a particular solution

$$Y = (A_1 x^{(m+r)} + A_2 x^{(m+r-1)} + \cdots + A_{m+1} x^{(r)})\beta^x.$$

We proceed to determine the A's as previously. Proof that this is always possible depends upon the formula

$$E^j(\beta^x \mu(x)) = \beta^x(\beta + \beta\Delta)^j \mu(x),$$

and consequently

$$F(E)\beta^x\mu(x) = \beta^x\left[F(\beta) + F'(\beta)\beta\Delta + \cdots + \frac{F^n(\beta)}{n!}\beta^n\Delta^n\right]\mu(x),$$

coupled with the fact that

$$\Delta x^{(j)} = jx^{(j-1)},$$

where

$$x^{(j)} = x(x-1)\cdots(x-(j-1)).$$

ON MY PROOF OF THE BUDAN-FOURIER THEOREM

M. F. SMILEY, Postgraduate School, U. S. Naval Academy

The author has given a proof of the Budan-Fourier theorem* which assumes a knowledge of the Fundamental Theorem of Algebra but is otherwise strictly elementary. The following theorem was assumed in our proof.

THEOREM. *If $f(x)$ is a polynomial with real coefficients and if α is a real number for which $f(\alpha)=0$, then there is a positive number ϵ such that*

$$f(x)f'(x) > 0, \qquad (\alpha < x \leqq \alpha + \epsilon),$$
$$f(x)f'(x) < 0, \qquad (\alpha - \epsilon \leqq x < \alpha).$$

Our proof of this theorem will rest on the even more fundamental

LEMMA. *If $f(x)$ is a polynomial with real coefficients and α is a real number for which $f(\alpha)>0$, then there is an interval $\alpha-\epsilon\leqq x\leqq\alpha+\epsilon$ on which $f(x)>0$.*

Remark. We might base a proof of our lemma on the fact that a polynomial is a continuous function. Such considerations are, however, not essential.

Proof of the lemma. Since $f(x)$ has real coefficients, its complex roots occur in conjugate imaginary pairs, and we have

$$f(x) = a_0(x - r_1)(x - r_2) \cdots (x - r_m)(x^2 + 2a_1x + b_1) \cdots (x^2 + 2a_px + b_p)$$

in which each r_j $(j=1, \cdots, m)$ is real, $r_1\leqq r_2\leqq \cdots \leqq r_m$, and each factor $x^2+2a_ix-b_i$ $(i=1, \cdots, p)$ arises from a pair of conjugate complex roots. Each of the factors $x^2+2a_ix+b_i$ is *positive* for all real x since $a_i^2-b_i<0$ and $x^2+2a_ix+b_i=(x+a_i)^2+b_i-a_i^2>0$. For convenience, we set $r_0=-\infty$, $r_{m+1}=+\infty$. Since $f(\alpha)>0$, we have $r_j<\alpha<r_{j+1}$ for some $j=0, \cdots, m$. Define ϵ to be the smaller of $|r_{j+1}-\alpha|/2$ and $|r_j-\alpha|/2$. We see that no factor of $f(x)$ can change sign on the interval $\alpha-\epsilon\leqq x\leqq\alpha+\epsilon$. Since $f(\alpha)>0$, the proof is complete.

Proof of the theorem. Using the Fundamental Theorem of Algebra, we may write $f(x)=(x-\alpha)^\mu\phi(x)$, in which μ is a positive integer and $\phi(\alpha)\neq 0$. We calculate† $f'(x)=\mu(x-\alpha)^{\mu-1}\phi(x)+(x-\alpha)^\mu\phi'(x)$. We then have

$$f(x)f'(x) = (x - \alpha)^{2\mu-1}\{\mu[\phi(x)]^2 + (x - \alpha)\phi(x)\phi'(x)\}.$$

Setting $g(x)$ equal to the polynomial inside the parentheses, we note that $g(\alpha)=\mu[\phi(\alpha)]^2>0$. Our theorem now follows immediately from our lemma.

We take this opportunity to correct two typographical errors in our original proof. The equations $g'(\alpha)=g''(\alpha)= \cdots =g^{(\nu+1)}(\alpha)=0$ should be replaced by $g'(\alpha)=g''(\alpha)= \cdots =g^{(\nu)}(\alpha)=0$. In the sentence beginning "If ν is even $\cdots$" replace $g^{(\nu)}(x)$ by $g^{(\nu+1)}(x)$.

* An inductive proof of Budan's theorem, this MONTHLY, vol. 49, 1942, pp. 112–113. *Cf.* N. B. Conkwright, An elementary proof of the Budan-Fourier theorem, this MONTHLY, vol. 50, 1943, pp. 603–605.

† Here we use the formula $(fg)'=f'g+fg'$ for *polynomials*. This may be proved *algebraically* as well as by the standard methods of the calculus.

A REPRESENTATION OF INTEGERS

MICHAEL WILENSKY, Cincinnati, Ohio

The solution for q of the two equations

$$q^2 - r = km, \tag{1}$$

$$(m - q)^2 - r = q'^2 - r = k'm, \tag{2}$$

where $(q, m) = 1$; $q < m/2$, $m > q' > m/2$, and therefore $k < m/4$, $k' > m/4$; $|r| < m$ and not a square, gives us $2q = m - (k' - k)$. By substituting this value of q in (1), we obtain the relation

$$m^2 - 2m(k' - k) + (k' - k)^2 - 4r - 4km = 0.$$

The solution of this quadratic equation for m gives us the representation

$$m = k + k' + 2\sqrt{kk' + r}. \tag{3}$$

(The minus sign before the radical should be omitted, for $q \geqq k$, $q' \geqq k'$ and $m = q + q' \geqq k + k'$.) The integer m is thus represented as a function of k, k', and r. Since every pair of integers which satisfies the congruence $x^2 \equiv r \pmod{m}$ leads to a representation (3), there are $2^{\lambda-1}$ representations of an integer m for a given r, if we denote by 2^λ the number of the distinct roots of the congruence. Particularly in case m is an odd prime or double an odd prime, when the congruence has two distinct roots, the representation of m for a given r is *unique*.

It must, however, be borne in mind that two congruent residues modulo m lead to *different* representations of this number, and that *every* r has a congruent quadratic residue within the interval we set for it: $|r| < m$, *viz.*, $r > 0$ and $-(m - r)$. If the representation which corresponds to r has the form of $m = k + k' + 2\sqrt{kk' + r}$, the representation by the other residue will be

$$m = (k + 1) + (k' + 1) + 2\sqrt{(k + 1)(k' + 1) - (m - r)}.$$

Representation (3) characterizes the integer m, for conversely, if an integer m has a representation (3), it has r as a quadratic residue. In fact, multiplying both sides of (3) by k, we obtain

$$(k + \sqrt{kk' + r})^2 - r = km.$$

Comparing this expression with (1) we have

$$q = k + \sqrt{kk' + r}. \tag{4}$$

Similarly multiplication of (3) by k' and comparison with (2) gives us

$$q' = k' + \sqrt{kk' + r}, \tag{4a}$$

so that (3) is the sum of (4) and (4a).

From (3) it is also evident that if m is odd, k and k' are of different parity, and if it is even, both are of the same parity.

The following table illustrates (3) for $r = -1$.

m	q	k	k'	$2\sqrt{kk'-1}$
13	5	2	5	6
65 = 5.13	8	1	50	14
"	18	5	34	26
1105 = 5.13.17	47	2	1013	90
"	242	53	674	378
"	268	65	634	406
"	463	194	373	538

We shall now consider only the special case, $r=-1$. As it is well known, only a number the prime divisors of which are congruent to 1 modulo 4, or double such a number, has minus 1 as a quadratic residue and that such a number can be represented as a sum of two squares $m=a^2+b^2$, where we may suppose $(a, b)=1$. It is also known that one can always find two relatively prime numbers $x\leqq a$ and $y\leqq b$ which satisfy the equation $ay-bx=\pm 1$. Putting $a=x_1+x_2$, $b=y_1+y_2$ and letting x and y assume the values x_2, y_2, respectively, we have $(x_1+x_2)y_2-(y_1+y_2)x_2=\pm 1$, or

$$x_1y_2 - y_1x_2 = \pm 1. \tag{5}$$

Since k and k' have, like m, minus 1 as a quadratic residue, each of them can also be represented as a sum of two squares. We assume $k=x_1^2+y_1^2$, $k'=x_2^2+y_2^2$ and write (3) accordingly

$$m = a^2 + b^2 = (x_1^2 + y_1^2) + (x_2^2 + y_2^2) + 2\sqrt{(x_1^2 + y_1^2)(x_2^2 + y^2) - 1}. \tag{6}$$

Now

$$(x_1^2 + y_1^2)(x_2^2 + y_2^2) = (x_1x_2 + y_1y_2)^2 + (x_1y_2 - y_1x_2)^2.$$

The last expression, according to (5), equals 1. Hence (6) may be written $m=(x_1+x_2)^2+(y_1+y_2)^2$; our assumption is thus proved. Accordingly

$$q = k + \sqrt{kk' - 1} = x_1(x_1 + x_2) + y_1(y_1 + y_2) = ax_1 + by_1, \tag{7}$$

and

$$q' = x_2(x_1 + x_2) + y_2(y_1 + y_2) = ax_2 + by_2. \tag{7a}$$

If x_1 and y_1 are given fixed values α, γ, respectively, so that $k=x_1^2+y_1^2$ is a fixed number, and moreover, if β, δ are the smallest integers x_2, y_2, respectively, which satisfy (5) and (3) or (6) so that $k<m/4$, all integers m having the same k can be represented as a function of t:

$$m = f(t) = (\alpha + \beta + \alpha t)^2 + (\gamma + \delta + \gamma t)^2,$$

for $\alpha(\delta+\gamma t)-\gamma(\beta+\alpha t)=\alpha\delta-\beta\gamma=\pm 1$. Expanding, we have

$$
(8) \quad \begin{aligned} f(t) &= (\alpha^2 + \gamma^2)t^2 + 2[(\alpha^2 + \gamma^2) + \alpha\beta + \gamma\delta]t + (\alpha + \beta)^2 + (\gamma + \delta)^2 \\ &= kt^2 + 2q(0)t + f(0), \end{aligned}
$$

the determinant of which, $[q(0)]^2 - kf(0)$, is indeed minus 1. We can thus represent the entire class of integers having the same k by the binary quadratic form

$$
kt^2 + 2q(0)tu + f(0)u^2,
$$

when α, β, γ, δ are given, in putting $u = 1$ and giving t all positive values, including 0. To every positive value of t corresponds a different value of m.

From (8) we learn that

$$
(9) \qquad f(t + 1) = f(t) + k + 2[kt + q(0)].
$$

Since $kf(t) - 1 = [kt + q(0)]^2$, we can write (9) in the form

$$
(9a) \qquad f(t + 1) = k + f(t) + 2\sqrt{kf(t) - 1}.
$$

Regarding k' as a function of t and comparing (9a) with (3), we have $k'(t+1) = f(t)$. Now the derivative

$$
(10) \qquad f'(t) = 2[kt + q(0)] = 2\sqrt{kf(t) - 1}.
$$

Therefore $f(t+1) = k + f(t) + f'(t)$. Since $[q(t)]^2 = kf(t) - 1$, we have $q(t) = \frac{1}{2}f'(t)$, and

$$
(11) \qquad q(t + 1) - q(t) = \tfrac{1}{2}[f'(t + 1) - f'(t)] = k.
$$

The q of the integers m with the same k thus form an arithmetic progression with the difference k, whereas m, q', and k'—each form a sequence with the second difference $2k$.

This line of reasoning, that r is also a quadratic residue of k, is applicable to every r. An expression of the form $kt^2 + 2q(0)t + f(0)$ with the determinant r represents the class of integers having r as a quadratic residue and k as coefficient.

Let us give some examples for $r = -1$:

Example I. Let $k = 1$, then $\alpha = 0$, $\beta = 1$, $\gamma = 1$, $\delta = 1 + t$; $m = (t+2)^2 + 1 = t^2 + 4t + 5$; $q = \frac{1}{2}f'(t) = t + 2$; $k' = (t+1)^2 + 1$.
Putting $x = t + 2$, we have $m = x^2 + 1$, $x \geqq 2$.

Example II. Let $k = 5$. There are two classes:

1. $\alpha = 1$, $\beta = 1 + t$, $\gamma = 2$, $\delta = 3 + 2t$; $\alpha\delta - \beta\gamma = 1$; $m = (t+2)^2 + (2t+5)^2 = x^2 + (2x+1)^2$, $x \geqq 2$.

2. $\alpha = 1$, $\beta = 2 + t$, $\gamma = 2$, $\delta = 3 + 2t$; $\alpha\delta - \beta\gamma = -1$; $m = (t+3)^2 + (2t+5)^2 = x^2 + (2x-1)^2$, $x \geqq 3$.

RECENT PUBLICATIONS

EDITED BY VIRGIL SNYDER, Cornell University, and H. P. EVANS, University of Wisconsin

All books for review should be sent directly to the editors of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y., and not to any of the other editors or officers of the Association.

Basic Mathematics for War and Industry. By P. H. Daus, J. M. Gleason, and W. M. Whyburn. New York, The Macmillan Company, 1944. 11+277 pages. $2.00.

This text provides in a single volume topics of elementary mathematics which are usually found in separate text-books. The book begins with a short chapter, 28 pages, on arithmetic which includes common and decimal fractions, approximations and significant figures, rules for extracting square root, and mensuration. This is followed by a chapter on each of the following topics: algebra, plane geometry, plane trigonometry, solid geometry, and spherical trigonometry. The book includes a table of squares, cubes, square roots, and cube roots of the integers 1 to 100; a table of four place logarithms, a table of four place trigonometric functions, both natural and logarithmic; and answers to all but a very few of the over 500 exercises which are given in the 59 sets. There is an adequate index as well as a very detailed table of contents. A paper protractor is supplied with each book.

The exposition is surprisingly detailed when one considers how much material is covered in the book. The typography is clear and the numerous figures are excellent. As might be expected in a condensed text, many proofs are omitted and some propositions from geometry are called axioms. However, some of the brevity is attained by presenting clever short proofs and by avoiding the formal superposition and statement-reason types of proof. The illustrative problems and the exercises throughout the book draw upon all fields of applications with special emphasis on aerial and naval navigation, physics, and geometry. An exceptional feature is the use of spherical trigonometry and spherical geometry in the solution of shop problems.

The authors state in the preface, "The book should be a suitable text for courses offered under the Engineering, Science, and Management War Training Program; for the various armed service courses, both within army schools and in colleges; for vocational and other specialized school courses." Furthermore, the reviewer agrees with the authors that the book is "thoroughly practical" and that the arrangement and treatment of the topics would permit selection of the special parts that a course may require.

LOUISE A. WOLF

Intermediate Differential Equations. By E. D. Rainville. New York, John Wiley and Sons, Inc., 1943. 6+213 pages. $2.75.

This text serves a useful purpose in providing a second course in differential equations for students who have finished a first year elementary course. Possibly a more descriptive title would be *Linear Differential Equations of Second Order*, since the treatment, except for Chapter II on the Riccati equation and a brief excursion in Chapter X, is devoted almost exclusively to this type.

The purpose of the text as set forth in Chapter I is three-fold. It is to bring into play certain mathematical tools useful in the solution of linear ordinary differential equations; to train the student to grasp the aims, methods and accomplishments of the theory so that he may proceed to advanced work; and to present material of interest for its own sake. After two chapters in which the linear equation of second order is introduced and attacked in an elementary manner, including an approach through the Riccati equation, the third chapter treats briefly of selected complex variable theory suitable for making use of series solutions. Chapter IV then gives the technique of power series solutions with numerical examples. This leads naturally to a treatment of equations of the Fuchsian type, to the hypergeometric equation and to Kummer's twenty-four solutions.

Chapters VII and VIII develop the method of confluence of singularities with the Fuchsian equation as a point of departure. Series solutions are obtained for Bessel's equation and for Whittaker's confluent hypergeometric equation. Chapter IX treats of some classical equations, and the final chapter gives a few applications, such as: a fourth order equation used in the design of large pipes; a problem of Timoshenko in deformation of circular plates; temperature in an infinite solid cylinder, with a brief word on other boundary value problems.

One question that naturally comes to mind is whether there are other ways of orienting an intermediate course. For example, one could take typical cases of differential equations from the engineering and physical applications, and develop one or more theories around each type case. Such a method might appeal strongly to engineers and physicists. Also, such a scheme might have the added advantage of flexibility: *e.g.*, the instructor could choose to go into operational theory, say (including the more modern phases of it), as far as the capabilities of the students and the allotted time permitted.

The treatment throughout is lucid and well illustrated with problems. The student who has had a first course will find this an excellent preparation for the more weighty discussions of advanced theory. Those who need a better command of differential equations than they obtain in an elementary course, can profit by reading the text and working the problems.

T. M. SIMPSON
B. F. DOSTAL

Principles of Air Navigation. By B. A. Shields. New York and London, McGraw-Hill Book Company, Inc., 1943. 7+451 pages. $2.20.

This book is the third in a series on introductory aviation. It is written primarily to be readable by secondary school students. Yet, at the same time it gives a very interesting account of the intimate problems of pilots and navigators. The range of its content is perhaps best shown by listing the titles of the chapters. These are nine in number, as follows:

1. Maps and Charts.—2. Map Reading.—3. Cross-Country Navigation.—4. Dead-Reckoning Navigation.—5. Practical Navigation Problems.—6. Radius of Action.—7. Computers and Calculators.—8. Radio Navigation.—9. Over-Water Navigation.

The first few chapters deal with the various types of maps, special discussion being given to aviation charts and special aeronautical charts. The interpretation of charts and maps is very carefully considered.

The next few chapters deal with the use of the compass and other navigation instruments in laying out the course, calculating the drift, solving problems of windage, and determining air speed. There then follows a very fine discussion on the factors which limit the radius of action in combat.

Chapter 7 treats primarily of the various types of calculators for solving graphically the usual problems of air navigation.

Chapter 8 is devoted to radio communications, signal systems, range finding, and radio beam navigation.

Chapter 9, which is entitled "Over-Water Navigation," considers only the fundamental aspects of this type of navigation. Many of the problems which have been discussed earlier in the text are here solved on a Mercator plotting sheet. The real problem of celestial navigation is discussed on pages 409–446. This is mainly an explanatory discussion, not complicated by the theory necessary for understanding fully such problems. This latter comment does not infer a lack of thoroughness on the part of the author, but is intended to show that he thoroughly understood the background of his probable readers and did not intend that this book should constitute a text on celestial navigation.

Throughout the book there are numerous reproductions of maps, drawings and photographs of instruments, charts showing the graphical solution of problems, and all of the work of illustrations is remarkably well done. The entire book is remarkably well written, and the reviewer recommends it highly to those who would approach the topic of aviation from a practical and elementary point of view.

G. E. Moore

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND H. S. M. COXETER

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to H. S. M. Coxeter, 24 Strathearn Boulevard, Toronto 10, Canada.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 636. *Proposed by P. R. Halmos, Syracuse University*

What is the least number of links that have to be cut, in a chain of n links, so that every integer between 1 and n can be represented as a sum of numbers of links in the disconnected chains obtained? (Cutting one link which is not at either end of a connected chain produces three pieces: the severed link and the two ends which it held together.)

E 637. *Proposed by V. Thébault, San Sebastián, Spain*

Locate a plane which touches four equal spheres inscribed in the trihedra at the respective vertices of a given tetrahedron.

E 638. *Proposed by C. H. Wolfe, Lakeside High School, Ohio*

Without the use of tables, find the smallest integer whose cube terminates in seven sevens.

E 639. *Proposed by Howard Eves, Syracuse University*

A *clothoid* (or *transition spiral*, used in highway engineering) is defined as a curve whose curvature varies directly with the arc length. Locate the geometrical pole of this spiral.

E 640. *Proposed by E. D. Schell, Arlington, Virginia*

Solve in integers the equation

$$x^y = y^x \qquad (x > y).$$

SOLUTIONS

Arrangements in a Circle

E 599 [1943, 634]. *Proposed by U. P. Davis, University of Florida*

In how many different ways can $3n$ objects be arranged in a circle, if there are n different kinds, 3 of each kind?

I. *Solution by J. B. Kelly, Princeton University.* Let any object of the kth kind be denoted by a_k; $k=1, \cdots, n$. We first determine the number of ways of arranging the $3n$ objects in a row with a particular object a_1 in the initial position; this is

$$\frac{(3n-1)!}{2!(3!)^{n-1}}.$$

Now separate out the $(n-1)!$ permutations of the form

$$a_1a_{i_2}a_{i_3}\cdots a_{i_n}a_1a_{i_2}a_{i_3}\cdots a_{i_n}\cdots a_1a_{i_2}a_{i_3}\cdots a_{i_n}.$$

For every permutation P not of this form there are two others, distinct from P, having a_1 in the initial position, and obtainable from P by cyclic permutation. Hence the required number of permutations is

$$\frac{1}{3}\left(\frac{(3n-1)!}{2\cdot 6^{n-1}} - (n-1)!\right) + (n-1)! = \frac{(3n-1)!}{6^n} + \frac{2}{3}(n-1)!.$$

The corresponding problem with 3 replaced by m is quite difficult, and seems to involve the decomposition of m into factors.

II. *Note by E. P. Starke, Rutgers University.* The required number of arrangements is

$$2\{(n-1)!\}/3 + (3n-1)!/6^n.$$

This is a special case of formulas developed by H. S. Grant in connection with his solution of problem no. 519 in the *National Mathematics Magazine*, April, 1944. He solves there the more general problem of finding the number of cyclic permutations of $x_1+\cdots+x_n$ letters of which x_1 are alike, x_2 others are alike, and so on. The present problem merely requires that we put $x_1=\cdots=x_n=3$.

Also solved by R. G. Blake, D. H. Browne, E. D. Schell, and the proposer. Blake and Browne give the expression

$$(p-1)\{(n-1)!\}/p + (pn-1)!/(p!)^n$$

for the case when $x_1=\cdots=x_n=p$, a prime. Schell refers to R. E. Allardice, *Proceedings of the Edinburgh Mathematical Society*, vol. 8, 1890.

A Square and a Cube with Ten Different Digits

E 602 [1944, 46]. *Proposed by V. Thébault, San Sebastián, Spain*

With the ten different digits, taken once each, form two numbers which are respectively the square and the cube of two consecutive multiples of 3.

Solution by Monte Dernham, San Francisco. In order that exactly ten digits be utilized, the integer to be cubed cannot be less than 48 nor greater than 99. Of the 18 multiples of 3 within this range, the cubes of only five, *viz.*, 66, 69, 75, 84 and 93, consist wholly of unrepeated digits. The first four of these must be discarded because, in each instance, both the square of the preceding and of the succeeding multiple of 3 would, if employed, involve a repetition of digits.

The square of 90 is similarly inadmissible. Hence the only solution is

$$9216 = 96^2, \qquad 804537 = 93^3.$$

Also solved by Colin Blyth, D. H. Browne, W. E. Buker, Walter Penney, E. D. Schell, E. P. Starke, Alan Wayne, and the proposer.

A Symmetrical Set of Five Points

E 605 [1944, 47]. *Proposed by L. M. Kelly, U. S. Coast Guard Academy*

If a set of four coplanar points has the property that the circumcircles of all subsets of three are equal (but not coincident), then the set is orthocentric. (See R. A. Johnson, *Modern Geometry*, p. 75.) Establish the existence of an analogous set of five points in space, *i.e.*, such that the circumspheres of all subsets of four are equal (but not coincident).

Solution by the Proposer. In his solution to E 540 [1943, 389], Eves established the existence of a tetrahedron whose sixteen-point sphere has a radius equal to one-half the circumradius. The vertices of such a tetrahedron, together with the isogonal conjugate of its circumcenter, form a set with the prescribed properties. For, the reflected images of this isogonal conjugate in the four faces can be shown to lie on the circumsphere. This follows from the fact that the images all lie on a sphere whose center is the circumcenter and whose radius is twice that of the sixteen-point sphere. (See N. A. Court, *Modern Pure Solid Geometry*, p. 244.) Thus the four spheres through the isogonal conjugate and the triads of vertices are all equal to the circumsphere.

Homothetic Orthocentric Tetrahedra

E 607 [1944, 88]. *Proposed by V. Thébault, San Sebastián, Spain*

Consider an orthocentric tetrahedron $ABCD$, of orthocenter H. Let O, A', B', C', D' be the circumcenters of the tetrahedra $ABCD$, $BCDH$, $CDAH$, $DABH$, $ABCH$. Prove that the tetrahedra $ABCD$ and $A'B'C'D'$ are homothetic from a center which divides OH in the ratio 3:2. Show also that the lines AA', BB', CC', DD' pass through the centers of gravity of the respective tetrahedra $BCDH$, $CDAH$, $DABH$, $ABCH$.

Solution by the Proposer. Let λ denote the power of H with respect to the circumsphere O. Then, if A'', B'', C'', D'' are the feet of the altitudes, we have

$$HA\cdot HA'' = HB\cdot HB'' = HC\cdot HC'' = HD\cdot HD'' = \lambda/3,$$

whence $OA' = \lambda/2HA'' = \frac{3}{2}HA, \cdots$ (See Thébault, *Annales de la Société Scientifique de Bruxelles*, 1924, p. 176). The tetrahedra $A'B'C'D'$ and $ABCD$ are thus inversely homothetic, in the ratio $-3/2$, and the homothetic center divides OH in the ratio 3/2.

The last part of the problem can be generalized as follows. For any point P, the lines joining the vertices A, B, C, D to the centroids of the respective tetrahedra $BCDP$, $CDAP$, $DABP$, $ABCP$ are concurrent.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4132. *Proposed by T. H. Matthews, McGill University*

If an aircraft travels at a constant airspeed, and traverses (with respect to the ground) a closed curve in a horizontal plane, the time taken is always less when there is no wind, than when there is any constant wind.

4133. *Proposed by A. L. Putnam, Yale University*

Let a, b, and c be integers with $b\neq 0$, and let d and f be the respective greatest common divisors of a and b and c and b. Then if

$$a \not\equiv \pm d \pmod{b} \quad \text{and} \quad d \not\equiv \pm f \pmod{b},$$

there is an infinite number of integers k for which the equation

$$ax + bxy + cy = k$$

has no solutions in integers.

4134. *Proposed by Hüseyin Demir, Columbia University*

Let $C_1^1C_2^1C_3^1$ be the inscribed triangle of a reference triangle $A_1A_2A_3$, and $C_1^2C_2^2C_3^2$ be that of $C_1^1C_2^1C_3^1$, and so on, obtaining a triangle $C_1^nC_2^nC_3^n$ after n steps. Denoting the angles of the nth triangle by C_i^n, prove that

1. $(C_i^n-\pi/3)/(A_i-\pi/3)=(-1)^n2^{-n}$.

2. The limit of the direction of $C_2^nC_3^n$ as $n\to\infty$, is the direction of one of the trisectrices of the angle $(A_2A_3,\ C_2^1C_3^1)$, and from that observe a method of trisecting an angle by ruler and compass in infinitely many steps.

4135. *Proposed by V. Thébault, San Sebastián, Spain*

With the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, taken one at a time, form two numbers of five digits each such that one is a multiple of the other.

SOLUTIONS

A Definite Gamma Integral

4066 [1943, 65]. *Revised. Proposed by Richard Bellman, Brooklyn College*

Prove that

$$\int_0^\infty \frac{dx}{\Gamma(x)} = \int_0^1 \left[1 + \frac{e}{x} - \frac{e}{1!(x+1)} + \frac{e}{2!(x+2)} - \cdots\right]\frac{dx}{\Gamma(x)}.$$

Solution by the Proposer. We have

$$\int_0^\infty \frac{dx}{\Gamma(x)} = \sum_{n=1}^{\infty} \int_{n-1}^{n} \frac{dx}{\Gamma(x)}. \tag{1}$$

In the nth integral, $n \geqq 2$, set $x' = x - (n-1)$, and using the relation $\Gamma(x+1) = x\Gamma(x)$, this integral becomes

$$\int_0^1 \frac{dx'}{x'(x'+1)\cdots(x'+n-2)\Gamma(x')}$$

and we have, after replacing x' by x

$$\int_0^\infty \frac{dx}{\Gamma(x)} = \int_0^1 \left[1 + \frac{1}{x} + \frac{1}{x(x+1)} + \frac{1}{x(x+1)(x+2)} + \cdots\right]\frac{dx}{\Gamma(x)}.$$

Using Prym's identity for the infinite series in the brackets, we obtain the desired result.

Editorial Note. The Prym's identity may be established by the method of finite differences, or by the resolution of the fractions $1/x(x+1)\cdots(x+n)$ into partial fractions. The summation is then performed in a different order, and this change may be justified.

The Euler Function

4068 [1943, 65]. *Proposed by James Singer, Brooklyn College*

If a and n are integers, each greater than unity, and r is any positive integer, and if $\phi(x)$ is the Euler function, show that $\phi(1+a^r+a^{2r}+\cdots+a^{nr})$ is a multiple of $2r(n+1)$.

Solution by N. G. Gunderson, Cornell University. Let $1+a^r+a^{2r}+\cdots+a^{nr} = (a^{r(n+1)}-1)/(a^r-1) = A$. Then by Theorems I and V of the paper, *On the Integral Divisors of a^n-b^n* by G. D. Birkhoff and H. S. Vandiver on page 173 of the Annals of Math., series II, vol. 5, 1903, there exists a prime divisor p of A such that $p \equiv 1 \bmod r(n+1)$, except when $r=n=a=2$. But $2\cdot 2\cdot 3 \mid \phi(1+2^2+2^4)$.

If both r and $n+1$ are odd, $p \equiv 1 \bmod 2r(n+1)$, and so, since $(p-1)\mid \phi(A)$, we have that $2r(n+1) \mid +\phi(A)$. If $n+1$ is odd and r is even, then $A = [(a^{r(n+1)/2}-1)/(a^{r/2}-1)]\,[(a^{r(n+1)/2}+1)/(a^{r/2}+1)]$. If $n+1$ is even, then $A = [(a^{r(n+1/2)}+1)]\,[(a^{r(n+1/2)}-1)/(a^r-1)]$. In each case each of these factors of A is greater that 2, and their greatest possible common divisor is 2. Hence either there exists an odd prime divisor q of A different from p, in which case $r(n+1)(q-1)\mid \phi(A)$, so that $2r(n+1)\mid\phi(A)$, or else $2^\alpha\cdot p \mid A$ where $\alpha>1$, and again $2r(n+1)\mid\phi(A)$.

Note. A paper by Gunderson entitled *Some Theorems on the Euler ϕ-Function* in the Bulletin of the American Mathematical Society, vol. 49, 1943, pp. 278–280 contains the proof of several facts closely related to this problem.

Tetrahedron of Excenters

4069 [1943, 65]. *Proposed by V. Thébault, San Sebastián, Spain*

Let I_a, I_b, I_c, I_d denote the centers of the spheres escribed in the truncated trihedral angles for the corresponding vertices of the tetrahedron $ABCD$, and A_1, B_1, C_1, D_1 the points where the straight lines AI_a, *etc.* meet the faces BCD, *etc.* Show that

$$(1) \qquad V'/V = -ABCD/(S-A)(S-B)(S-C)(S-D),$$

$$(2) \qquad V_1/3V = -ABCD/(B+C+D)(C+D+A)(D+A+B)(A+B+C),$$

where V, V', V_1 denote the volumes of the tetrahedrons $ABCD$, $I_aI_bI_cI_d$, $A_1B_1C_1D_1$; A, *etc.* denote the areas of the faces BCD, *etc.* and $2S=A+B+C+D$.

Note. The formula (2) is due to Genty, *Nouvelles Annales de Mathématiques*, 1880, p. 528 and 1881, p. 341.

Editorial Note. The proposer gave the following information: The straight lines AI_a, BI_b, CI_c, DI_d are concurrent in I, the incenter of the tetrahedron $ABCD$; and, since the vertices of the latter are external centers of similitude of the pairs of the inscribed sphere (I) with the escribed spheres (I_a), (I_b),)I_c), (I_d), we have four equations such as

$$(1) \qquad II_a = -\frac{(r_a - r)}{r} IA,$$

where r, r_a, r_b, r_c, r_d are the respective inradius and four exradii. There are four equations such as

$$(2) \qquad \frac{\text{vol. } IBCD}{\text{vol. } II_bI_cI_d} = -\frac{IB\cdot IC\cdot ID}{II_b\cdot II_c\cdot II_d},$$

which lead to four corresponding equations

$$(3) \qquad V_a = \text{vol. } II_bI_cI_d = -8A\cdot B\cdot C\cdot Drr_br_cr_d/81V^3,$$

where $V' = V_a + V_b + V + V_d$. Then

$$(4) \qquad V' = -16A\cdot B\cdot C\cdot Dr_ar_br_cr_d/81V^3,$$

from which the desired equation for V'/V results.

Since the lines AI, BI, CI, DI pass respectively through A_1, B_1, C_1, D_1, it suffices to proceed similarly to obtain

$$(5) \qquad V_1/3V = -A\cdot B\cdot C\cdot D/(2S-A)(2S-B)(2S-C)(2S-D).$$

which is the second desired result. The equality (5), as stated in the problem

note, is due to Genty; and it was deduced in a manner different from the above by Faure, *l. c.* 1881, p. 341.

Since the derivations are rather tedious if carried out in a random manner, we shall give some formulas which may be used to shorten the labor:

$$3V = 2rS = 2r_a(S - A) = 2r_b(S - B) = \cdots;$$

$$\text{(6)} \qquad \frac{r - r_a}{r} = -\frac{2}{3}\frac{Ar_a}{V}; \qquad \frac{II_a}{IA} = -\frac{2}{3}\frac{Ar_a}{V} = -\frac{A}{S - A};$$

$$\text{vol. } IBCD = rA/3.$$

The first line of equations give the known result

$$\frac{2}{r} = \frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} + \frac{1}{r_d}.$$

The equations (6) with (2) give

$$V_a = -\frac{A\cdot B\cdot C\cdot Dr(S - A)}{3(S - A)(S - B)(S - C)(S - D)},$$

and from this we find at once the desired result for V'/V.

We now consider V_1 and we have equations of the type

$$\frac{IA_1}{IA} = \frac{r - r_a}{r + r_a} = -\frac{A}{(2S - A)};$$

$$\frac{\text{vol. } IB_1C_1D_1}{\text{vol. } IBCD} = \frac{IB_1\cdot IC_1\cdot ID_1}{IB\cdot IC\cdot ID} = -\frac{B\cdot C\cdot D}{(2S - B)(2S - C)(2S - D)};$$

$$\text{vol. } IB_1C_1D_1 = -\frac{rA\cdot B\cdot C\cdot D(2S - A)}{3(2S - A)(2S - B)(2S - C)(2S - D)}.$$

This gives at once the desired expression for $V_1/3V$.

The triangle ABC may be considered in the same way. If S and S' denote the areas of the triangle ABC and of $I_aI_bI_c$, we have

$$\frac{S'}{S} = \frac{abc}{2(s - a)(s - b)(s - c)}, \qquad 2s = a + b + c,$$

where a, b, c are lengths of the sides. This gives

$$S' = 2Rs,$$

where R is the circumradius.

Orthological Tetrahedrons

4075 [1943, 203]. *Proposed by N. A. Court, University of Oklahoma*

If the radical center of four spheres coincides with the Monge point of the tetrahedron (T) determined by their centers, the tetrahedron (S) formed by the

four radical planes of the given spheres with their orthogonal sphere is orthological to the twin tetrahedron (T') of (T) (*i.e.*, the perpendiculars dropped from the vertices of (S) upon the corresponding faces of (T') are concurrent).

Solution by the Proposer. The radical plane α of the given sphere (A) and the orthogonal sphere (M) of the four given spheres (A), (B), (C), (D) is perpendicular to the line AM joining the centers A, M of (A), (M). Now if M is the Monge point of the tetrahedron $(T)=ABCD$, the line AM is parallel to the line OA' joining the circumcenter O of (T) to the vertex A' of the twin tetrahedron (T') which corresponds to the vertex A of (T).

Thus the perpendicular from A' upon α passes through (O), and it may be shown in a like manner that the same holds for the other vertices of (T'). Hence the perpendiculars from the vertices of the tetrahedron (S) upon the corresponding faces of (T') are also concurrent (see the proposer's *Modern Pure Solid Geometry*, p. 147, art. 460. Macmillan, 1935).

Editorial Note. Since T' is the twin tetrahedron for T, their common centroid G is their homothetic center with ratio $(1:-1)$. Let (M) be any chosen sphere with arbitrarily chosen center M and radius r; then the polar reciprocal of T with respect to (M) is a tetrahedron S, and T and S are orthological with M as the common point of concurrency of the two sets of perpendiculars. In the homothetic transformation $T\leftrightarrow T'$, $S\leftrightarrow S'$ with corresponding faces parallel, and $(M)\leftrightarrow(M')$. Hence the perpendiculars from the vertices of T' to corresponding faces of S are concurrent in M', whereas those from vertices of S to corresponding faces of T' are concurrent in M. For the special case where M is the Monge point of T, the point M' is the circumcenter of T.

Prime Integers

4083 [1943, 330]. *Proposed by P. Erdös, Princeton, N. J.*

Let $a_1<a_2< \cdots <a_x\leqq n$ be an arbitrary sequence of positive integers such that no a_i divides the product of the others, then $x\leqq\pi(n)$, where $\pi(n)$ denotes the number of primes not exceeding n.

Solution by Whitney Scobert, University of Oregon. Any number $\leqq n$ is a product of powers of one or more of the $\pi(n)$ primes which are $\leqq n$. In order that this number not divide the product of the others it is necessary that it have at least one prime factor raised to a higher power than the sum of the powers to which it is raised in all of the other numbers of the sequence. This implies that it must be to a higher power than it is in any other number of the sequence. Since any prime can have its greatest power in only one of the numbers this allows for at most $\pi(n)$ numbers in the sequence.

In this connection, there is a statement on page 4 of Hardy and Wright's *Theory of Numbers* which has puzzled me for several years. The book states that there are 50,847,478 primes less than one billion and that these are not individually known. I would like to have a reference to the method of obtaining

this figure. It is used on page 9 to check the asymptotic prime number theorem so that evidently it was not found by the theorem.

Solved also by S. H. Gould, N. G. Gunderson, J. B. Kelly, G. Pall, and the proposer.

Editorial Note. The proposer's solution is similar to the above, and the remaining solutions use different methods. Can some reader supply the desired reference?

Regular Polygons

4086 [1943, 391]. *Proposed by P. Erdös, Princeton, N. J.*

Let $A_1, A_2, \cdots, A_{2n+1}$ be the vertices of a regular polygon, and O any point in its interior. Show that at least one of the angles A_iOA_j satisfies the relation.

$$\pi\left(1 - \frac{1}{2n+1}\right) \leqq A_iOA_j < \pi.$$

Editorial Note. R. K. Allen, L. M. Kelly, and C. R. Phelps state that the theorem of the problem is incorrectly stated as is shown by the regular pentagon. The point O is taken as the intersection of diagonals A_1A_3 and A_2A_4. The greatest angle $A_iOA_j<\pi$ is given by $A_5OA_3=126°$, which is less than the required 144°.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to B. W. Jones, White Hall, Cornell University, Ithaca, New York.

Dr. R. E. von Mises of Harvard University and Professor John von Neumann of the Institute for Advanced Study have been elected to membership in the American Academy of Arts and Sciences.

Professor Oscar Zariski of The Johns Hopkins University has been elected to membership in the National Academy of Sciences.

Dr. O. S. Adams of the U. S. Coast and Geodetic Survey has retired.

Dr. A. E. Basch of the College of the City of New York has been appointed to an assistant professorship in Mechanics at Rensselaer Polytechnic Institute.

Dr. Helen P. Beard of Newcomb College has been promoted to an assistant professorship.

Dr. R. A. Beaumont of the University of Washington has been promoted to an assistant professorship.

Assistant Professor E. M. Beesley of the University of Nevada has been promoted to an associate professorship. He is acting head of the department.

Assistant Professor J. R. Britton of the University of Colorado has been promoted to an associate professorship.

Dr. R. H. Bruck of the University of Wisconsin has been promoted to an assistant professorship.

Associate Professor J. W. Cell of North Carolina State College has been granted a leave of absence to serve as a mathematician at Aberdeen Proving Grounds.

Dr. Esther Comegys of the University of Maine has been promoted to an assistant professorship.

W. S. Dawkins has been appointed to a professorship at Loyola College, Baltimore, Maryland.

Assistant Professor L. A. Dye of The Citadel, Charleston, South Carolina, has been promoted to an associate professorship.

Dr. C. J. Everett of the University of Wisconsin has been promoted to an assistant professorship.

Dr. Mariano García of the University of Richmond has been appointed to an assistant professorship at the College of Agriculture and Mechanic Arts of the University of Puerto Rico.

Dr. Hilda P. Geiringer of Bryn Mawr College has been appointed to a professorship at Wheaton College, Norton, Massachusetts.

Professor G. A. Hedlund of the University of Virginia has been granted a leave of absence to serve with the applied mathematics group at Columbia University.

Assistant Professor S. A. Jennings of the University of British Columbia has been promoted to an associate professorship.

Dr. Wilfred Kaplan of the University of Michigan has been promoted to an assistant professorship.

Professor N. J. Lennes of Montana State University has retired.

Assistant Professor Norman Levinson of Massachusetts Institute of Technology has been promoted to an associate professorship.

Assistant Professor Eugene Lukacs of Illinois College has been appointed to an associate professorship at Berea College.

Assistant Professor M. S. Macphail of Acadia University has been promoted to an associate professorship.

Dr. C. T. McCormick of Fort Hays Kansas State College has been appointed to an associate professorship at the Illinois State Normal University.

Assistant Professor W. C. McDaniel of Southern Illinois Normal University has been promoted to an associate professorship.

Assistant Professor A. N. Milgram of the University of Notre Dame has been promoted to an associate professorship.

Assistant Professor David Moskovitz of the Carnegie Institute of Technology has been promoted to an associate professorship.

Assistant Professor D. C. Murdoch of the University of Saskatchewan has been appointed to an associate professorship at the University of British Columbia.

Assistant Professor Abba V. Newton of Smith College has been appointed to an assistant professorship at Vassar College.

Dr. K. L. Nielsen of Louisiana State University has been promoted to an assistant professorship. He is on leave with the Chance Vought Aircraft Research Division.

Assistant Professor H. L. Olson of Southwestern University, Georgetown, Texas, has been appointed head of the mathematics department at Western Union College, Le Mars, Iowa.

Assistant Professor E. L. Post of the College of the City of New York has been promoted to an associate professorship.

Associate Professor W. T. Reid of the University of Chicago has been appointed to a professorship at Northwestern University.

Associate Professor L. V. Robinson of Marshall College has been appointed to an associate professorship at Emory University, Georgia.

Assistant Professor W. J. Robinson of Allegheny College has been appointed to a professorship at Centre College of Kentucky.

Assistant Professor R. G. Sanger of the University of Chicago has been appointed assistant dean of students of the Division of the Physical Sciences.

Dr. Seymour Sherman of Curtiss Wright Corporation has been appointed research associate in the Allegany Ballistics Laboratory at Cumberland, Maryland.

Professor Gertrude Smith of Vassar College has retired with the title professor emeritus.

Dr. R. G. Sturm has been appointed to a professorship of engineering mechanics at Purdue University.

Assistant Professor E. G. Swafford of Fort Hays Kansas State College has been granted a leave of absence to accept a temporary appointment to an assistant professorship at Park College, Parkville, Missouri.

Assistant Professor E. C. Varnum of the University of Wyoming has been appointed mathematical physicist in the development laboratory of the Barber-Colman Company of Rockford, Illinois.

Assistant Professor J. F. Wardwell of Colgate University has been promoted to an associate professorship.

Professor W. M. Whyburn of the University of California at Los Angeles has been appointed to the presidency of Texas Technological College.

Dr. Frantisek Wolf of the University of California has been promoted to an assistant professorship.

Dr. Alexander Wundheiler of the College of the City of New York has been appointed to an assistant professorship in physics at the University of Rochester.

The following appointments to instructorships are announced:

Columbia University: P. B. Norman
Hollins College (instructor in education): Frances E. Falvey
Rice Institute: Dr. Albert Newhouse
Russell Sage College: Dr. Helene Reschovsky
Tulane University: Dr. R. L. Swain, Dr. Bernard Vinograde, Dr. Nelson Van de Luyster
Wellesley College: Ellen E. Fedder
Williams College: Dr. Abraham Seidenberg

Professor Emeritus F. G. Reynolds of the College of the City of New York died June 8, 1944. He was a charter member of the Mathematical Association.

WAR INFORMATION

EDITED BY C. V. NEWSOM

Send news reports upon the utilization of mathematicians or mathematics in war activities to C. V. Newsom, Oberlin College, Oberlin, Ohio.

THE ASBURY PARK PRE-V-12 PROGRAM

Previous to the term of the Navy V-12 Program starting July 1, 1944, approximately 2,500 men, who had been on active duty in the Navy, reported to the U. S. Naval Reserve Pre-Midshipmen's School at Asbury Park, New Jersey,

for refresher courses in mathematics, English, and physics. The Pre-V-12 curriculum was designed for a period of two months, and classes were to be held three times a week in each of the three subjects. In general, it was anticipated that there would be a minimum of two hours of preparation for each hour of class work. Many of the men were unable to arrive for the start of the program on May 1; in fact, the average period of refresher training proved to be about six weeks.

No grades were assigned, and no screening was done by the academic officers in charge of the program. The motivation was simply: "Here is an opportunity to learn, or re-learn, as much as you can before you go to college." The great majority of men responded well to this approach, and seemed to profit from the opportunity given them.

Virtually all men in the Pre-V-12 Program were given the C-2 and the C-3 tests developed by the Navy Test Research Unit at Princeton; these are the same tests which have been given to civilian applicants for V-12. The C-2 test was given to the men upon their arrival at Asbury Park, and the C-3 test was given at the end of the course of study. It is noteworthy that those trainees who were able to be present for virtually the entire program were able to improve their scores upon the mathematics sections of these tests by an average of 10.5 points; the total score possible in mathematics is 60 points. Even the large group of men arriving after May 17 made an average improvement of 8.1 points. On the basis of the showing made at Asbury Park, a report was drawn up for each trainee recommending V-12 placement in mathematics, English, and physics.

The program is to be repeated this fall previous to the term starting November 1. It is planned again to spend about two months upon refresher training; moreover, the general character of the curriculum will not be changed. Approximately 900 men have been assigned to Asbury Park for the second program.

Lieutenant Jackson B. Adkins, USNR, is head of the mathematics department of the Pre-Midshipmen's School at Asbury Park, and he organized the mathematics curriculum in the Pre-V-12 Program. All texts employed are reprints made for the United States Armed Forces Institute.

Early in the program, it is emphasized to trainees that the refresher course in mathematics has two major objectives, namely, "(1) to enable the student to acquire in an intelligent fashion the vocabulary of the subject, (2) to help the student to develop a skill in the technical processes that will be sufficient for the V-12 courses to build upon." The meaning of these phrases to the student is explained by the syllabus of the course in the following paragraphs.

"The first objective implies more than could be acquired by studying a dictionary of the terms. It means that the processes of the subject will be described in words so that an understanding of them can be acquired. By this procedure the student will not learn, for example, that $2x-3=5$ is solved by 'transposing' the 3 and dividing by 2, but will learn instead: 'Add 3 to both sides, then divide both sides of the equation by 2, the coefficient of the unknown term.' This

apparently cumbersome procedure is the only way to acquire an intelligent understanding that will be useful in successive lessons. Nothing more needs to be learned for more complicated problems. Nothing needs to be unlearned. The student will defeat himself unless he learns the words that describe the processes. The importance of 'talking' the subject cannot be overemphasized.

"Skill in the technical processes cannot be acquired until the processes are understood in the way indicated in the first objective. This means that the correct way to work on the technical processes is to 'make haste slowly.' Great skill and facility will not be acquired by some students in this brief course. But, if the course is properly studied, all students will be able to *do* any of the processes involved in the course. They may not be able to do them very rapidly, but they *will* be able to *do* them, and to get the right answers."

The course itself has been organized in twenty lessons, starting with simple arithmetic and ending with a brief study of the conic sections. A student assignment sheet of considerable detail has been developed for each lesson. Also, instructor's sheets pertaining to the various lessons were given to each Pre-V-12 instructor. An outline of the course is given below.

1. The Fundamental Operations with Decimal Fractions. How to read a decimal; place-value notation; sum; remainder; difference; product; multiplicand; multiplier; quotient; divisor; dividend; fundamental principle of fractions; multiples of 10; meaning of percent; rounding off; involution; evolution; square root table; order of operations; parentheses.

2. Common Fractions. Fundamental principle of fractions; meaning of fractions; prime number; factor; factoring; rules for reducing to lowest terms; adding, subtracting, dividing, multiplying; proper and improper fractions; mixed number; percent; ratio.

3. Algebraic Notation and Language. A study of $A=bh$; function; constant; variable; ratio; direct and inverse variation; locus; graphs; coordinates; axes; directed numbers; survey of operations with directed numbers.

4. The Fundamental Operations with Directed Numbers. Literal numbers; binomials; trinomials; polynomials; exponents; powers of signed numbers; special products; long division of polynomials.

5. Simple Equations. Rule for solving linear equations; literal equations; percentage problems.

6. Factoring. Term *vs* factor; a^2-b^2; $(a\pm b)^2-c^2$; ax^2+bx+c; long division; primes; perfect square and perfect cube factors.

7. Fractions. The fundamental operations; binomial denominators; the signs of a fraction; complex fractions.

8. Fractions. Fractional equations with binomial denominators; literal fractional equations; ratio; reciprocal.

9. Powers, Roots, and Radicals. Fundamental operations; binomial denominators; rationalizing the denominator; arithmetical square root.

10. Powers, Roots, and Radicals. Checking roots of $ax^2+bx+c=0$; negative exponents; inequalities.

11. Quadratic Equations. Completing the square; formula; factoring.
12. Equations. Final generalization of procedure.
13. Sets of Equations. Elimination by substitution.
14. The Right Triangle. Sum of angles; inscription in a circle; altitude on hypotenuse; Pythagorean Theorem; area; Hero's Formula.
15. The Measurement of Plane and Solid Figures. Polygons; circle; cylinder; cone; sphere; area; volume; variation; functions.
16. Numerical Trigonometry. Tables of natural functions.
17. Logarithms.
18. Trigonometric Tables.
19. The Oblique Triangle. Law of sines; law of cosines.
20. Conic Sections.

RECIRCULARIZATION OF REGISTRANTS IN THE NATIONAL ROSTER

The National Roster of Scientific and Specialized Personnel is now over four years old. From its inception, one of its procedures has been the continued recircularization of all registrants in its files in order to keep the file up to date with regard to addresses and added fields of specialization that any individual registrant may wish to indicate. The recent recircularization is a part of this continuing policy. Mathematicians not yet registered with the Roster, including recent graduates, should take steps to have their names included in the record.

Both the Army and the Navy have adopted the Roster's numerical coding system for describing the qualifications of officers and certain enlisted personnel in their Services. This fact, together with the continued vital demand for scientists in certain areas in connection with the war, make it seem extremely important that the records of the Roster be kept exactly up to date.

There are, of course, many thousands of individuals in the Armed Forces with professional and scientific competencies of a high order. The Roster has been requested to participate in the development of procedures which will assist these individuals in returning to civilian life when full demobilization begins. In this connection, the Roster is already working upon the problem of the most effective placement of returning service men who are registered with it.

THE SECTION ON EDUCATION IN THE G.I. BILL

To assist mathematicians who may be giving consideration to postwar plans, important parts of the education section of the G.I. Bill (S. 1767), as finally passed, are quoted below.

"Any person who served in the active military or naval service on or after September 16, 1940, and prior to the termination of the present war, and who shall have been discharged or released therefrom under conditions other than dishonorable, and whose education or training was impeded, delayed, interrupted, or interfered with by reason of his entrance into the service, or who desires a refresher or retraining course, and who either shall have served ninety days or more, exclusive of any period he was assigned for a course of education

or training under the Army specialized training program or the Navy college training program, which course was a continuation of his civilian course and was pursued to completion, or as a cadet or midshipman at one of the service academies, or shall have been discharged or released from active service by reason of an actual service-incurred injury or disability, shall be eligible for and entitled to receive education or training under this part: Provided, That such course shall be initiated not later than two years after either the date of his discharge or the termination of the present war, whichever is the later: Provided further, That no such education or training shall be afforded beyond seven years after the termination of the present war: And provided further, That any such person who was not over 25 years of age at the time he entered the service shall be deemed to have had his education or training impeded, delayed, interrupted, or interfered with.

"Any such eligible person shall be entitled to education or training, or a refresher or retraining course, at an approved educational or training institution, for a period of one year (or the equivalent thereof in continuous part-time study), or for such lesser time as may be required for the course of instruction chosen by him. Upon satisfactory completion of such course of education or training, according to the regularly prescribed standards and practices of the institutions, except a refresher or retraining course, such person shall be entitled to an additional period or periods of education or training, not to exceed the time such person was in the active service on or after September 16, 1940, and before the termination of the war, exclusive of any period he was assigned for a course of education or training under the Army specialized training program or the Navy college training program, which course was a continuation of his civilian course and was pursued to completion, or as a cadet or midshipman at one of the service academies, but in no event shall the total period of education or training exceed four years.

"Such person shall be eligible for and entitled to such course of education or training as he may elect, and at any approved educational or training institution at which he chooses to enroll whether or not located in the State in which he resides, which will accept or retain him as a student or trainee in any field or branch of knowledge which such institution finds him qualified to undertake or pursue: Provided, That, for reasons satisfactory to the Administrator, he may change a course of instruction: And provided further, That any such course of education or training may be discontinued at any time, if it is found by the Administrator that, according to the regularly prescribed standards and practices of the institution, the conduct or progress of such person is unsatisfactory.

"From time to time the Administrator shall secure from the appropriate agency of each State a list of the educational and training institutions (including industrial establishments), within such jurisdiction, which are qualified and equipped to furnish education or training (including apprenticeship and refresher or retraining training), which institutions, together with such additional ones as may be recognized and approved by the Administrator, shall be deemed

qualified and approved to furnish education or training to such persons as shall enroll under this part.

"The Administrator shall pay to the educational or training institution, for each person enrolled in full time or part time course of education or training, the customary cost of tuition, and such laboratory, library, health, infirmary, and other similar fees as are customarily charged, and may pay for books, supplies, equipment, and other necessary expenses, exclusive of board, lodging, other living expenses and travel, as are generally required for the successful pursuit and completion of the course by other students in the institution: Provided, That in no event shall such payments, with respect to any person, exceed $500 for an ordinary school year: And provided further, That if any such institution has no established tuition fee, or if its established tuition fee shall be found by the Administrator to be inadequate compensation to such institution for furnishing such education or training, he is authorized to provide for the payment, with respect to any such person, of such fair and reasonable compensation as will not exceed $500 for an ordinary school year.

"While enrolled in and pursuing a course under this part, such person, upon application to the Administrator, shall be paid a subsistence allowance of $50 per month, if without a dependent or dependents, or $75 per month, if he has a dependent or dependents, including regular holidays and leave not exceeding thirty days in a calendar year.

"No department, agency, or officer of the United States, in carrying out the provisions of this part, shall exercise any supervision or control, whatsoever, over any State educational agency, or State apprenticeship agency, or any educational or training institution: Provided, That nothing in this section shall be deemed to prevent any department, agency, or officer of the United States from exercising any supervision or control which such department, agency, or officer is authorized by existing provisions of law, to exercise over any Federal educational or training institution.

"The Administrator of Veterans' Affairs is authorized and empowered to administer this title, and, insofar as he deems practicable, shall utilize existing facilities and services of Federal and State departments and agencies on the basis of mutual agreements with them.

"The Administrator may arrange for educational and vocational guidance to persons eligible for education and training under this part. At such intervals as he deems necessary, he shall make available information respecting the need for general education and for trained personnel in the various crafts, trades, and professions: Provided, That facilities of other Federal agencies collecting such information shall be utilized to the extent he deems practicable.

"As used in this part, the term 'educational or training institutions' shall include all public or private elementary, secondary, and other schools furnishing education for adults, business schools and colleges, scientific and technical institutions, colleges, vocational schools, junior colleges, teachers colleges, normal schools, professional schools, universities, and other educational institutions."

THE MATHEMATICAL ASSOCIATION OF AMERICA

THE ANNUAL MEETING OF THE NEBRASKA SECTION

The twenty-first meeting of the Nebraska Section of the Mathematical Association of America was held at the University of Nebraska in Lincoln, Nebraska, on Saturday, May 6, 1944. Professor W. A. Dwyer, Chairman of the Section, presided.

The attendance was twenty-one, including the following fifteen members of the Association: M. A. Basoco, A. K. Bettinger, W. C. Brenke, C. C. Camp, A. R. Congdon, H. M. Cox, W. A. Dwyer, J. M. Earl, M. G. Gaba, C. B. Gass, F. S. Harper, W. N. Huff, H. W. Linscheid, H. L. Rice, Lulu L. Runge.

At the business meeting the following officers were elected for the next year: Chairman, F. S. Harper, University of Nebraska; Secretary, Lulu L. Runge, University of Nebraska; Member of Executive Committee, W. A. Dwyer, Creighton University. It was voted to hold the next meeting at Lincoln, Nebraska, on a date to be announced later.

The following papers were presented:

1. *A class of orthogonal polynomials*, by Professor W. C. Brenke, University of Nebraska.

A class of polynomials $y_n(x)$ was considered as derived from the generating function $e^t f(xt)$, where $f(\theta)$ can be expanded as a formal power series in θ without missing terms. It was shown that, if the polynomials $y_n(x)$ form an orthogonal system, the reciprocals of the coefficients of θ in this power series form a "moment sequence." The result was extended to the more general type of generating function $\theta(t)f(xt)$.

2. *Particular instances of the uniform convergence of power series on an infinite interval*, by Professor J. M. Earl, Municipal University of Omaha.

Professor Earl remarked that if $f(x)$ is a given function of x whose power series expansion converges at each point of the infinite interval $(-\infty, \infty)$, if $S_n(x)$ is the nth partial sum of that series, and if n is fixed, then there usually exists an x such that the magnitude of the difference $f(x)-S_n(x)$ exceeds any preassigned positive number. On the other hand, if $r(x)$ is a certain non-negative weight function whose properties depend on those of $f(x)$, it can be shown that

$$r(x)\,|\,f(x) - S_n(x)\,| < \epsilon_n$$

uniformly on the interval $(-\infty, \infty)$, where ϵ_n depends on n but not on x, and tends toward zero as n becomes infinite. The first member of the inequality above is called the "weighted error."

Upper bounds ϵ_n for a few familiar functions and series were exhibited, and the order of the weighted uniform convergence was obtained. For example, if $r(x)=e^{-x^2}$, $f(x)=e^x$, and the series is the Maclaurin series for e^x, then the upper

bound ϵ_n for the weighted error on the infinite interval $(-\infty, \infty)$ is a constant multiple of e^{-n}.

3. *A theorem on unit groups of simple algebras*, by Professor Ralph Hull, University of Nebraska.

This paper was read by title.

4. *Problem in Diophantine analysis*, by Professor O. C. Collins, University of Nebraska, introduced by Professor W. A. Dwyer.

In this paper it was pointed out that a positive integer n can be represented in the form a^2-ab+b^2, where a and b are positive integers, only if $4n-3(a-b)^2$ is a perfect square. By consideration of $f(x+r)$, where $f(x)=x(x-a)(x-b)$, it was shown that the number of distinct representations is less than $\sqrt{n}/3$.

5. *A program of high school and college testing*, by Professor H. M. Cox, University of Nebraska.

The speaker stated that achievement examinations in mathematics and English were given this spring in one hundred and thirty Nebraska high schools to more than six thousand pupils. Ten colleges in Nebraska cooperated in the use of one or more guidance examinations administered as a part of the registration procedure for new students. The results of these examinations in high schools and colleges can be expressed in terms of *scaled scores* which represent approximately equal linear displacements in a normalized distribution. Thus the achievement of a student can be compared not only with the achievement of others in his own school and class, but also with the normative distribution composed of entering freshmen at the University of Nebraska.

6. *Correlation between machine grades and others*, by Professor C. C. Camp, University of Nebraska.

Professor Camp found that, for A.S.T.P. students in analytics, integral calculus, and algebra, the correlation between machine grades and others ranged from 0.41 to 0.72. Multiple choice tests were compared with each other, with final G. I. examinations, with class grades, or with term grades independent of such tests. In six of these comparisons 225 or more students were involved. In integral calculus, the grades of 145 students were studied. A limited number of civilian classes were also studied, and the mean of the correlation from quiz to quiz was 0.52. The longer the test and the more advanced the students, the higher the correlation was found to be.

7. *Panel discussion: Review of the army program*, by Professors W. C. Brenke, A. K. Bettinger, F. S. Harper, and A. R. Congdon.

8. *Panel discussion: Mathematics for returned service men*, by Professors M. G. Gaba and H. M. Cox.

9. *A photo-electric photometer of novel design*, by Professor O. C. Collins, University of Nebraska.

LULU L. RUNGE, *Secretary*

THE ANNUAL MEETING OF THE KENTUCKY SECTION

The twenty-seventh annual meeting of the Kentucky Section of the Mathematical Association of America was held at the University of Kentucky, Lexington, Kentucky, on Saturday, April 29, 1944, in conjunction with the annual meeting of the Kentucky Academy of Science. Professor Charles Hatfield, Chairman of the Section, presided.

There were twenty in attendance, including the following eleven members of the Association: P. P. Boyd, M. C. Brown, H. H. Downing, Charles Hatfield, Aughtum S. Howard, W. R. Hutcherson, C. G. Latimer, F. Elizabeth LeStourgeon, Sallie E. Pence, D. E. South, H. A. Wright.

At the business meeting the following officers were elected for the coming year: Chairman, Aughtum S. Howard, Kentucky Wesleyan College; Secretary, M. C. Brown, University of Kentucky.

The following program was presented:

1. *Algebraic solutions of linear differential equations of the first order with linear coefficients*, by Professor H. H. Downing, University of Kentucky.

In this discussion the solutions of the differential equation

$$(ax + by + c)dx + (a'x + b'y + c')dy = 0$$

were obtained by the usual methods, and conditions in terms of the coefficients a, b, c, a', b', c' were found for which the solutions could be reduced to rational integral functions of x and y. Three cases were treated, namely: (1) $a/a' = b/b' = c/c'$; (2) $a/a' = b/b' \neq c/c'$; (3) $a/a' \neq b/b'$. Conditions in terms of the coefficients were found for which the algebraic solutions would reduce to quadratic form.

2. *The Tchebycheff inequality*, by Professor D. E. South, University of Kentucky.

Professor South discussed the proof of the Tchebycheff inequality for discrete and continuous variates. The Bernoulli limit theorem was then derived from the Tchebycheff criterion.

3. *Linkages*, by Dean P. P. Boyd, University of Kentucky.

Dean Boyd discussed the history of linkages from the time of Scheiner, the various linkages designed to approximate the drawing of a straight line, and the discoveries of Peaucellier and Hart. Emch's work in his *Kinematische Gelenksysteme* demonstrating the possibility of realizing a collineation by means of linkages was also described.

4. *Mathematics in the navy V-12 program*, by Professor W. R. Hutcherson, Berea College.

The speaker explained that every freshman in the program is required to take ten credit hours of mathematics. This includes algebra, trigonometry, and analytic geometry. Practically all sophomores (except premedics, predentals, and prechaplains) are required to take additional mathematics. Berea College

requires the weaker students to attend a remedial class in mathematics. Students, faculty, and navy officials have found that this extra work was justified by the results.

5. *Code writing*, by Professor Sallie E. Pence, University of Kentucky.

This paper dealt with the two principal methods of code writing, namely the substitutional and the transpositional methods. Examples of the four substitutional and three transpositional types were given, with some mention of the methods used in attempting to decipher them.

6. *Codes and matrices*, by Professor C. G. Latimer, University of Kentucky.

The speaker described certain ciphers based upon linear homogeneous transformations of n-tuples, with particular reference to involutory transformations. The connections between this paper and the preceding one were pointed out.

M. C. BROWN, *Secretary*

CALENDAR OF FUTURE MEETINGS

Twenty-eighth Annual Meeting, Chicago, Ill., November 24–26, 1944.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN
ILLINOIS
INDIANA, Indianapolis, November 10, 1944
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA
NORTHERN CALIFORNIA, San Francisco, January 27, 1945
OHIO, Columbus, April 5, 1945
OKLAHOMA
PHILADELPHIA, Philadelphia, November, 1944
ROCKY MOUNTAIN
SOUTHEASTERN
SOUTHERN CALIFORNIA, Los Angeles, March 10, 1945
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE
WISCONSIN, Milwaukee, May, 1945

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THE FOUNDATIONS OF PROBABILITY

P. R. HALMOS, Syracuse University

1. Introduction. Probability is a branch of mathematics. It is not a branch of experimental science nor of armchair philosophy, it is neither physics nor logic. This is not to say that the experimenter and the philosopher should not discuss probability from their points of view. They should, and they do. The situation is analogous to that in geometry. No one denies that the physicist and the philosopher have made valuable contributions to our understanding of the space concept, nor, in spite of this, that geometry is a rigorous part of modern mathematics.

Like Euclidean geometry, and for that matter like most mathematical theories, probability has four aspects: axiomatization, development, coordinatization, and application. We proceed to explain our use of these words.

"Axiomatization" is clear. We all know that the study of geometry begins with a list of undefined terms and a list of postulates. It is important in this connection to remember two facts. First: the selection of the list of terms and postulates is not entirely arbitrary, but is derived only after a thorough examination of our intuitive notions of the subject. Second: the selection of terms and postulates is not uniquely determined. When several different axiomatizations of the same subject exist then only extra mathematical considerations, such as practical convenience or personal prejudice, can lead us to prefer one among the many. The greater part of this paper is devoted to a prepostulational examination of probability. The axiomatic system to which this examination leads is not the only possible approach to probability, but it is the approach which has been adopted by the majority of workers in this field.

By "development" we mean simply the main part of the theory, the definitions and theorems which chiefly occupy the professional mathematician. "Coordinatization" is a general process the most familiar instance of which is the proof of the equivalence of the synthetic and analytic aspects of Euclidean geometry. The isomorphism of a finite group to a group of permutations and the representation of an algebra by matrices are further examples of this process. Properly speaking coordinatization is just one of the theorems belonging to development, but a theorem of such fundamental implications that it effects basic changes in the appearance, methods, and results of the entire theory.

The hardest philosophical problem in geometry as well as in probability is the problem of "application." Do the theorems derived from the postulates reflect any light on the physical world which suggested them, and if so, how and why?

The purpose of this paper is exposition, exposition intended to convince the professional mathematician that probability is mathematics. To this end we shall discuss the four features just enumerated. The paper contains almost no proofs, very few precise definitions and theorems, and many heuristic derivations. Despite however the small number of rigorous statements, they form the

foundation on which the remainder is built. For the convenience of the reader they are italicized. If these italicized statements are lifted from their context and read consecutively, they will furnish at least a partial answer to the question "what is probability?"

2. Boolean algebra. The principal undefined term in probability theory is "event." Intuitively speaking an event is one of the possible outcomes of some physical experiment.

To take a rather popular example consider the experiment of rolling an ordinary six-sided die and observing the number $v(=1, 2, 3, 4, 5, \text{or } 6)$ showing on the top face of the die. "The number v is even"—"it is less than 4"—"it is equal to 6"—each such statement corresponds to a possible outcome of the experiment. From this point of view there are as many events associated with the experiment as there are combinations of the first six positive integers taken any number at a time. If for the sake of aesthetic completeness and later convenience we consider also the impossible event, "the number v is not equal to any of the first six positive integers," then there are altogether 2^6 admissible events associated with the experiment of the rolling die. For the purpose of studying this example in more detail let us introduce some notation. We write $\{246\}$ for the event "v is even," $\{123\}$ for "v is less than 4," and so on. The impossible event and the certain event $(=\{123456\})$ deserve special names: we reserve for them the symbols o and e respectively.

Everyday language concerning events uses such phrases as these: "two events a and b are incompatible or mutually exclusive," "the event a is the opposite of the event b or complementary to b," "the event a consists of the simultaneous occurrence of b and c," "the event a consists of the occurrence of at least one of the two events b and c." Such phrases suggest that there are relations between events and ways of making new events out of old that should certainly be a part of their mathematical theory.

The notion of complementary event is probably closest to the surface. If a is an event we denote the complementary event by a': an experiment one of whose outcomes is a will be said to result in a' if and only if it does not result in a. Thus if $a=\{246\}$ then $a'=\{135\}$. We may also introduce combinations of events suggested by the logical concepts of "and" and "or." With any two events a and b we associate their "join" $a\cup b$ (also called union or sum and often denoted by $a+b$), and their "meet" $a\cap b$ (or intersection or product, often denoted by ab). Here $a\cup b$ occurs if and only if at least one of the two events a or b occurs, while $a\cap b$ occurs if and only if both a and b occur. Thus if $a=\{246\}$ and $b=\{123\}$ then $a\cup b=\{12346\}$ and $a\cap b=\{2\}$.

The operations a', $a\cup b$, and $a\cap b$ satisfy some simple algebraic laws. It is clear for example that both the expressions $a\cup b$ and $a\cap b$ are independent of the order of the terms (commutative law), and that neither of the expressions $a\cup b\cup c$ and $a\cap b\cap c$ depends on the order in which the two indicated operations are performed (associative law). These facts are intuitively obvious from the verbal definition of the operations and are easily verified in any finite case such

as the rolling die. There are many other similar identities satisfied by these methods of combining events: the following is a list of the most important ones.

$$o' = e \qquad (a')' = a \qquad e' = o$$

$$(a \cap b)' = a' \cup b' \qquad (a \cup b)' = a' \cap b'$$

$$a \cap a' = o \qquad a \cup a' = e$$

$$o \cap a = o \qquad o \cup a = a$$

$$e \cap a = a \qquad e \cup a = e$$

$$a \cap b = b \cap a \qquad a \cup b = b \cup a$$

$$(a \cap b) \cap c = a \cap (b \cap c) \qquad (a \cup b) \cup c = a \cup (b \cup c)$$

$$a \cap (b \cup c) = (a \cap b) \cup (a \cap c) \qquad a \cup (b \cap c) = (a \cup b) \cap (a \cup c)$$

A system B of elements $o, a, b, \cdots, e$ in which operations a', $a \cup b$, and $a \cap b$ are defined in such a way that each of the above list of identities is satisfied is called a "Boolean algebra." For the traditional theory of probability, concerned with simple gambling games such as the rolling die, in which the total number of possible events is finite, the above heuristic reduction of events to elements of a Boolean algebra is adequate. For situations arising in modern theory and practice, and even for the more complicated gambling games, it is necessary to make an additional assumption. This assumption, in descriptive terms, is that the operations $\cup$ and $\cap$, assumed defined for two elements and immediately extended by mathematical induction to any finite number, should make sense also for an infinite sequence. In other words it is desirable to have an interpretation for symbols such as $a_1 \cup a_2 \cup \cdots$ and $a_1 \cap a_2 \cap \cdots$. In order to phrase precisely this assumption of infinite operations it is necessary to use a few simple facts from the theory of Boolean algebras.

If a and b are any two elements of the Boolean algebra B which satisfy the relation $a \cup b = b$ (or the equivalent relation $a \cap b = a$) we shall write $a \subset b$ and say that "a is smaller than b" or "a is contained in b" or "a implies b." The intuitive interpretation of this relation is as follows: the event a implies the event b, or is contained in the event b, if the occurrence of a is a sub-case of the occurrence of b. Thus in the example of the die $\{123\} \subset \{1234\}$ and "$v=2$"$\subset$"v is even." The technical significance of the relation $\subset$ is that the operations $\cup$ and $\cap$ may be defined in terms of it. For example $a \cup b$ is the smallest of all elements which contain both a and b. In more detail: given a and b, consider all c's for which both $a \subset c$ and $b \subset c$. The assertion concerning $a \cup b$ is two fold: first, $a \cup b$ is an admissible c, and second, for any admissible c we have $a \cup b \subset c$. As an example consider $a = \{12\}$ and $b = \{24\}$. The elements $\{1234\}$, $\{1246\}$, $\{12456\}$, $\cdots$ all have the property of containing both a and b. However the element $\{124\}$, which also has that property, is smaller than any other such element, and it is in fact true that $\{12\} \cup \{24\} = \{124\}$.

Motivated by the relation between $\cup$ and $\subset$ we now proceed as follows. Let B be a Boolean algebra. If for every infinite sequence $a_1, a_2, \cdots$ of elements of

B there exists among the elements containing all the a_n a smallest one, say a, we say that B is a σ-algebra and we write $a = a_1 \cup a_2 \cup a_3 \cup \cdots$. Not every Boolean algebra is a σ-algebra; the assumption that B is one (the hypothesis of countable additivity) is an essential restriction.

Perhaps an example, though a somewhat artificial one, might illustrate the need for the added assumption. Suppose that a player determines to roll a die repeatedly until the first time that the number showing on top is 6. Let a_n be the event that the first 6 appears only on the nth roll. The event $a = a_1 \cup a_2 \cup a_3 \cup \cdots$ occurs if and only if the game ends in a finite number of rolls. The occurrence of the opposite event a' is at least logically (even if not practically) conceivable and it seems reasonable to want to include a discussion of it in a general theory of probability. Numerous examples of this kind together with some rather deep lying technical reasons justify therefore the following statement.

The mathematical theory of probability consists of the study of Boolean σ-algebras.

This is not to say that all Boolean σ-algebras are within the domain of probability theory. In general statements concerning such algebras and the relations between their elements are merely qualitative: probability theory differs from the general theory in that it studies also the quantitative aspects of Boolean algebras. In the next section we shall describe and motivate the introduction of numerical probabilities.

3. Measure algebra. When we ask "what is the probability of a certain event?" we expect the answer to be a number, a number associated with the event. In other words probability is a numerically valued function P of events a, that is of elements of a Boolean σ-algebra B, $P = P(a)$. On intuitive and practical grounds we demand that the number $P(a)$ should give information about the occurrence habits of the event a. If in a large number of repetitions of the experiment which may result in the event a we observe that a actually occurs only a quarter of the time (the remaining three quarters of the experiments resulting therefore in a') we may attempt to summarize this fact by saying that $P(a) = 1/4$. Even this very rough first approximation to what is desired yields some suggestive clues concerning the nature of the function P.

If, to begin with, $P(a)$ is to represent the proportion of times that a is expected to occur, then $P(a)$ must be a positive real number, in fact a number in the unit interval $0 \leqq P(a) \leqq 1$. The extreme value 0 has a special significance. Since the impossible event o will never occur, it is clear that we must write $P(o) = 0$. Conversely however if an event a refuses ever to occur, we are tempted to declare its occurrence impossible and thus from the relation $P(a) = 0$ to deduce $a = o$. The other extreme value of $P(a)$ has of course a similar interpretation: $P(a) = 1$ if and only if $a = e$.

The relation between proportion and probability has further consequences. Suppose that a and b are mutually exclusive events—say $a = \{1\}$ and $b = \{246\}$ in the example of the die. (In the algebraic theory mutually exclusive events correspond to "disjoint" elements of the Boolean algebra B, that is to elements

a and b for which $a \cap b = o$.) In this case the proportion of times that the join $a \cup b (= \{1246\}$ for the example) occurs is clearly the sum of the proportions associated with a and b separately. If an ace shows up one-sixth of the time and an even number half the time, then the proportion of times in which the top face is either an ace or an even number is $\frac{1}{6} + \frac{1}{2}$. It follows therefore that the function P cannot be completely arbitrary—it is necessary to subject it to the condition of additivity, that is to require that if $a \cap b = o$ then $P(a \cup b)$ should be equal to $P(a) + P(b)$.

We are now separated from the final definition of probability theory only by a seemingly petty (but in fact very important) technicality. If $P(a)$ is an additive function of the sort just described on a Boolean σ-algebra B, and if $a_1, a_2, \cdots, a_n$ is any finite set of pairwise disjoint elements of B (this means that for $i \neq j$, $a_i \cap a_j = o$) then it's easy to prove by mathematical induction that $P(a_1 \cup a_2 \cup \cdots \cup a_n) = P(a_1) + P(a_2) + \cdots + P(a_n)$. If however $a_1, a_2, a_3, \cdots$ is an infinite sequence of pairwise disjoint elements then it may or may not be true that $P(a_1 \cup a_2 \cup a_3 \cup \cdots) = P(a_1) + P(a_2) + P(a_3) + \cdots$. The general condition of countable (that is, finite or enumerably infinite) additivity is a further restriction on the probability measure P—a restriction without which modern probability theory could not function. It is a tenable point of view that our intuition demands infinite additivity just as much as finite additivity. At least however infinite additivity does not contradict any of our intuitive ideas and the theory built on it is sufficiently far developed to assert that the assumption is justified by its success. We shall therefore adopt this assumption as our final postulate.

Numerical probability is a measure function, that is a finite, nonnegative, and countably additive function P of elements in a Boolean σ-algebra B, such that if the null and unit elements of B are o and e respectively then $P(a) = 0$ is equivalent to $a = o$ and $P(a) = 1$ is equivalent to $a = e$.

In the next section we shall discuss a general method of constructing examples of probability measures.

4. Measure space. Let $\omega_j (j = 1, \cdots, 6)$ be the point on the real axis whose directed distance from the origin is j, and let Ω be the set whose elements are these six points. Consider the system B^* of all subsets of Ω. (The empty set o and the full set $e = \Omega$ are counted as belonging to B^*.) With any element a of B^* (that is, with any subset of Ω) we may associate the complementary element (set) consisting of exactly those points ω_j which do not belong to a. Similarly with any two subsets a and b of Ω we may associate their union (the set of points belonging to either a or b or both), and their intersection (the set of points belonging simultaneously to a and b). It is easy to verify that under the operations of complementation (a'), formation of unions $(a \cup b)$, and formation of intersections $(a \cap b)$, the system B^* forms a Boolean algebra, in fact, though somewhat vacuously, a σ-algebra. Suppose moreover that for each $j = 1, \cdots, 6$, p_j is a positive number such that $p_1 + \cdots + p_6 = 1$. Then we may define $P(a)$ for any subset a of Ω, to be the sum of those p_j whose ω_j belongs to a. Thus if

$a=\{135\}$ then $P(a)=p_1+p_3+p_5$; if $a=o$ then $P(a)=0$. The function P and the algebra B^* satisfy all the assumptions of probability theory and the reader has doubtless recognized that this B^* and P were implicit in our earlier discussion of the rolling die. It is often customary on philosophical and practical grounds to discuss only the case $p_1=\cdots=p_6=\frac{1}{6}$. We shall say a word about this special case later; for the moment it is sufficient to point out that any other choice of the p_j furnishes an equally acceptable probability structure and does in fact constitute the mathematical theory of some carefully loaded die.

The above example of a Boolean algebra can be generalized: we attempt next to obtain a similar but more geometrical example. For this purpose we again choose a set Ω, but, instead of a finite set, we choose a set with infinitely many points, in fact all the points of a continuum. To be specific let us choose for Ω the points ω of a square of unit area in the Cartesian plane. In analogy with the preceding example we consider the system B^* of all subsets of Ω and define complement, union, and intersection as before. Once more B^* is a Boolean σ-algebra; it is not however the one on which we shall base our probability theory. (It can be shown that it is not possible to define a probability measure P with the desired properties on B^*.) We shall instead consider a certain subsystem (sub-algebra) of B^*, constructed as follows:

We begin with the system R of all rectangles contained in Ω (where for the sake of definiteness we consider closed rectangles, that is sets consisting of the interior plus the perimeter of a rectangle). The system R is not closed under the Boolean operations: in general not even a finite (let alone a countably infinite) union or intersection of rectangles is itself a rectangle, and similarly the complement of a rectangle isn't one. We have therefore to enlarge the system R to a system R' including all complements and countable unions and intersections of elements of R. It turns out that even this is not enough: R' is still not a Boolean algebra, and the extension process has to be continued. If however the extension process is continued sufficiently (and this happens to mean transfinitely) often, we reach eventually a Boolean σ-algebra B of subsets of Ω. (The algebra B is important in analysis: sets of B are called the Borel sets of the square.)

We face next the task of defining P. For those familiar with the theory of Lebesgue measure it will suffice to say that we define $P(a)$, for each a in B, to be the Lebesgue measure of the set a. It is not difficult to get an intuitive idea of how P is defined. If a is a rectangle (that is an element of R) we define $P(a)$ to be the area of a. If a is an element of R' we proceed to determine $P(a)$ in accordance with the requirement of countable additivity. Thus for example if b is the complement of a rectangle a, we write $P(b)=1-P(a)$, and if b is the union of a finite or infinite sequence of disjoint rectangles $a_1, a_2, \cdots$ we define $P(b)=P(a_1)+P(a_2)+\cdots$. By repeating this extension process ad transfinitum we succeed eventually in defining $P(a)$ for every a in B.

There is an objection to the construction just described. If the set a consists of a single point then it is intuitively obvious (and follows easily from the rigorous definition of P) that $P(a)$ ($=$the area of a) is zero. More generally if a

consists of any finite or enumerably infinite set of points we still have $P(a)=0$, and it is even possible (if for example a is a line segment) to have $P(a)=0$ for sets a containing uncountably many points. This definitely contradicts our explicitly formulated axiom that $P(a)=0$ should happen if and only if $a=o$. The customary way to get around this difficulty is by redefining the notion of equality that occurs in the equation $a=o$. It is proposed that we agree to consider as identical two subsets of Ω whose difference has probability zero. (In technical language, we consider, instead of the sets a, equivalence classes of sets modulo the class of sets of probability zero.) Through this agreement we are committed in particular to identifying any set of probability zero with the empty set o, and it follows therefore that in the reduced algebra B (that is, the algebra obtained from B by making the suggested identifications) all the axioms of probability are valid.

The long and tortuous process just described is very general. If Ω is any space (such as an interval or a cube) on a certain σ-algebra B of subsets of which a countably additive measure P is defined (such as length or volume), subject only to the restriction that the measure of all Ω is equal to 1, we obtain from B and P a system satisfying all the axioms of probability theory by the process of identification according to sets of measure zero. Thus there are as many probability systems as there are examples of "measure spaces."

The reason for the introduction of measure spaces into a discussion of probability theory is not merely to give examples. It can in fact be shown that the two theories (measure and probability) are coextensive. More precisely:

If B is any Boolean σ-algebra and P a probability measure on B, then there exists a measure space Ω such that the system B is abstractly identical with an algebra of subsets of Ω reduced by identification according to sets of measure zero, and the value of P for any event a is identical with the values of the measure for the corresponding subsets of Ω.

Hence measure is probability and probability is measure and, in virtue of the theorem just stated, the entire classical theory of measure and integration may be and has been carried over and used to give rigorous proofs of probability theorems.

5. Measure vs. probability. Having discussed the extent to which probability and measure are the same, we now dedicate a few words to describing the extent to which they are different. One feature that differentiates the two theories is that in the general theory of measure it is usual to admit the possibility that the measure of the entire space is infinite. This possibility is not admissable in probability theory. As long, however, as the measure of the whole space is finite it is always possible to introduce a scale factor which makes it equal to 1, and hence it is always possible to think of it (even if somewhat artificially) as a "probability space." Thus for example the language and notation of probability may be and have been used in such seemingly widely separated parts of mathematics as ergodic theory, topological groups, and integral geometry.

Even however if the infinite case is ruled out, it is a conspicuous fact that most theorems in which the word measure is used (rather than the word probability) have a very different appearance from the theorems of probability theory. The best way to explain the difference between measure and probability is to liken it to the difference between analytic and synthetic geometry. It isn't stretching a point too far to say that the representation of a probability algebra by a measure space is similar to the introduction of coordinates into geometry. Synthetic and analytic geometry are of course abstractly identical in the sense that any theorem in the one domain may be stated and proved in the language and machinery of the other—may be, but isn't. The theorems in the two fields differ in their intuitive content. It is natural to discuss linear transformations in analytic geometry and the nine point circle in synthetic geometry—and even though the interchange is possible, it isn't desired. The abstract identity of the two fields is however an extremely useful fact, exploited mostly by the synthetic side which often finds it convenient to lean on the analytic crutch. Similarly, probability is measure, and research in the field would be very greatly hampered if we were not permitted to use this analytic crutch—but the notions suggested by probability, the notions which are important and intuitive and natural inside the field, appear sometimes extremely special and artificial in the frame work of general measure theory.

In this section and the preceding ones we have treated axiomatization and coordinatization. We proceed now to development. In the following sections we shall define the basic concepts of probability theory, and discuss in particular those which serve in the sense described above to give to probability its distinguishing flavor.

6. Independent events. In order to motivate the definitions of the concepts to be studied in the sequel we return to the example of the die. For simplicity we make the classical assumption that any two faces are equally likely to turn up and that consequently the probability of any particular face showing is $\frac{1}{6}$. Consider the events $a=\{246\}$ and $b=\{12\}$. The first notion we want to introduce, the notion of conditional probability, can be used to answer such questions as these: "what is the probability of a when b is known to have occurred?" In the case of the example: if we know that v is less than 3, what can we say about the probability that v is even? The adjective "conditional" is clearly called for in the answer to a question of this type: we are evaluating probabilities subject to certain preassigned conditions.

To get a clue to the answer consider first the event $c=\{2\}$ and ask for the conditional probability of a, given that c has already occurred. The intuitive answer is perfectly clear here, and is independent as it happens of any such numerical assumptions as the equal likelihood of the faces. If v is known to be 2 then v is certainly even, and the probability must be 1. What made the answer easy was the fact that c implied a. The general question of conditional probability asks us to evaluate the extent (measured by a numerical probability or propor-

tion) to which the given event b implies the unknown event a. Phrased in this way the question almost suggests its own answer: the extent to which b is contained in a can be measured by the extent to which a and b are likely to occur simultaneously, that is by $P(a \cap b)$. Almost—not quite. The trouble is that $P(a \cap b)$ may be very small for two reasons: one is that not much of b is contained in a, and the other is that there isn't very much of b altogether. In other words it isn't merely the absolute size of $a \cap b$ that matters: it's the relation or proportion of this size to the size of b that's relevant.

We are led therefore to define the conditional probability of a, given that b has occurred, in symbols $P_b(a)$, as the ratio $P(a \cap b)/P(b)$. For $a = \{246\}$ and $c = \{2\}$ this gives the answer we derived earlier, $P_c(a) = 1$; for $a = \{246\}$ and $b = \{12\}$ we get the rather reasonable figure $P_b(a) = \frac{1}{2}$. In other words if it's known that v is either 1 or 2 then v is even or odd (that is equal to 1 or equal to 2) each with probability $\frac{1}{2}$.

Consider now the following two questions: "b happened, what is the chance of a?" and simply "what is the chance of a?." The answers of course are $P_b(a)$ and $P(a)$ respectively. It might happen, and does in the example given above, that the two answers are the same, that in other words knowledge of b contributes nothing to our knowledge of the probability of a. It seems natural in this situation to use the word "independent": the probability distribution of a is independent of the knowledge of b. This motivates the precise definition: two events a and b are independent if $P_b(a) = P(a)$. The definition is transformed into its more usual form and at the same time gains in symmetry if we recall the definition of $P_b(a)$. In symmetric form: a and b are independent in the sense of probability (statistically or stochastically independent) if and only if $P(a \cap b) = P(a)P(b)$.

7. Repeated trials. Suppose next that we wish to make two independent trials of the same experiment—say, for example, to roll an honest die twice in succession. We shall presently exploit the precise definition of independence to clarify the notion of independent trials; first however it's worth while to remark on the intuitive content of the concept. Suppose that in a crude attempt to even things up we resolve on the following procedure: if the first die shows an even number we choose for the second experiment a die on which all the numbers are odd, and vice versa. The two experiments are not independent of each other in this case: whereas the a priori probability of getting an even number with the second die is $\frac{1}{2}$, the conditional probability of getting an even number with the second die, given that the first one showed an odd number, is one. We say that the two experiments are performed independently of each other only if the conditions under which the second experiment is to be performed are unaffected by the outcome of the first experiment.

If an experiment consists of two rolls of a die we don't expect the reported outcome of the experiment to be a number v, but rather a pair of numbers (v_1, v_2). The measure space Ω associated with the two-fold experiment consists

not of 6 but of 36 points. (It is convenient to imagine these points laid out along the regular pattern of a 6×6 square.) The problem is to determine how the probability is distributed among these points. For a clue to the answer consider the events $a=$ "$v_1<3$" and $b=$ "$v_2<4$." We have $P(a)=\frac{1}{3}$ and $P(b)=\frac{1}{2}$; hence if we interpret the independence of the trials to mean the independence of any two events such as a and b we should have $P(a\cap b)=\frac{1}{6}$. If in the suggested diagram for the measure space associated with this discussion we encircle the points belonging to $a\cap b$ we get the following figure.

v_2 \ v_1	1	2	3	4	5	6
1	⊙	⊙	·	·	·	·
2	⊙	⊙	·	·	·	·
3	⊙	⊙	·	·	·	·
4	·	·	·	·	·	·
5	·	·	·	·	·	·
6	·	·	·	·	·	·

We see therefore that the formula $P(a\cap b)=P(a)P(b)$ appears analogous to the fact that the area of a rectangle is the product of the lengths of its sides.

We say therefore, if the analytic description of an experiment is given by a measure space Ω with a Boolean σ-algebra B of subsets on which a probability measure P is defined, that the analytic description of the experiment consisting of two independent trials of the given experiment is as follows. The space of points ω is replaced by the space of pairs of points (ω_1, ω_2) (the so called product space $\Omega\times\Omega$), B is replaced by the Boolean σ-algebra generated by the "rectangular" sets of the form $\{\omega_1 \text{ is in } a_1, \omega_2 \text{ is in } a_2\}$ where a_1 and a_2 belong to B, and the probability measure on this space of pairs is determined by the requirement that its value for rectangular sets of the kind described should be given by the product $P(a_1)P(a_2)$. The ideas involved in this procedure are not essentially original nor characteristic of probability theory: they are the same as the ideas involved in defining the area of plane sets in terms of the length of linear sets. There is of course a theorem hidden in this definition—a theorem which asserts that a probability measure satisfying the stated product requirement indeed exists and is in fact uniquely determined by this requirement.

What we can do once, we can do again. Just as two repetitions of an experiment gave rise to ordered pairs (ω_1, ω_2), similarly any finite number of repetitions (say n) give rise to the space of ordered n-tuples $(\omega_1, \omega_2, \cdots, \omega_n)$, with a multiplicatively determined probability measure. The procedure can be extended also to infinity: the analytic model of an infinite sequence of independent repetitions of an experiment is a measure space Ω whose points ω are infinite sequences $\{\omega_1, \omega_2, \omega_3, \cdots\}$. Even if an actually infinite sequence of repetitions of an experiment is practically unthinkable, there is a point in considering the infinite dimensional space Ω. The point is that many probability statements are asser-

tions concerning what happens in the long run—assertions which can be made precise only by carefully formulated theorems concerning limits. Hence even if practice yields only approximations to infinity, it is the infinite sequence space Ω that is the touchstone whereby the mathematical theory of probability can be tested against our intuitive ideas. The first and most important such long run statement is described in the following paragraphs.

Suppose that an experiment is capable of producing an event a with probability p, and suppose that an infinite sequence of independent trials of this experiment is performed. We consider therefore the space of all sequences $\omega = \{\omega_1, \omega_2, \omega_3, \cdots\}$ where for each n, ω_n may or may not belong to a. Once the experiments have been performed so that we are given a particular point ω we may start asking numerical questions. We may ask for example: out of the first n trials of the basic experiment how many resulted in a? This means: out of the first n coordinates $\omega_1, \omega_2, \cdots, \omega_n$ of ω how many belong to a? The answer to this question depends obviously on n and just as essentially on the particular sequence ω—let us denote it by $m_n(\omega)$.

Now what does out intuition say? The usual statement (one which we have already exploited in our heuristic derivation of the notion of probability) is that the ratio of the number of successes to the total number of trials should be approximately equal to the probability of the event being tested. In our notation this seems to mean that for large n the ratio $m_n(\omega)/n$ should be close to the constant $p=P(a)$. The question arises: for which ω's should this be true? Not surely for all of them. For the sequence space Ω contains sequences none of whose coordinates belong to a, and for such a sequence ω, $m_n(\omega)$ is zero for all n. The best that we have a right to demand is that the ω's for which our statement is not true should be equivalent to the empty set of ω's in the sense of probability—that is that their totality should have probability zero. And this is true.

To sum up: we have just derived the statement (not the proof) of the most important special case of the so called strong law of large numbers. In mathematical language the assertion of this law is that as $n \to \infty$, $\lim m_n(\omega)/n$ exists and is equal to $p(=P(a))$ except for a set of ω's of measure zero. In more classical terms: it is almost certain that the "success ratios" converge to the probability of the event being tested.

8. Random variables. In order to gain a more thorough understanding of the law of large numbers and at the same time to introduce the language in which most of the theorems of probability theory are stated, we proceed to discuss the notion of a random variable.

"A random variable is a quantity whose values are determined by chance." What does that mean? The word "quantity" is meant to suggest magnitude—numerical magnitude. Ever since rigor has come to be demanded in mathematical definitions it has been recognized that the word "variable," particularly a variable whose values are "determined" somehow or other, means in precise language a function. Accordingly a random variable is a function: a function

whose numerical values are determined by chance. This means in other words that a random variable is a function attached to an experiment—once the experiment has been performed the value of the function is known. The spatial model of probability is extremely well adapted to making this notion still more precise. If the analytic correspondent of an experiment is a measure space Ω then any possible outcome of the experiment is by definition represented by a point ω in this space. Hence a function of outcomes is a function of ω's: a random variable is a real valued function defined on a probability space Ω.

The preceding sentence does not yet constitute our final definition of a random variable. For suppose that $x=x(\omega)$ is a function on the space Ω. We shall call x a random variable only if probability questions concerning the values of x can be answered. An example of such a question is: what is the probability that x is between α and β? In measure theoretic language: what is the measure of the set of those ω's for which the inequality $\alpha \leqq x(\omega) \leqq \beta$ is satisfied? In order for such questions to be answerable it is necessary and sufficient that the sets that occur in them belong to the basic σ-algebra B of Ω. A function $x(\omega)$ for which this is true for every interval (α, β) is called "measurable." Accordingly we make the following definition:

A random variable is a measurable function defined on a measure space with total measure 1.

Instances of random variables can be found even in that part of our discussion which preceded their definition. The quantity v associated with the rolling die is an example, as are also the quantities v_1 and v_2 associated with the two fold repetition of this experiment. To obtain some further examples, consider any fixed event a which may result from an experiment and let the random variable x be the number of times that a actually occurs. If the experiment is performed only once then x has only two possible values: 1 if a occurs and 0 otherwise. More generally if the experiment is repeated n times the random variable x becomes the function $m_n(\omega)$ introduced in the discussion of the law of large numbers.

9. Expectation, variance, and distribution. Let us consider in detail the random variable v associated with an honest die. The possible values of v are the first six positive integers. The arithmetic mean of these values, that is the number $(1+\cdots+6)/6$, is of considerable interest in probability theory. It is called the average, or mean value, or expectation of the random variable v and it is denoted by $E(v)$. If the die is loaded so that the probability p_j associated with j is not necessarily $\frac{1}{6}$ then the arithmetic mean is replaced by a weighted average: in this case $E(v)=1\cdot p_1+\cdots+6\cdot p_6$. It is well known that the analogs of such weighted sums in cases where the number of values of the function (random variable) need not be finite are given by integrals. The kind of integral that enters into probability theory is similar in every detail to the Lebesgue integral and we shall not reproduce its definition here.

If the measurable function $x(\omega)$ is integrable then its expectation $E(x)$ is by definition the value of its integral extended over the entire domain Ω.

As a useful though extremely special case we mention that if x is a counting variable of the sort mentioned in the preceding paragraph ($x=1$ if a certain even a occurs and $x=0$ otherwise) then $E(x)=P(a)$.

It is obviously of interest to ask not only what is the expected value of a random variable x but also how closely the values of x are clustered about its expected value. The customary measure of clustering of a random variable x is one inspired by the method of least squares and called the "variance" or "dispersion" of x.

The variance of x is the expression $\sigma^2(x)=E(x-\alpha)^2$, where $\alpha=E(x)$.

(The square root of the variance is called the "standard deviation.") In words: take the square of the deviation of x from its expected value α, and use the sum (weighted sum, integral) of these squared deviations as a measure of clustering. Since a sum of squares vanishes only if each term does, the vanishing of the variance indicates that x is identically equal to its expected value (except perhaps for a set of probability zero). In general, the smaller the variance the closer the values of x lie to $E(x)$.

Such numbers as $E(x)$ and $\sigma^2(x)$ yield partial information about the distribution of the values of x. Complete information would mean an answer to every question of the form "what is the probability that x lies in the interval (α, β)?" In order to deal with such questions we introduce the notion of distribution function.

The distribution function $F_x(\lambda)$ of a random variable x is a function of a real variable λ defined for each λ to be the probability that $x<\lambda$.

These functions can be used to answer every probability question concerning random variables; for example the expression $F_x(\beta)-F_x(\alpha)$ represents the probability that x belong to the (half open) interval $\alpha\leqq x<\beta$, and the Stieltjes integrals $\int_{-\infty}^{\infty}\lambda dF_x(\lambda)$ and $\int_{-\infty}^{\infty}\{\lambda-E(x)\}^2dF_x(\lambda)$ represent the expectation and variance of x respectively. Distribution functions are useful because being comparatively simple real functions of real variables they are amenable to treatment by the methods of classical analysis. It is the whole purpose of a large part of probability theory to find the distribution functions of certain random variables.

10. Independent variables. Let us consider next two random variables x and y which are comparable in the sense that they are both represented by measurable functions on the same measure space Ω, so that $x=x(\omega)$ and $y=y(\omega)$. It is easy to see that the function $E(x)$, being defined by an integral, is homogeneous of degree 1 and additive, that is $E(\lambda x)=\lambda E(x)$ for every real constant λ and $E(x+y)=E(x)+E(y)$. Similarly the variance $\sigma^2(x)$ is homogeneous of degree 2, that is $\sigma^2(\lambda x)=\lambda^2\sigma^2(x)$. One way to prove this latter fact is to make use of the following identity connecting σ^2 and E:

$$\sigma^2(x) = E(x)^2 - E^2(x), \tag{1}$$

(where for later convenience we write $E(x)^2$ for $E(x^2)$ and $E^2(x)$ for $\{E(x)\}^2$). This identity in turn follows from the definition of σ^2. Since $\sigma^2(x)=E(x-\alpha)^2$

where $\alpha = E(x)$, we have $\sigma^2(x) = E(x^2 - 2\alpha x + \alpha^2) = E(x^2) - 2\alpha E(x) + \alpha^2$. (We used here the fact that the expected value of a constant is equal to that constant.) The identity (1) follows by substituting for α its value $E(x)$. Letting the formalism guide us we may inquire whether σ^2 is additive, that is whether or not the identity

$$\sigma^2(x + y) = \sigma^2(x) + \sigma^2(y) \tag{2}$$

is valid. The answer in general is no. In order to investigate conditions under which (2) is true we proceed to a brief discussion of some possible relations between pairs of random variables.

Let a and b be two independent events and let x and y be the associated counting random variables (so that x for example is 1 if and only if a occurs and $x = 0$ otherwise). The product random variable xy in this case can be equal to 1 if and only if both a and b occur, so that xy is the counting variable of $a \cap b$. Since $E(x) = P(a)$, $E(y) = P(b)$, and similarly $E(xy) = P(a \cap b)$, we have in this special case

$$E(xy) = E(x)E(y). \tag{3}$$

The validity of this formula is sufficiently important in the applications of probability to bear a name of its own: two random variables, not necessarily the counting variables of a pair of independent events, satisfying it are called "uncorrelated." The reason for the terminology is that the coefficient of correlation $r = r(x, y)$ of two random variables x and y is defined by $r = \{E(xy) - E(x)E(y)\}/\sigma^2(x)\sigma^2(y)$; this coefficient vanishes if and only if (3) holds.

It is now easy to state the facts concerning the formula (2): it is valid if and only if (3) is. In other words the variance is additive for a pair of random variables if and only if the expectation is multiplicative, that is if and only if they are uncorrelated. For the proof we merely expand the left member of (2), thus:

$$\begin{aligned} \sigma^2(x + y) &= E(x + y)^2 - E^2(x + y) \\ &= \{E(x)^2 - 2E(xy) + E(y)^2\} - \{E^2(x) - 2E(x)E(y) + E^2(y)\} \\ &= \sigma^2(x) + \sigma^2(y) - 2\{E(xy) - E(x)E(y)\}. \end{aligned}$$

Let us now return to the pair of counting variables x and y associated with two independent events a and b. Because of the independence of a and b, any probability statement concerning y is unaffected by our knowledge of ignorance of the value of x. More precisely, any two events defined by x and y, for example the events "$x = 0$" and "$y = 1$," are independent. If in general any two events by two random variables x and y respectively, that is any two events defined by inequalities of the form $\alpha \leqq x \leqq \beta$ and $\gamma \leqq y \leqq \delta$, are independent events, no matter what α, β, γ and δ are, we say that x and y are independent random variables. It is not too difficult to generalize what we proved about the special case of counting variables: for independent random variables the expectation, if it exists, is multiplicative and consequently the variance is additive. In still

other words: independence implies absence of correlation—a proposition which certainly sounds natural enough.

One word of caution before we leave this brief introduction to the notion of independence for random variables. What we defined was the independence of two random variables. It would be natural to try to define the independence of a finite or infinite sequence of random variables $x_1, x_2, \cdots$, by the requirement that any pair be independent. Natural, but as it happens, not very useful. The correct definition replaces two-term products by many-term products in the following way.

The random variables $x_1, x_2, \cdots$, are independent if the probability of the simultaneous occurrence of any finite number of the events defined by $\alpha_n \leqq x_n \leqq \beta_n$ is the product of the separate probabilities, no matter what real constants the α's and β's are.

It is easy to construct examples to show that this notion is indeed different from the notion of pairwise independence.

11. Law of large numbers. We are now in a position to reformulate and generalize the strong law of large numbers in terms of random variables. Let the sequence space of points $\omega = \{\omega_1, \omega_2, \cdots\}$ be the analytic model of the infinite repetition of an experiment one of whose possible outcomes is the event a. Let a_n be the event "ω_n belongs to a" or equivalently the event "the nth experiment results in a," and let $x_n = x_n(\omega)$ be the counting variable associated with a_n. In this context that means that $x_n(\omega)$ has the value 1 for all those sequences $\omega\{\omega_1, \omega_2, \cdots\}$ for which the nth coordinate ω_n belongs to a, and $x_n(\omega)$ has the value 0 otherwise. What significance has the sum $x_1 + \cdots + x_n$? Since a particular term x_j contributes one unit to this sum if and only if the jth experiment results in a, it is clear that the value of the sum, for any sequence ω, is the number of those coordinates among the first n coordinates of ω which do belong to a. But this is exactly the function we denoted above by $m_n(\omega)$. Hence our version of the law of large numbers is equivalent to the assertion that the averages $(x_1 + \cdots + x_n)/n$ converge (except possibly for a set of ω's of probability zero) to the constant $p = P(a)$. For the generalization of this result that we are about to formulate it is worth while to observe that $p = E(x_n)$ is also equal to the common value of the expectations of the x's.

The sequence of random variables $x_1, x_2, \cdots$ has two important properties which are sufficient to ensure the validity of the law of large numbers. One of these properties is independence. It follows very easily from the fact that the experiments yielding the values of the various x's are independently performed, that the variables $x_1, x_2, \cdots$ are indeed independent. The other essential property of the sequence is usually expressed by the statement that the random variables x_n all have the same distribution. The definition of this concept is as follows.

Two random variables x and y have the same distribution if for every interval (α, β) the probabilities of the two events $\alpha \leqq x \leqq \beta$ and $\alpha \leqq y \leqq \beta$ are equal, or equivalently if the distribution functions $F_x(\lambda)$ and $F_y(\lambda)$ are identical.

In our particular case it is the fact that the probability that ω_n belong to a is the same for all n (namely $P(a)$) that implies that the x_n all have the same distribution. That independence and equidistribution are indeed the crucial hypotheses for the law of large numbers is shown by the following general formulation of that law.

If $x_1, x_2, \cdots$ is a sequence of independent random variables with the same distribution, and if the expectations $E(x_n)$ exist and have the value α (necessarily the same for all n) then the averages $x_1+ \cdots +x_n/n$ converge as $n\to\infty$ (except perhaps on a set of probability zero) to the constant α.

12. Central limit theorem. Sums (such as $x_1+ \cdots +x_n$) of independent random variables with the same distribution occur very often in probability theory. It is of considerable practical importance to investigate the precise distribution of such sums and if possible the limiting behavior of these distributions. We assume concerning the x's that their expectations and variances both exist and write $E(x_j)=\alpha$, $\sigma^2(x_j)=\beta$. It follows from the independence and equidistribution of the x's that $E(x_1+ \cdots +x_n)=n\alpha$ and $\sigma^2(x_1+ \cdots +x_n)=n\beta$. At first sight this seems like a discouraging phenomenon: if both the expectation and the variance become infinite, how can we expect a reasonable asymptotic behavior from the much more delicate distribution function? But the way out of the difficulty is easy: by a translation and a change of scale (different to be sure for each n) it is possible to normalize the sum $x_1+ \cdots +x_n$ so that its expectation is 0 and its variance 1 for every positive integer n. To get the expectation to be 0 we merely subtract its actual value, $n\alpha$, from the sum—the additivity of the expectation ensures the desired result. To get the variance to be 1 we divide by a constant factor. It is important to recall that the variance is homogeneous of degree 2, so that the constant factor will be not $n\beta$ but $\sqrt{n\beta}$. We arrive thus at the normalized sums

$$\frac{x_1 + \cdots + x_n - n\alpha}{\sqrt{n\beta}}$$

and inquire again after the distribution function of this random variable and the limit of such distribution functions. The answer here is known and is embodied in the so called central limit theorem (or Laplace-Liapounoff theorem) stated as follows.

If $x_1, x_2, \cdots$ is a sequence of independent random variables with the same distribution, expectation α, and variance β, then the distribution functions of the modified sums $(x_1+ \cdots +x_n-n\alpha)/\sqrt{n\beta}$ converge as $n\to\infty$ to a fixed distribution function, the same no matter what the original distribution of the x's is. In more detail, the limit as $n\to\infty$ of the probability of the event defined by the inequality

$$\frac{x_1 + \cdots + x_n - n\alpha}{\sqrt{n\beta}} < \lambda$$

exists and is equal to

$$G(\lambda) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\lambda} e^{-u^2/2}du.$$

The distribution function $G(\lambda)$ is called the Gaussian or normal distribution.

With this statement we end our discussion of the development of probability theory and turn to a few remarks connected with the problem of application.

13. Determination of initial probabilities. When the mathematician announces that the probability of an event is a certain number, he is immediately faced with two questions. First the practical man asks what is the practical meaning of a probability statement? How should one act on it? If the mathematician succeeds in answering this question then the philosopher wants to know the reason for the answer. What establishes the connection between mathematical theory and practice? Our remarks in what follows will bear on these very old and very difficult questions only incidentally—they are dedicated mainly to a smaller problem of the theory, but one which frequently worries the layman.

The problem is how the probability of concretely given events is really defined. It is all very well to talk about Boolean algebras and measure theory, but what is the probability that a coin will fall heads up? What the layman realizes and what we now wish to emphasize is that the mathematician has not answered any such questions. He cannot. He can no more say that the probability of obtaining two heads in succession with a coin is $\frac{1}{4}$ than he can say that the volume of a cube is 8. The volume of a cube is given by a formula. If the hypotheses under which the formula applies are verified and if the variables entering into the formula are given specific values then the volume of a cube can be calculated. In exactly the same sense the mathematical theory of probability is a collection of formulae which enable us to calculate certain probabilities assuming that certain other ones are given. If we know that the probability of obtaining heads with a certain coin is $\frac{1}{2}$ and if we know that two successive tosses of the coin were performed independently then we can assert that the probability of getting two heads is $\frac{1}{4}$.

Despite the fact that probability theory shares with all other mathematical theories its inability to state a conclusion without hypotheses, the above answer to the layman's question will probably seem unsatisfactory to many readers. There must be some reason why most people believe that the probability of heads is $\frac{1}{2}$. It is often even proved. The usual proof is based on symmetry arguments, or equivalently on the principle of sufficient reason. (Why should heads have any greater likelihood of appearing than tails?) Do these proofs have any mathematical validity?

The answer is definitely yes. In some cases it is more pleasing to the intuition or more convenient for practice to formulate our hypotheses purely qualitatively. In almost all such cases the hypotheses take the form of invariance—the probabilities entering into the problem are required to be invariant under a certain group of transformations. It often turns out then that an existence and uniqueness theorem is true, that is it can be proved that there exists one and only one probability measure satisfying the stated hypotheses. Theorems of this type are certainly a part, an increasingly important part, of the theory of proba-

bility, and as long as their hypotheses are clearly formulated and recognized as hypotheses, the professional mathematician is the last person to sneer at them. Their advantage at the level of elementary pedagogy seems to lie in the fact that the statement "heads and tails are equally likely" is easier to grasp intuitively than the statement "the probability of heads is $\frac{1}{2}$."

We see thus that a mathematical statement on probability has to have certain either explicitly or implicitly given probabilities to begin with. In practice the physicist (or actuary, or anyone else interested in applying the theory) obtains these initial numbers experimentally. If he wants to know what is the probability of a coin falling heads up, he tosses the coin a large number of times and then uses the law of large numbers to assure himself that he may use the obtained frequency ratio as an approximation to the correct value of the probability. Or he may observe that the values of a random variable are obtained as the sum of a large number of independent variables each with a negligible variance and thus be led to introduce the normal distribution. Such approximative procedures are of course common to all parts of applied mathematics.

14. Conclusion. Our exposition is finished. If the reader has been patient enough to read this far he may be curious enough to read farther. Our scanty bibliography will furnish a basis for such reading. For certainly not all probability theory is contained in this paper, nor as yet in any collection of books or papers. There is still much room in the field for the exercise of the analytic ingenuity and abstract generality of both classical and modern mathematics. If this paper will be instrumental in persuading mathematicians that probability is mathematics, and in causing some to look into the subject more deeply than they had previously thought worth while, it will have more than accomplished its purpose.

BIBLIOGRAPHY

G. Birkhoff, Lattice Theory, New York, 1940.
H. Cramér, Random Variables and Probability Distributions, Cambridge, 1937.
A. Khintchine, Asymptotische Gesetze der Wahrscheinlichkeitsrechnung, Berlin, 1933.
A. Kolmogoroff, Grundbegriffe der Wahrscheinlichkeitsrechnung, Berlin, 1933.
S. Saks, Theory of the Integral, Warsaw, 1937.
J. V. Uspensky, Introduction to Mathematical Probability, New York, 1937

NON-ANALYTIC FUNCTIONS

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The triumph of the theory of analytic functions lies in the fact that it has wide applications not only in other branches of mathematics but also in many physical investigations. In regard to the latter, it is possible merely because many physical quantities are distributed like the values of a harmonic function,

the real part of an analytic function. Since the likeness is actually approximate, we are compelled to ask the question whether such applications may not lead to serious mistakes and if not, what is the limit of the errors?

It is the purpose of this note to investigate non-analytic functions [1] which are "approximately analytic" and find the necessary modifications for some fundamental theorems in the theory of analytic functions. This furnishes an indirect answer of the question suggested.

1. Functions of non-analyticity r. If $f(z)=X(x, y)+i\, Y(x, y)$ is an analytic function of z in a region R, it has, at each point of R, a unique derivative

$$f'(z) = \lim_{\delta\to 0} \frac{f(z+\delta)-f(z)}{\delta}.$$

The derivative will not exist if $f(z)$ is non-analytic. We have, however, the following theorem.

THEOREM 1. *Let $X(x, y)$ and $Y(x, y)$ be continuous and have continuous partial derivatives of the first order near $z_0=x_0+iy_0$ and let*

$$w(\lambda) = u(\lambda) + iv(\lambda) = \lim_{\delta\to 0}{}^{\lambda} \frac{f(z_0+\delta)-f(z_0)}{\delta}.$$

where $\lim^{\lambda}_{\delta\to 0}$ denotes that $\delta\to 0$ along a line of slope λ. Then the point $w(\lambda)$ lies on the circle [2]

$$[u - \tfrac{1}{2}(X^0_x + Y^0_y)]^2 + [v - \tfrac{1}{2}(Y^0_x - X^0_y)]^2 = r^2(z_0) \tag{1}$$

where

$$r(z_0) = \tfrac{1}{2}[(X^0_x - Y^0_y)^2 + (Y^0_x + X^0_y)^2]^{1/2}$$

and

$$X^0_x = \left[\frac{\partial}{\partial x} X(x, y)\right]_{x=x_0, y=y_0} \quad etc.$$

Proof. Let $\delta=h+ki$. Then, by law of the mean,

$$\frac{f(z_0+\delta)-f(z_0)}{\delta} = \frac{X(x_0+h, y_0+k)-X(x_0, y_0)+i(Y(x_0+h, y_0+k)-Y(x_0, y_0))}{h+ki}$$

$$= \frac{hX^{\theta}_x + kX^{\theta}_y + i(hY^{\theta'}_x + kY^{\theta'}_y)}{h+ki}$$

where $0<\theta<1$, $0<\theta'<1$ and

$$X^{\theta}_x = X_x(x_0+\theta h, y_0+\theta k), \qquad Y^{\theta'}_x = Y_x(x_0+\theta' h, y_0+\theta' k), \quad etc.$$

Let $k/h=\lambda$, then

$$\frac{f(z_0+\delta)-f(z_0)}{\delta}=\frac{X_x^{\theta}+\lambda X_y^{\theta}+i(Y_x^{\theta'}+\lambda Y_y^{\theta'})}{1+\lambda i}$$

$$=\frac{1}{1+\lambda^2}\left[X_x^{\theta}+\lambda(X_y^{\theta}+Y_x^{\theta'})+\lambda^2 Y_y^{\theta'}\right]$$

$$+\frac{i}{1+\lambda^2}\left[Y_x^{\theta'}+\lambda(Y_y^{\theta'}-X_x^{\theta})-\lambda^2 X_y^{\theta}\right].$$

Since X_x, X_y, Y_x and Y_y are continuous near (x_0, y_0), we have

$$u(\lambda)=\frac{1}{1+\lambda^2}\left[X_x^0+\lambda(X_y^0+Y_x^0)+\lambda^2 Y_y^0\right],$$

$$v(\lambda)=\frac{1}{1+\lambda^2}\left[Y_x^0+\lambda(Y_y^0-X_x^0)-\lambda^2 X_y^0\right].$$

The theorem follows by a direct computation.

We define (1) as the *derivative circle* and its center the *derivative* of $f(z)$ at $z=z_0$. We write [3]

$$f'(z)=\tfrac{1}{2}(X_x+Y_y)+\tfrac{1}{2}i\,(Y_x-X_y).$$

It is interesting to note that if $r(z)=0$, the Riemann-Cauchy differential equations are satisfied and the function $f(z)$ is analytic. Thus we may define $r(z_0)$ as the *non-analyticity* of $f(z)$ at $z=z_0$, and its least upper bound in R, the non-analyticity of $f(z)$ in R.

THEOREM 2. *If $f(z)$ is of non-analyticity $r\geqq 0$ at $z=z_0$, then for any given $\epsilon>0$, we can find $\delta_0>0$ such that*

$$\left|\frac{f(z)-f(z_0)}{z-z_0}-f'(z_0)\right|=r+\epsilon\eta \tag{2}$$

provided that $0<|z-z_0|\leqq\delta_0$ where $-1\leqq\eta\leqq 1$.

Proof. As in the proof of Theorem 1,

$$\frac{f(z_0+\delta)-f(z_0)}{\delta}=\frac{1}{1+\lambda^2}\left[X_x^{\theta}+\lambda(X_y^{\theta}+Y_x^{\theta'})+\lambda^2 Y_y^{\theta'}\right]$$

$$+\frac{i}{1+\lambda^2}\left[Y_x^{\theta'}+\lambda(Y_y^{\theta'}-X_x^{\theta})-\lambda^2 X_y^{\theta}\right].$$

Since X_x, X_y, Y_x and Y_y are continuous, having given $\epsilon>0$ we can find $\delta_0>0$ such that

$$|X_x^{\theta}-X_x^0|<\frac{\epsilon}{4}, \quad etc.,$$

provided that $|\delta| \leqq \delta_0$. Using the notations $u(\lambda)$ and $v(\lambda)$ of theorem 1, we have, then,

$$\frac{f(z_0+\delta)-f(z_0)}{\delta} = u(\lambda) + iv(\lambda) + \eta'\epsilon$$

where $|\eta'| \leqq \frac{1}{2}(1+|\lambda|)^2/(1+\lambda^2) \leqq 1$. Therefore

$$\frac{f(z_0+\delta)-f(z_0)}{\delta} - f'(z_0) = \left[u(\lambda) - \tfrac{1}{2}(\overset{0}{X_x} + \overset{0}{Y_y})\right] + i\left[v(\lambda) - \tfrac{1}{2}(\overset{0}{Y_x} - \overset{0}{X_y})\right] + \eta'\epsilon$$

$$= \xi r + \eta'\epsilon = \xi(r + \eta'\xi^{-1}\epsilon)$$

where $|\xi| = 1$, by theorem 1.

The theorem follows immediately, since

$$r + \epsilon \geqq |r + \eta'\xi^{-1}\epsilon| \geqq r - \epsilon.$$

In the preceding treatment, we have assumed that $X(x, y)$ and $Y(x, y)$ have continuous partial derivatives of the first order. It is better to generalize our definitions as follows.

DEFINITIONS. Let $f(z) = X(x,y) + i\,Y(x,y)$ be continuous in the region R and $X(x, y)$ and $Y(x, y)$ have partial derivatives of the first order in R. Let

$$f'(z) = \tfrac{1}{2}(X_x + Y_y) + \tfrac{1}{2}i\,(Y_x - X_y). \tag{3}$$

Let z_0 be a point in R. If for any $\epsilon > 0$ we can find $\delta_0 > 0$, such that

$$|f(z_0+\delta) - f(z_0) - \delta f'(z_0)| \leqq |\delta|(s+\epsilon), \qquad s \geqq 0$$

provided that $|\delta| \leqq \delta_0$, then $f(z)$ is called a function of *non-analyticity* $\leqq s$ at z. The greatest lower bound of s is called the *non-analyticity* of $f(z)$ at $z = z_0$. We denote it by $r(z_0)$. The least upper bound of $r(z)$ in R is called the non-analyticity of $f(z)$ in R.

Evidently we may put $s = r(z_0)$.

By the two-dimensional form of the modified Heine-Borel theorem,* we have

THEOREM 3. *Let $f(z)$ be of non-analyticity r in a region containing the closed region R. Then given $\epsilon > 0$ we can divide R into a finite number of parts (squares with sides parallel to the axes and their interiors or portions of such squares which do not exist when R is a square) such that inside or on the boundary of any part there is one point z_0 such that the inequality*

$$|f(z) - f(z_0) - (z - z_0)f'(z_0)| \leqq (r+\epsilon)(z - z_0)$$

is satisfied by all points z inside or on the boundary of that part.

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* See Whittaker and Watson, A Course of Modern Analysis, first edition. §3.6.

From the definitions we derive immediately the following rules

THEOREM 4. *Let* $f_x(z)=X_x+iY_x, f_y(z)=X_y+iY_y$, *then*

(a) $$f'(z) = \tfrac{1}{2}(f_x - if_y),$$

(b) $$(f(z) + \phi(z))' = f'(z) + \phi'(z),$$

(c) $$(f(z)\phi(z))' = f(z)\phi'(z) + f'(z)\phi(z).$$

THEOREM 5. *Let* $f(z)$ *and* $\phi(z)$ *be functions of non-analyticity* r_1 *and* r_2 *respectively at* $z=z_0$ *(in a region containing the closed region* R*). Then*

(a) $f(z) + \phi(z)$ *is of non-analyticity* $\leqq r_1 + r_2$ *at* z_0 *(or in* R*)*

(b) $f(z)\phi(z)$ *is of non-analyticity* $\leqq |f(z_0)|\, r_2 + |\phi(z_0)|\, r_1$ *at* z_0

$$(\text{or} \leqq r_2 \max_{z \text{ in } R} |f(z)| + r_1 \max_{z \text{ in } R} |\phi(z)| \text{ in } R).$$

2. A generalization of Cauchy's fundamental theorem.

THEOREM 6. *If* $f(z)$ *is of non-analyticity* r *in a simply connected region* R *and if* C *is a rectifiable simple closed curve lying entirely within* R, *then*

$$\left| \int_C f(z)dz \right| \leqq 4\sqrt{2}\Omega r,$$

where Ω *is the area enclosed by* C. [4].

The proof of this theorem is a simple modification of the ordinary proof of Cauchy's fundamental theorem. It runs as follows.

Proof. Let us divide the whole plane into equal squares with sides of length d and parallel to the real and the imaginary axes respectively. Consequently the interior A of C is divided into a number of sub-regions. Let us re-divide each of the sub-regions in the manner of theorem 3. Then A is finally divided into a number of regions whose boundaries are squares $p_1, \cdots, p_M$ and regions whose boundaries $q_1, \cdots, q_N$ are portions of sides of squares and parts of C. Then

(1) $$\int_C f(z)dz = \sum_{i=1}^{M} \int_{p_i} f(z)dz + \sum_{j=1}^{N} \int_{q_j} f(z)dz$$

where the paths of integration are taken counter-clockwise. According to theorem 3, we can find a point u_i within p_i and a point v_j within q_j such that

$$f(z) - f(u_i) = f'(u_i)(z - u_i) + \eta_i(r + \epsilon)(z - u_i), \qquad |\eta_i| \leqq 1, \quad \text{for} \quad z \text{ on } p_i,$$

$$f(z) - f(v_j) = f'(v_j)(z - v_j) + \zeta_j(r + \epsilon)(z - v_j), \qquad |\zeta_j| \leqq 1, \quad \text{for} \quad z \text{ on } q_j,$$

where ϵ is a given positive number.

Then

$$\int_{p_i} f(z)dz = f'(u_i)\int_{p_i}(z-u_i)dz + (r+\epsilon)\int_{p_i}\eta_i(z-u_i)dz$$
$$= (r+\epsilon)\int_{p_i}\eta_i(z-u_i)dz$$

and

$$\left|\int_{p_i} f(z)dz\right| \leqq 4\sqrt{2}\, c_i^2(r+\epsilon)$$

where c_i is the length of the sides of p_i. Similarly

$$\left|\int_{p_i} f(z)dz\right| \leqq (4d_j + l_j)\sqrt{2}d_j(r+\epsilon),$$

where l_j is the length of the curved part of q_j and d_j is the side-length of the square consisting a part of q_j. By (1),

$$\left|\int_C f(z)dz\right| \leqq 4\sqrt{2}\,(r+\epsilon)\left(\sum_{i=1}^{M} c_i^2 + \sum_{j=1}^{N} d_j^2 + ld\right),$$

where $l = \sum_{j=1}^{N} l_j$ is the length of C.

Let $d \to 0$, $\epsilon \to 0$, then

$$\sum_{i=1}^{M} c_i^2 + \sum_{j=1}^{N} d_j^2 \to \Omega$$

and the theorem follows.

3. The approximation of a non-analytic function by an analytic function. It is well-known that a real continuous function can be approximated by a real continuous function, as accurately as we please. A non-analytic function of a complex variable, however, cannot be approximated by an analytic function with arbitrary degree of accuracy. Before handling this problem, we have to generalize Cauchy's fundamental formula.

THEOREM 7. *If $f(z)$ is of non-analyticity r in a simply connected region containing the rectifiable simple closed curve C, we have for any z inside C,*

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(w)dw}{w-z} + kr, \qquad k \leqq 4\sqrt{2}\left(1+\frac{\Omega}{2\pi}\right),$$

where Ω is the area enclosed by C. [5].

Proof. 1) Suppose C lies within the unit circle about z. (A part or the whole of C may coincide with the circle.) Let us describe, within C, a small circle about z, and join c and C by a cut l so that the region S bounded by c, C and l is simply connected. Let Γ be the boundary. Then

$$\int_C \frac{f(w)-f(z)}{w-z}dw = \int_\Gamma - \int_c = I_1 - I_2,$$

say, where the paths of integration are taken counter-clockwise.

We may choose the radius δ of c so small that, on c,

$$f(w) - f(z) = (w - z)f'(z) + \eta(r + \epsilon)(w - z), \qquad |\eta| \leqq 1.$$

Then

$$I_2 = \int_c f'(z)dw + (r + \epsilon)\int_c \eta dw = (r + \epsilon)\int_c \eta dw,$$

$$|I_2| \leqq 2\pi\delta(r + \epsilon).$$

By theorem 5, the non-analyticity of $(f(w)-f(z))/(w-z)$ at any point within Γ is $\leqq r/|w-z|$, since the non-analyticity of $(w-z)^{-1}$ within Γ is 0.

Dividing the interior S of Γ into any number of sub-regions with boundaries $c_1, \cdots, c_N$, we have

$$I_1 = \sum_{i=1}^{N}\int_{c_i} \frac{f(w) - f(z)}{w - z}dz$$

where the paths of integration are taken counter-clockwise. By theorem 6,

$$|I_1| \leqq 4\sqrt{2}\sum_{i=1}^{N}\Omega_i\delta_i r$$

where Ω_i is the area enclosed by c_i and δ_i is the least upper bound of $1/|w-z|$ within and on c_i. By the definition of double integral,

$$|I_1| \leqq 4\sqrt{2}\,r\iint_S \frac{dxdy}{\sqrt{(x-a)^2 + (y-b)^2}} \leqq 4\sqrt{2}\,r\int_\delta^1 d\rho\int_0^{2\pi} d\phi \leqq 8\sqrt{2}\,\pi r$$

where (a, b) are the coordinates of z.

Making $\delta \rightarrow 0$, we have $\epsilon \rightarrow 0$ and

$$\left|\int_C \frac{f(w) - f(z)}{w - z}dw\right| \leqq 8\sqrt{2}\,\pi r.$$

2) Now consider the case that C does not lie entirely within the unit circle described about z. The arc of the circle, which lies within C, will divide the region enclosed by C into two parts. Let Γ_1 and Γ_2 be their boundaries. Let Γ_1 include z. By theorem 5, $(f(w)-f(z))/(w-z)$ is of non-analyticity $\leqq r$ inside and on Γ_2. By 1) and theorem 6, we have

$$\left|\int_C \frac{f(w) - f(z)}{w - z}dw\right| \leqq \left|\int_{\Gamma_1}\right| + \left|\int_{\Gamma_2}\right| \leqq 8\sqrt{2}\,\pi r + 4\sqrt{2}\,\Omega r.$$

The theorem follows immediately.

The theorem gives an analytic approximation of $f(z)$, since the function $\int_C f(w)dw/(w-z)$ is analytic within C. The error is kr which depends on the non-analyticity r of $f(z)$ and the area Ω of the domain of validity of the formula. From this theorem, we can deduce an approximate Taylor's expansion of $f(z)$ with the same error kr.

REFERENCES

This paper was received from the author without suitable references or indications of which portions of the paper are presentations of the work of other mathematicians and which portions are due to the author. Since it was not feasible to communicate with the author, it was decided to ask some American mathematician to supply suitable notes. This work was done by Professor John De Cicco of Illinois Institute of Technology.

THE EDITOR

1. A complex function $w=X(x, y)+iY(x, y)$ of the complex variable $z=x+iy$, where the dependent real functions (X, Y) possess continuous partial derivatives with respect to the real variables (x, y) but do not necessarily satisfy the Cauchy-Riemann equations, has been termed a *polygenic function* of z by Kasner. If the Cauchy-Riemann equations are satisfied, then the function is, of course, monogenic in z. See Kasner, A theory of polygenic (or non-monogenic) functions. Science, 66, 581–582, 1927.

2. This derivative circle has been called the *Kasner circle* by Hedrick and others. Kasner first obtained this circle in 1927 and has since developed a very interesting geometry of the derivative of a polygenic function. See Hedrick, Non-analytic functions of a complex variable, Bull. Amer. Math. Soc. 39, 75–96, 1933.

3. What the author defines as the derivative of a polygenic function is really the mean derivative. As an operator, the *mean derivative* is

$$\mathfrak{D} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

and the *phase derivative* is

$$\mathfrak{P} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

If $w=X+iY$ is an analytic polygenic function, that is, if X and Y are each Taylor's Series in (x, y), then the mean derivative is the formal partial derivative of w with respect to $z=x+iy$, and the phase derivative is the formal partial derivative of w with respect to $\bar{z}=x-iy$.

4. This result is intimately related to Pompieu's areolar derivative theorem which states that

$$\lim_{C\to 0} \int_C \frac{f(z)dz}{\sigma} = 2i\mathfrak{P}[f(z)]$$

where σ is the area enclosed by C. See D. Pompieu, Rendiconti di Palermo, Vol. 33, 108–113, 1912 and Vol. 35, 277, 1913.

5. This may be obtained as a consequence of Pompieu's representation of a polygenic function, which is the following

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(w)dw}{w-z} + \frac{1}{\pi}\int\int_R \frac{\mathfrak{P}[f(w)]dxdy}{z-w}.$$

See also the following papers which contain a bibliography of other papers on polygenic functions.

Kasner, The second derivative of a polygenic function, Trans. Amer. Math. Soc. 30, 803–818, 1928.

Kasner and DeCicco, The derivative circular congruence representation of a polygenic function, Amer. Journal Math. 61, 995–1003, 1939.

Finally see a forthcoming paper by DeCicco, Survey of polygenic functions, Scripta Mathematica, 1945.

CLUBS AND ALLIED ACTIVITIES

EDITED BY J. S. FRAME

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to J. S. Frame, Michigan State College, East Lansing, Michigan.

CLUB REPORTS 1943–44

Kappa Mu Epsilon, Central Michigan College of Education

Meetings are held twice a month. Eighteen new members were initiated in the fall of 1943, and thirteen in the spring of 1944. Due to the Navy V-12 program, some members have been transferred to advanced training. By July 1 many more will have left. The membership at the present time totals thirty-three. Although the membership has been varying every four months due to transfer of the V-12 students to advanced training, there has not been a noticeable decline in the activities while they have been with us. The necessity of mathematics is known to all of these men who have left us.

Discussions at two meetings in March and April dealt primarily with the publication of a pamphlet. The main object of this activity is to discuss the mathematical requirements in various fields (science, business, teaching, engineering, *etc.*). Upon the completion of the pamphlet it will be distributed among high school students and it is expected to clarify and help them in the selection of mathematical courses while in high school.

The following talks, from one-half to one hour in length, were presented during the winter and spring.

The Mercator chart, by Paul Decess

Mathematical recreations, by Fran Teel

Cryptanalysis, by Dr. Cleon Richtmeyer

History of calculus and its application to physics, by Bob Murch.

The annual picnic was held early in June. Officers for the fall term of 1943 were: President, Paul Brown; Vice-President, Emma Skinner; Secretary, Betty Sack; Treasurer, Charles Turner. Officers for the spring term of 1944 were: President, Emma Skinner; Vice-President, George Yerganian; Secretary, Betty Sack; Treasurer, Joyce Sherwood. Officers for next fall are: President, Joyce Sherwood; Vice-President, Pauline Nelson; Secretary, Martha Toles; Treasurer, Margaret Beck; Corresponding Secretary, L. C. Gray.

Delta-x, University of the City of Toledo

At the first meeting of *Delta-x* for 1943–44, the members of the club amended a requirement that all members must be studying or have studied calculus, so that students taking college algebra, trigonometry or analytic geometry are eligible for associate membership. This action has resulted in an active membership nearly as large as that of other years despite the large number of members

called to the armed forces. Papers of interest to students of mathematics were presented at each of the regular monthly meetings:

Series, by Tom Jarrett
Geometrics of paper folding, by Margaret Kauffmann
Special properties of the conics, by Ellsworth Shinavar
Computation of pi, by Larry Muttart
Computation of e, by Lois Martin
Curve fitting, by William Bergman
Sun dials, by William Lunn
Geometric proofs of trigonometric formulas, by Mildred Schalkhauser

The annual *Get-Acquainted Roast* opened social activities of the club. The *Hard Times Party* was held in November and a Christmas party was held after the regular December meeting. In January the members enjoyed an ice skating party. The Annual *Delta-x* Banquet was held in May. The speaker was Dean John Brandeberry, a faculty member of the club. His topic was:

Imaginary branches of real curves.

The last function of the club for the year was a picnic. *Delta-x* yearbooks were presented to all members. The co-editors were Margaret Kauffmann and Lois Martin.

Officers elected for the coming year were installed after the banquet meeting by the retiring officers and Professor Dancer, Faculty Adviser. They are: President, William Lunn; Vice-President, Lois Martin; Secretary-Treasurer, Margaret Kelley.

Mathematics Club, Montana State University

The following talks were presented during 1943–44:
The life and work of Fermat, by Phyllis Biddle
Mathematician's apology, by Grayce Miller
Mathematics in big game hunting, by Dr. Roy Dubisch
Anecdotes on mathematicians, by Dr. Harold Chatland
Take-off on mathematics, by Mrs. Mel-Iden Chambers
Logarithms, by Kenneth Axvig
Navigation, by Dr. Harold Chatland
Mathematical tricks, by Mark Jacobsen
Amusements in mathematics, by Anna Harwick
Mathematical nuts, by Elizabeth Nelson
The basis of geometry, by Dr. N. J. Lennes.

The annual party for members of the Mathematics Club and their guests was held at the home of Dr. Chatland on February 18th. After Miss Nelson's talk on May 4th, the Club, together with the university chapter of *Pi Mu Epsilon* decided to have pictures taken of our two retired professors of mathematics, Dr. N. J. Lennes and Dr. E. F. A. Carey, to be framed and hung in the Mathematics Library of Craig Hall. The next afternoon the annual Math-Chem Clubs picnic was held at Greenough Park in Missoula. Officers of the club were: President, Phyllis Biddle; Secretary, Anna Harwick; Treasurer, Grayce Miller.

Mathematics Club and Kappa Mu Epsilon, College of Wooster

Kappa Mu Epsilon held only two meetings during the year, because of the limited number of advanced mathematics students. At the last meeting five new members were elected and it is hoped that there will be enough members to warrant regular meetings next year.

The *Mathematics Club* has met regularly every two weeks during the college year and considered the subjects of *Navigation and Meteorology* as they pertain to flying. Professor C. O. Williamson, who is a licensed pilot and has taught navigation in the Navy Preflight School, did most of the lecturing and illustrated with interesting experiences. A picnic was held in September and a Christmas party in December. The officers elected for 1944–45 are: President, Anne Widener; Vice-President, Janet Thompson; Secretary-Treasurer, Lois Hayenga.

Junior Mathematical Club, University of Chicago

During the year 1943–44 the *Junior Mathematical Club* of the University of Chicago continued its fortnightly meetings and teas, and supplemented this schedule with a party in the Winter Quarter and a picnic in the Spring Quarter. At the regular meetings, the following papers were presented to the Club:

Topological spaces, by Robert Kates

Mathematics in dynamic meteorology, by Lawrence Markus of the Institute of Meteorology

Mathematical paradoxes, by Professor E. P. Northrup

Functional equations, by Daniel Zelinsky

A mathematical problem in traffic control, by Professor L. R. Wilcox, Illinois Institute of Technology

Fourier coefficients, by Marie Wurster

Soap films and the calculus of variations, by Professor W. T. Reid

Dynamics of rigid bodies, by Daniel Zelinsky

Some problems in contour integration, by Sidney Dancoff of the Metallurgical Laboratory

Product complexes, by Hyman Zimmerberg

Relative motion and the accelerations of Coriolis, by Professor E. P. Lane

A matrix method in curvilinear coordinates, by George Platzman of the Institute of Meteorology

Transformations of surfaces, by Mary Petty.

The student paper adjudged to have the best mathematical content and presentation was that on *Dynamics of rigid bodies*: the prize awarded to Mr. Zelinsky for this paper was a copy of Weyl's *Classical Groups*.

Officers of the Club for 1943–44 were: President, Daniel Zelinsky, Social Chairman, Mary Petty; Treasurer, Hyman Zimmerberg; Committee, Charles Nichols and Marie Wurster. Officers-elect for 1944–45 are: President, Charles Nichols; Social Chairman, Marie Wurster; Treasurer, Betty Alexander; Committee. Verna LaMantia and Herman Rubin.

RECENT PUBLICATIONS

EDITED BY VIRGIL SNYDER, Cornell University, and H. P. EVANS; University of Wisconsin

All books for review should be sent directly to the editors of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y., and not to any of the other editors or officers of the Association.

Analytic Geometry. By K. B. Patterson and A. O. Hickson. New York, F. S. Crofts and Company, 1944. 10+187 pages. $2.10.

To quote from the preface, "this book contains topics usually treated in a brief course on analytic geometry and can be discussed thoroughly in from forty to fifty class periods. . . . it contains many examples and more figures than are usual in a brief course in the subject." The format is pleasing and the figures are clear though somewhat small. A list of the chapter headings should indicate well enough the general contents of the book: 1. Cartesian Coordinates. 2. The Loci of Equations. 3. The Equation of a Locus. 4. The Straight Line. 5. The Circle. 6. Conic Sections. 7. Translation and Rotation of Axes. 8. Polar Coordinates. 9. Parametric Equations. 10. Tangents and Normals. 11. Analytic Geometry of Three Dimensions. The problems are of the type usually found in elementary American textbooks and seem adequate for the purpose. In chapter 10 the notions of limit and derivative are introduced and applied to the theory of tangents.

Although the book makes no attempt to give a true axiomatic approach to analytic geometry, it illustrates a recent tendency in mathematical texts which deserves encouragement. (It seems to the reviewer that the titles of chapters 2 and 3 are very revealing as to the spirit of the authors.) Naturally the book assumes a knowledge of the elements of algebra and trigonometry, but the authors have tried to supply carefully explicit definitions at appropriate points in the development of the subject. As the following examples should indicate, theorems are explicitly stated, usually with great care: "Theorem 16. The equation of every straight line is of the first degree." "Theorem 17. Every equation of the first degree in x and y represents a straight line."

It would appear that this book could be materially improved by the addition of perhaps five pages, in the form of qualifying sentences, additional remarks and definitions, and a few expanded or corrected proofs. The reviewer finds himself in disagreement with about forty items scattered throughout the text, most of which represent comparatively trivial slips easy of correction. (We are not speaking of typographical errors; of the latter we found only two, both perfectly obvious.) As a typical example we note on page 155 the italicized false statement that "the slope of the parabola $y^2=4ax$ at any point (x, y) on the parabola is $dy/dx=2a/y$." Despite the fact that the authors are careful at more than one point to ban division by zero, a number of theorems and proofs are false because

of the neglected possibility of zero denominators. As a further example, Theorem 24 (page 56) gives a false "necessary and sufficient condition" (in determinantal form) that three straight lines should have a common point. The criterion fails in the case of three parallel lines, and the "proof" moreover tacitly assumes (in the absence of any explanatory remark) that no two of the lines are parallel. An equally serious matter (page 15) is a "description," not formally presented as a definition, of an algebraic equation; this description would embrace among others the equation $y = \sin x$. Again, on pages 34–35 the authors raise the important question as to whether the graph of an equation derived from a geometrically defined locus will necessarily represent all of that locus and nothing else; but the example they give appears to be nicely calculated to quiet any doubts which a student might entertain. It seems unfortunate moreover that this remark should be recalled several times in connection with the standard equations of the conic sections.

Lest there be any suggestion of bias in this review we hasten to mention an excellent section on horizontal and vertical asymptotes, a well-phrased discussion of families of lines, and a good chapter on transformation of axes. At the end of the chapter on conics there are in addition some geometrical proofs of the fact that conics may be obtained as plane sections of a cone. (The last of these proofs seems to contain an obvious slip.)

If a revised edition of this book should be brought out it could easily be made into a first rate example of its kind.

R. H. BRUCK

Tables of Lagrangian Interpolation Coefficients. Prepared by the Mathematical Tables Project, Work Projects Administration of the Federal Works Agency, conducted under the sponsorship of the National Bureau of Standards. New York, Columbia University Press, 1944. 36+392 pages. $5.00.

TABLE I (extent 100 pages) gives three-point interpolation coefficients $p(p+1)$, $(p-1)(p+1)$ and $p(p-1)$ to 9, 8 and 9 decimals from $p=0.0000$ to $p=1.0000$. TABLES II to IX give four-point to eleven-point coefficients. Extent of TABLE IX is 2 pages (10 decimals). In the reviewer's opinion, 2 pages for TABLE I and 100 pages for TABLE IX would offer more. TABLE X gives interpolation coefficients from 3 to 8 points for p at intervals of 0.1. TABLE XI does the same for p in multiples of 1/12. TABLES XIIa and XIIb give Lagrangian integration polynomials and integration coefficients. A 22-page introduction by A. N. Lowan gives a concise and thorough exposition of Lagrangian interpolants with an analysis of the error and applications to numerical integration. This publication definitely is an excellent help for the specific purpose of interpolation and numerical integration. The cover size is 11″ by 8″, the photo-offset reproduction is immaculately distinct and clear.

M. A. SADOWSKY

Basic Air Navigation. By E. E. Blackburn. New York, McGraw-Hill Book Company, Inc., 1944. 7+296 pages. $3.00.

Here is a text in which are presented in orderly manner all the problems that the air navigator must deal with in planning a flight; in flying his airplane at the proper height, with the right speed, and in the correct direction; and, finally, in bringing his craft safely to its destination. The author states in the Preface that the book is written to assist the pilot in his rigorous training in navigation, and that he has endeavored to keep the subject simple. The author's years of experience in flying and in teaching navigation, and his ability to express in simple, understandable language the various phases of the problems confronting the navigator prove that he has successfully accomplished the task he set for himself.

The few examples involving numerical calculations require the ability to do the simple arithmetic operations. The triangle of velocities is solved graphically or by means of a computer. There is one reference (page 179) to a sine, but no knowledge of the trigonometric functions is required, for tables are provided giving necessary data where such values are needed. The slide rule (especially, the circular slide rule) and the ground speed computer are recommended for use where possible. Charts, maps, and graphs are used freely in solving problems involving direction, distance, gas load, gas consumption, altitude, speed, time, *etc.*

Problems of flying by day and by night, over land and water, with and without land or water marks, with radio aids, and with celestial bodies as guides, are all treated carefully. Methods of determining lines of position and fixes are explained. Two chapters are devoted to the principles and practices of celestial navigation.

The tools of the navigator are pictured and their uses are explained in connection with the subject as it progresses. These tools include sample pages from the tables H. O. 208 and H. O. 214; a list of fifty-five navigational stars, and diagrams for locating them; sample pages from the *American Air Almanac* giving appropriate data on the solar-system bodies, sun, moon, Venus, Mars, and Jupiter; and such instruments as the computer, altimeter, compass, air-speed indicator, drift indicator, drift bombs, octant, and chronometer.

In the final chapter the author gives general instructions to the student navigator. He emphasizes the points to be considered in preparing for the flight, in the course of the flight, and upon approach of the terminal point of the flight. He advises careful planning of the flight and alertness at all times.

The text contains a few errors and defects of which the most noticeable are: (1) The last part of the first sentence on page 137 " · · · , he will either add or subtract 90° from this direction" should read something like this, " · · · , he will either add 90° to or subtract 90° from this direction." (2) Commas should follow the words "altitude" and "temperature" on page 190, third line from the bottom of the page. (3) The sentence beginning on page 260, fourth line from the

bottom, is misleading. The sentence should state clearly that when the sun, planets, and stars are viewed from two points, one on the earth's surface and the other at the earth's center, the rays of light from any one of these bodies may be considered parallel. (4) On page 203, eighth line from the bottom, the distance of the closest star is mentioned as "so many million light-years." This, of course, is an exaggeration, for the distance of the nearest star is slightly more than four light-years. (5) The statements in the last paragraph, page 217, concerning the earth's rate of travel in miles per minute and the apparent motion of the sun are not accurate. (6) In the lower part of FIG. 253, page 215, the star shown as Al N'air is correctly spelled Al Na'ir. (7) The two stars in FIGS. 241 and 248 designated by the Greek letter δ (delta) should be ϵ (epsilon). (8) The footnote, page 247, contains the statement that "FIGURE 275 is also contained in the pocket in the back of the book." The list of Chart Problems on the pocket did not include this as one of the figures in the pocket and it was not found there. It is given, however, on page 248 of the text. (9) In FIGS. 291a and 298, and in the text below FIG. 291a the bubble knob is designated by a large "B," while in the last paragraph on page 265 the knob is referred to by a small "b." (10) The data on the extreme right margin of FIG. 320 are incomplete, probably due to faulty printing.

The errors and defects listed above should not be weighed too heavily against the value of the book, for the reviewer believes, after a careful examination of this text, that the author and publishers of *Basic Air Navigation* have combined in giving to the student navigator a text of very high class, and in no case will any of the faults discovered lead the pilot to any error in the preparation for, or in the execution of, the flight. The text should prove valuable to the student navigator.

H. H. DOWNING

Navigation. By L. M. Kells, W. F. Kern, and J. R. Bland. New York & London, McGraw-Hill Book Company, Inc., 1943. 20+479 pages. $3.75.

This textbook on navigation is a complete exposition of the information necessary to determine one's location on the surface of the earth and of the "rules of the road" in going from place to place. The chief emphasis is laid on detailed procedure in making measurements, recording the data, manipulating the numerical values and interpreting the numerical results. Mathematics, as such, is distinctly avoided and the more difficult portions are relegated to an appendix which may be consulted by those students who have the curiosity and the desire to justify the methods.

The first chapter explains most of the mathematical terms and operations and it shows the student how to insert values in a formula, to use a slide rule or a table of logarithms. The rest of the book is devoted to a description and use of the various instruments, to charts, almanacs, signals, navigation laws and rules, range finding, the elements of astronomy and star identification. The basic formulas of spherical trigonometry are given in a ten-page chapter.

The various methods of piloting are carefully outlined. These include the astronomical methods, dead reckoning, sighting on lighthouses, buoys and landmarks and radio beams. A large selection of exercises and answers are very helpful for both the student and the instructor.

There are many excellent line diagrams. In addition, the book is profusely illustrated with inspirational pictures, principally official Navy photographs. Most of these have explanatory notes below them although a number of them have no necessary connection with the adjacent text. This book is definitely intended primarily for Navy personnel. It emphasizes Navy methods, charts and ship and plane maneuvers.

The thoughtful arrangement of the text and sketches, the use of heavy type for emphasis and the high quality of the typography makes this book very readable.

MICHAEL GOLDBERG

Basic Marine Navigation. By B. J. Bok and F. W. Wright. Boston, Houghton Mifflin Co., 1944. 8+422 pages. $4.50.

The major portion of this text is devoted to geo-navigation, with special emphasis on coastwise navigation. Piloting, Dead Reckoning, and the compass are treated thoroughly, but the sailings are passed over lightly. The only chart that receives much attention is the Mercator projection, and here the development is intuitive. The chapter on tides and currents and the chapter on meteorology make interesting and instructive reading. In the treatment of celestial navigation, the authors have tried to use only terms that are essential. Nothing is said of the real motion of the planets, the whole treatment is from the viewpoint of apparent motion. The chapter on the sun is well done, but the problem of finding a fix from the stars, planets, and moon is rather hastily passed over. The solution of the astronomical triangle is mainly made through the use of the H. O. 214 tables. However, a short chapter is devoted to the discussion of H. O. 208 and H. O. 211. The last chapter of the book is devoted to the problem of navigating a lifeboat.

The print is good, and the authors have an interesting style. There are, however, very few proofs as the reader is not assumed to have a knowledge of trigonometry. The general plan of attack throughout the book is to state the problem and how to solve it, followed by a well chosen illustrative example.

A *Kit* containing maps and practice material is published separately from the text, but is needed for the solution of some of the problems of the book. While the text was drafted by the first-named author, Bok, and the problems by Wright, the problems correlate nicely with the material of the book and are well stated and instructive.

The authors state that "the book and the *Kit* together provide a complete outfit for a self-taught course in navigation." However, many teachers will feel that more proofs and more problems are needed when the book is used as a textbook for a course in navigation.

F. G. DRESSEL

Mathematics for Exterior Ballistics. By G. A. Bliss. New York, John Wiley and Sons, Inc., 1944. 7+128 pages. $2.00.

This small book is based upon material used several times in courses given by the author at the University of Chicago. Professor Bliss, widely known and respected for his research, teaching and direction of graduate training in mathematics, was one of the small but enthusiastic band of research mathematicians in the Range Firing Section at Aberdeen Proving Ground during part of World War I. Professor Bliss contributed notably to the handling of differential corrections by applying the notion of functions of lines and introducing into practical ballistic computations appropriate adjoint equations.

The book is written in a clear and interesting style with material suitable for undergraduate study. After an opening chapter largely concerned with military maps, separate chapters treat of: II. The differential equations of a trajectory, III. The Siacci theory, IV. The approximate integration of the equations of exterior ballistics, V. Differential corrections, VI. Bombing from airplanes. A supplement consists of Tables for Computations. There are a few short sets of exercises headed "Examples" scattered through the book.

There seems to be no question that this is one of the most attractive brief American texts on this now timely subject.

One mild comment has been made by several local readers. It does not affect the suitability of the text for preliminary study:

To anyone currently engaged in ballistic computation under circumstances analogous to those to which the author of this text was introduced during World War I, there comes a curious sense of suspended animation on reading in this text the numerous unqualified assertions purportedly describing present practice. The reader, an unwilling Rip van Winkle, is inclined to turn to the cover and gaze incredulously at the printed date "1944." The terms and the practices here reported often sound unrealistically quaint to the present staff of young computers. They never use the term "range table"; the "standard trajectory" of today is not that described in the text. The corrections for sphericity are not now made as here described, and the corrections for the rotation of the earth are at present handled in a much simpler manner. Other routine corrections, much more important than these, are missing completely in this text. Even in such a static matter as map reading this text is out of date. The recent basic field manuals on map reading have freed themselves from some of the French overlay of terminology whose only excuse for original adoption lay in the special adjustment problems incident to American participation with an actively engaged army of France. More specific and detailed data might be reasonably expected from the reviewer to support these unenthusiastic comments, except for the fact cited by the author himself in the preface: "Not much can be said here .... They are closely guarded secrets, naturally not available to a civilian writer."

A. A. BENNETT, Major, Ord. Chief, Computing Branch
Ballistic Research Laboratory, Aberdeen Proving Ground, Md.

Celestial Navigation. A Problem Manual. By Walter Hadel. New York and London, McGraw-Hill Book Company, 1944. 13+261 pages. $2.50.

"This collection of practical problems in celestial navigation, including various calculations in simple dead-reckoning navigation, has been put together in the form of a text, or guide, to a course in celestial navigation." (From the preface.)

In the many texts on navigation which have appeared lately and are still appearing frequently, each author has his own ideas about what should be included and what a text-book should contain. Some go to the extreme of simplicity and others give in great detail every phase of the subject. This author goes on the assumption that the student should learn by practice. He believes that the student should be able to work the necessary problems with accuracy first and speed second in importance. He bases his course on an orderly succession of "Celestial Quizzes" which covers the necessary material with a minimum of theory and a maximum of practice.

There are 46 Celestial Quizzes, including four in Appendix A, arranged as lessons would be in an ordinary text. Each quiz begins with a page (more or less) of explanatory notes, then states the problems to be solved and solves a few, leaving the remainder to be worked out by the student.

For example, the first quiz begins with the averaging of time and angle. This must be done in reducing all observations with the sextant. Next is immediately introduced the standard practice of determining LHA (Local Hour Angle) and changing from Local Standard Time to Greenwich Civil Time, and the first lesson ends with the method of drawing azimuths. It seems likely that a student would have some difficulty in beginning his study of navigation from this book by himself and that he would need some help from an instructor.

In the succeeding lessons or quizzes, the author takes up such subjects as charts, lines of position, running fixes, corrections to observations and compass corrections by the method of short explanations and problems, some worked out and some left for the student.

One desirable feature of the book is the establishment of time limits for working the problems and a speeding-up of work as the student becomes more proficient. The whole idea is attractive to the instructor who wants a series of exercises for his students either for use with an explanatory text or with classroom lectures on the theory of navigation. The mathematics is reduced to addition and subtraction, although the development of the formulas for H. O. 211 is included in the Appendix. This development is based on spherical trigonometry, the details of which are left to the instructor.

All the work is based on modern methods, using the American Air Almanac and H. O. 214 and H. O. 211. Sufficient extracts are given in the Appendix B to enable the student to work all problems. Also there are a few examples from Weems' Star Altitude Curves—a rapid but somewhat inaccurate system of obtaining pre-computed altitudes.

It is obvious that the book has been prepared from the viewpoint of the air navigator. However, the air methods are being adopted, in part at least, by mariners, even to the extent of using the Air Almanac.

For a problem manual supplementing a course in the theory of navigation or where the practice is given without the theory, this book should be very useful and is recommended for careful consideration by all instructors in the field of navigation.

C. M. HUFFER

Riddles in Mathematics. A Book of Paradoxes. By E. P. Northrop. New York, D. Van Nostrand Company, 1944. 8+262 pages. $3.00.

This volume comprises a pleasantly informal collection of illustrative examples and problems illustrating common errors, interspersed with comments of historical and philosophical nature. Topics range from divison by zero to Russell's theory of types. The material might furnish entertaining supplementary reading for high school and undergraduate mathematics students, and persons with slight knowledge of the subject could enjoy at least the grammatical and topological "paradoxes."

After the two introductory chapters with paradoxes "for everybody," comes a chapter on arithmetic including the growth of powers of 2, Fermat numbers, division of the circle, and the game of Nim. Chapter IV, "paradoxes in geometry," starts with simple optical illusions and brings up among other things the Fibonacci series, curves of constant breadth, the Konigsberg bridge problem, Klein's bottle, and the four color problem. Chapters V and VI on "algebraic fallacies" and "geometric fallacies" demonstrate incorrect proofs of things such as $-1=+1$ and any triangle is isosceles. Chapter VII, "paradoxes of the infinite," brings in a bit of cardinal number theory, conditionally convergent series, and pathological curves. In Chapter VIII, on probability, some right, wrong, and uncertain methods of computing probability are included with a discussion of the theory and applications. Chapter IX on "paradoxes in logic" presents some of the difficulties in the foundations of mathematics, describing briefly the three schools of thought and discussing paradoxes like those of Richard and Burali-Forti. Chapter X on "paradoxes in higher mathematics" contains a brief and odd assortment of examples involving more formal mathematics, the most interesting being a demonstration of how in the computation of the surface area of a cylinder as the limit of inscribed polyhedra one may obtain different results.

An appendix gives brief explanations or discussions for problems posed by the author in the body of the book. A chapter by chapter bibliography and an index end the volume.

BERNARD VINOGRADE

Calculus Refresher for Technical Men. By A. A. Klaf. New York, McGraw-Hill Book Company, Inc., 1944. 8+431 pages. $3.00.

This book represents the extreme in the method of teaching by rote and example. As its title indicates, it is intended as a review of calculus rather than as a first presentation. In unique fashion it presents the subject catechistically, posing questions and following them with answers, which consist of brief unjustified statements usually followed by illustrations.

Most of the customary topics of the standard calculus text are covered, and numerous examples from diverse engineering fields are given. A notable feature is the complete detail with which the illustrative problems are solved, even to minute numerical calculations. A substantial portion of the book is devoted to formal processes, some sixty pages being spent, for example, on formal integration. The entire subject of series is omitted, and differential equations are barely introduced.

There are numerous mathematical errors and rhetorical peculiarities which cannot be overlooked because of their cumulative effect. For example, the term *function* is used to denote an equation connecting variables, rather than what is usually meant; no hint is given that functions are not always defined by analytic formulas. Discontinuous functions are those which become infinite, according to the author; certainly even technical men should know about finite discontinuities. Differentials are used before they are defined. Integrals are defined as antiderivatives, thus rendering a logical presentation of the fundamental theorem of integral calculus impossible. Indeed, the statement of this theorem is nothing short of nonsense. The author's motive for differentiating rational functions by first decomposing into partial fractions and then claiming the method superior must be obscure. In sequences of mathematical statements or equations the author omits punctuation, even final periods, apparently unaware that even mathematical discourse consists of sentences.

But more serious than such errors is the complete lack of exact ideas, careful definitions and proofs. The inevitable "explanations by example" must become extremely monotonous and irritating to any but the most stupid of students. The mathematician who sees this book cannot help feeling that his subject is being systematically robbed of its content and left only with the dregs of formal manipulations—all of which can benefit no one, least of all technical men, whose need for more not less, stress on fundamental concepts is admittedly desperate.

The only apparent legitimate use of the book is as a reference source for a few lesser known applications of calculus.

L. R. Wilcox

NEW BOOKS RECEIVED

Handbook of Air Navigation. By W. J. Vanderkloot. New York and London, McGraw-Hill Book Company, 1944. 13+333 pages. $3.50.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND H. S. M. COXETER

ELEMENTARY PROBLEMS

Send communications concerning Elementary Problems and Solutions to H. S. M. Coxeter, 24 Strathearn Boulevard, Toronto 10, Canada.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTIONS

E 641. *Proposed by F. M. Garnett, Savannah, Ga.*

A lighter, twenty by thirty feet, travels upstream at a uniform speed of 2 m.p.h., against a current of $1\frac{1}{2}$ m.p.h. A swimmer, whose rate of swimming is 4 m.p.h. in still water, swims around the lighter. How long does he take to complete the circuit?

E 642. *Proposed by V. Thébault, San Sebastián, Spain*

In the duodenary scale, a certain number does not end in 0, but its square does. Show that the square ends in 30, and the cube in 60.

E 643. *Proposed by W. E. Buker, Pittsburgh Public Schools*

The sides a, b, c, d of a plane quadrangle being given in order and also the area A, find the length of the longer diagonal.

E 644. *Proposed by A. W. Goodman, Republic Aviation Corporation*

Do there exist determinants of order $n>2$, such that all the $n!$ terms of the expansion are positive?

E 645. *Proposed by C. D. Olds, Purdue University*

Let P_1, P_2, P_3 be any three points on a plane curve C, and O a point in the same plane. If the areas of the three triangles OP_2P_3, O_3PP_1, OP_1P_2 are connected by a relation independent of the coordinates of P_1, P_2, P_3, prove that C is a central conic and O its center.

SOLUTIONS

A Naval Encounter

E 606 [1944, 88]. *Proposed by H. T. R. Aude, Colgate University*

An enemy ship A is proceeding on a course directly east at its top speed of 25 miles per hour when it is spied by a more powerful ship B of a speed of only 20 miles per hour. At the instant of discovery the ship B is 10 miles directly south of A. On account of the shore line the enemy ship A must continue its course, while the course of B is altered to $N\theta E$ to bring A within range. If the effective range is not more than 7 miles, find the course of B that will bring

about the least distance AB, and the interval of time (while following this course) during which A will be within range.

Solution by Frank Hawthorne, Allegheny College. Consider the motion of B relative to A. Construct a vector of length 25 (miles per hour) west from B to G. With G as center strike an arc of radius 20, and draw the tangent from B to this arc, the point of tangency being C. Then GC is in the direction B should sail. Triangle BGC is the common 3, 4, 5 right triangle; hence

$$\theta = \text{arc tan } \tfrac{4}{3}.$$

Thus the motion of B relative to A is 15 miles per hour in the direction BC.

Construct a vector of length 10 (miles) north from B to A. With A as center strike an arc of radius 7 to cut BC in the points D and E. Let F be the midpoint of DE. Now, since triangles AFB and BCG are similar, we have $AF/AB = BC/BG = 3/5$. But $AB = 10$; so $AF = 6$. Also $AD = 7$; hence $FD = \sqrt{13}$ and $DE = 2\sqrt{13}$. Thus the distance within range is $2\sqrt{13}$ miles, and the time within range is $2\sqrt{13}/15$ hours or $8\sqrt{13} = 28.85$ minutes.

Also solved by E. F. Allen, Murray Barbour, Colin Blyth, W. B. Campbell, L. R. Chase, Howard Eves, A. E. Gault, H. M. Gehman, R. W. Hamming, L. M. Kelly, D. A. Oliver, Jr. (using a real "maneuvering board"), E. K. Paxton, E. P. Starke, C. W. Topp, and W. Unterberg.

Barbour remarks that it is unreasonable to expect B to continue on the same course until A is out of range. By changing to a more easterly course as soon as A is within range, B could increase the time within range to 39.5 minutes.

A Number Divisible by the Sum of its Digits

E 608 [1944, 88]. *Proposed by C. H. Wolfe, Lakeside High School, Ohio*

A certain three-digit number yields a quotient of 26 when divided by the sum of its digits. If the digits are reversed, the quotient is 48. What is the smallest three-digit number for which this is possible?

Solution by Irving Kaplansky, New York, N. Y. We make use of only the *first* of the conditions. By casting out nines, the sum of the digits is a multiple of nine, *viz.*, 9, 18, or 27. Thus the smallest such number is $9 \cdot 26 = 234$ (and the only other is $18 \cdot 26 = 468$, as $27 \cdot 26$ has four digits).

Also solved by F. A. Alfieri, E. F. Allen, Murray Barbour, C. B. Barker, Colin Blyth, W. H. Bradford, D. H. Browne, W. E. Buker, H. N. Carleton, R. E. Crane, Marian E. Daniells, Monte Dernham, William Douglas, Howard Eves, A. E. Gault, R. W. Hamming, Frank Hawthorne, T. W. Jackson, Aida Kalish, R. J. Koch, Helen McDevitt, Walter Penney, P. W. A. Raine, W. C. Rufus, E. D. Schell, M. J. Sheehy, E. P. Starke, P. D. Thomas, W. Unterberg, Jeanette Van Os, Alan Wayne, Michael Wilensky, Hazel Schoonmaker Wilson, W. R. Talbot and the proposer.

Starke remarks that the problem has the same solution if we interpret the

hypothesis as requiring divisions where there is a quotient and a remainder, the quotient only being considered.

Diagonals of Faces of a Parallelepiped

E 609 [1944, 88]. *Proposed by Frank Hawthorne, Allegheny College*

Show that the diagonals of three faces of a parallelepiped, drawn from the same vertex and prolonged half their length, determine three points which are coplanar with the opposite vertex.

Solution by R. C. Buck, Cambridge, Mass. With the selected vertex as origin, take vectors a, b, c along the edges of the parallelepiped. The extended diagonals are then

$$3(b + c)/2, \qquad 3(c + a)/2, \qquad 3(a + b)/2.$$

The mean of these three vectors is $a+b+c$, which leads to the opposite vertex. Thus, not only is the opposite vertex coplanar with the three constructed points, but it is their centroid.

Also solved by E. F. Allen, W. H. Bradford, D. H. Browne, Howard Eves, A. E. Gault, R. W. Hamming, Irving Kaplansky, E. P. Starke, P. D. Thomas, J. A. Zilber, and the proposer.

Kaplansky remarks that the following generalization can be proved similarly. The diagonals of n cells of an n-dimensional parallelotope, drawn from the same vertex and prolonged by $1/(n-1)$, determine n points whose centroid is the opposite vertex.

Curves of Constant Width

E 610 [1944, 88]. *Proposed by Howard Eves, Syracuse University*

(a) Show that all closed curves of the same constant diameter, d, have the same perimeter, πd.

(b) What is the least area that a closed curve of constant diameter d may have?

(Such a "curve of constant width" touches two parallel lines, distant d apart, drawn in any direction. See, *e.g.*, H. Steinhaus, *Mathematical Snapshots*, 1938, p. 51.)

Partial Solution by the Proposer. By infinitesimal geometry we can very readily show that if r_1 and r_2 are the radii of curvature at a pair of opposite points on the closed curve, then $r_1+r_2=d$.

Let the intrinsic equation of the curve be $s=f(\phi)$, where $f(0)=0$. Then, if p is the perimeter of the curve, $f(2\pi)=p$. Since $f'(\phi)=r$, the radius of curvature, we have

$$f'(\phi + \pi) + f'(\phi) = d.$$

Integrating with respect to ϕ, we get

$$f(\phi + \pi) + f(\phi) = \phi d + c.$$

To determine the constant of integration c, we observe that when $\phi=0, f(\phi)=0$. Hence $c=f(\pi)$, and we have

$$f(\phi+\pi)+f(\phi) = \phi d + f(\pi).$$

Putting $\phi=\pi$ we get $p=f(2\pi)=\pi d$; which is part (a).

Part (b) seems to bear some similarity to the minimal problems of Besicovitch and Kakeya. See W. W. R. Ball, *Mathematical Recreations and Essays* (11th edition), p. 99.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known text-books or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4136. *Proposed by H. S. M. Coxeter, University of Toronto*

The equation

$$\cos\pi x+\cos\pi y+\cos\pi z = 0, \qquad 0\leqq x\leqq \tfrac{1}{2}\leqq y\leqq z\leqq 1,$$

has the trivial solutions $y=\frac{1}{2}$, $z=1-x$, and $y=2/3-x$, $z=2/3+x$. It has also the non-trivial solution $x=1/5$, $y=3/5$, $z=2/3$. Prove that it has no other rational solutions.

4137. *Proposed by P. Erdös, Purdue University*

Given an integer $x\leqq n^2/4$ which has no prime factor greater than n, show that $n!\equiv 0 \bmod x$.

4138. *Proposed by G. Pólya, Stanford University*

Given two positive integers p and q define

$$S_{p,q}=\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2p}-\frac{1}{3}-\frac{1}{5}-\cdots-\frac{1}{2q+1}+\frac{1}{2p+2}+\cdots$$
$$+\frac{1}{4p}-\frac{1}{2q+3}-\cdots-\frac{1}{4q+1}+\cdots,$$

$$P_{p,q}=\left(1+\frac{1}{2}\right)\left(1+\frac{1}{4}\right)\cdots\left(1+\frac{1}{2p}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\cdots$$
$$\cdot\left(1-\frac{1}{2q+1}\right)\left(1+\frac{1}{2p+2}\right)\cdots.$$

We obtain the series in which blocks of p positive terms alternate with blocks of q negative terms by rearranging the terms of the well known series

$$\tfrac{1}{2}-\tfrac{1}{3}+\tfrac{1}{4}-\tfrac{1}{5}+\cdots=S_{1,1}=1-\log 2,$$

and the product by rearranging correspondingly

$$(1+\tfrac{1}{2})(1-\tfrac{1}{3})(1+\tfrac{1}{4})(1-\tfrac{1}{5})\cdots=P_{1,1}.$$

Show directly that

$$P_{p,q}=(p/q)^{1/2}$$

and hence derive the well known result

$$S_{p,q}-S_{1,1}=\log (p/q)^{1/2}.$$

4139. *Proposed by F. L. Miksa, Aurora, Ill.*

The distinct pairs of n objects, n odd, are arranged in n columns so that each column contains $(n-1)/2$ pairs and no one object occurs more than once in the same column, without regard to the order of the columns or of pairs in a column. Determine the number of different ways in which this can be done.

4140. *Proposed by V. Thébault, San Sebastián, Spain*

Find an integer of two digits such that the difference between its cube and that of the integer whose digits are those of the first in reversed order is a perfect square.

SOLUTIONS

Generalization of the Orthopole

4081 [1943, 264]. *Proposed by Otto Dunkel, Washington Univ.*

Through the vertices of the triangle ABC parallels $A\alpha$, $B\beta$, $C\gamma$ of arbitrary direction are drawn meeting the transversal Δ in the points α, β, γ; and through the latter points straight lines are drawn parallel respectively to BC, CA, AB rotated through the angle θ, thus forming a triangle $A_1B_1C_1$ similar to ABC. Prove that: (1) As the direction of the parallels varies the vertices A_1, B_1, C_1 describe straight lines concurrent in a point $\phi(\theta)$. (2) For the particular set of parallels $A\alpha$, $B\beta$, $C\gamma$ which have the direction of Δ rotated through the angle $-\theta$, the triangle $A_1B_1C_1$ reduces to the point $\phi(\theta)$. (3) The locus of $\phi(\theta)$ is a unicursal cubic passing through the circular points at infinity, the point at infinity of the Newton line (ABC, Δ), the orthopole of Δ with respect to ABC, and A_0, B_0, C_0, the points of intersection of the sides of ABC with Δ.

Solution by the Proposer. Two sets of parallels through A, B, C meet Δ in the two sets of points α_i, β_i, γ_i; α_j, β_j, γ_j. From the similar triangles β_iA_0B, γ_iA_0C we have $A_0\beta_i/A_0\gamma_i=A_0B/A_0C=A_0\beta_j/A_0\gamma_j$ and $A_0\beta_i/A_0\beta_j=A_0\gamma_i/A_0\gamma_j$. The second set of straight lines through α_i, β_i, γ_i form a triangle $A_iB_iC_i$ whose sides are parallel to those of ABC turned through the angle θ. Hence the triangles $\beta_i\gamma_iA_i$, $\beta_j\gamma_jA_j$ have parallel sides and it follows from the above that A_i, A_j, A_0

are collinear, and similarly for B_i, B_j, B_0 and C_i, C_j, C_0. From this follows that the triangles $A_iB_iC_i$, $A_jB_jC_j$ have a homothetic center $\phi(\theta)$ which is fixed for all directions of the parallels through A, B, C and the given θ. Hence there exists one direction for the parallels $A\alpha$, $B\beta$, $C\gamma$ for which the straight lines $\alpha_iB_iC_i$, $\beta_iC_iA_i$, $\gamma_iA_iB_i$ meet in the point $\phi(\theta)$. We show next that $A\alpha$, $B\beta$, $C\gamma$ in this case make the angle $-\theta$ with Δ.

Let the x-axis of rectangular coordinates be along Δ and let the abscissas of α, β, γ be denoted by these letters, where here $A\alpha$, $B\beta$, $C\gamma$ make the angle $-\theta$ with Δ, $m=\tan\theta$. It will suffice to prove that αB_1C_1, βC_1A_1, γA_1B_1 which make the angle θ with BC, CA, AB are concurrent in a point, which must be $\phi(\theta)$. Let the coordinates of A be x_1, y_1 and the slope of BC be m_1, and similarly for the other two vertices and sides. We then have $m\alpha=y_1+mx_1$, and

$$\begin{aligned}(1)\qquad &\alpha B_1C_1\colon\ m(1-mm_1)y-m(m+m_1)x+(m+m_1)(y_1+mx_1)=0,\\ &m[x_3-x_2-m(y_3-y_2)]y-m[m(x_3-x_2)+y_3-y_2]x\\ &\qquad+m^2x_1(x_3-x_2)+m[y_1(x_3-x_2)+x_1(y_3-y_2)]+y_1(y_3-y_2)=0,\end{aligned}$$

where the two remaining equations are obtained by cyclic permutations of subscripts and of the letters in the designations of the equations. It is easily verified that the sum of the left members of the three equations is identically zero, and this proves that the three straight lines meet in a point which must be $\phi(\theta)$.

The point $\phi(\pi/2)$ is the known orthopole of Δ with respect to ABC, and $\phi(0)$ is the point at infinity in the directions of the Newton line (ABC, Δ), see the solution of 3890 [1943, 267]. If $m_1=\tan\psi_1$, then $\phi(-\psi_1)=A_0$; for, let the parallels $A\alpha$, $B\beta$, $C\gamma$ make the angle ψ_1 with Δ, then $\beta=\gamma=A_0$ and αB_1C_1, which makes the angle $-\psi_1$ with BC, must coincide with Δ. Thus Δ cuts the locus of $\phi(\theta)$ in A_0, B_0, C_0, and if we use m as a parameter for x, y of this locus, y must have the factors $m+m_1$, $m+m_2$, $m+m_3$. The relation for y turns out to be

$$(2)\quad 2m(1+m^2)Sy=-(m+m_1)(m+m_2)(m+m_3)(x_1-x_2)(x_2-x_3)(x_3-x_1),$$

where S is the area of ABC. The equation of the locus has the form

$$\begin{aligned}(3)\qquad &L_1C-L_2^2=0,\\ L_1&\equiv(m_2y_2-m_1y_1)y+m_1m_2(y_1-y_2)x+(m_2-m_1)y_1y_2\\ &\qquad+m_1m_2(x_1y_2-x_2y_1)\\ L_2&\equiv m_1m_2(y_2-y_1)y+(m_2y_2-m_1y_1)x+m_1y_1x_2-m_2y_2x_1,\\ C&\equiv(m_2-m_1)(x^2+y^2)+[y_1-y_2+m_1x_2-m_2x_1+m_1x_1-m_2x_2]x\\ &\qquad+[x_1-x_2+m_2y_1-m_1y_2+m_1m_2(x_1-x_2)]y\\ &\qquad+(m_2-m_1)x_1x_2+x_1y_2-x_2y_1.\end{aligned}$$

The straight lines $L_1=0$, $L_2=0$ are perpendicular, and the cubic is tangent to the circle $C=0$ and to $L_1=0$ at their intersections with $L_2=0$.

Maximum Area of a Variable Triangle

4084 [1943, 330]. *Proposed by Otto Dunkel, Washington University*

On the sides A_jA_k of a given triangle $A = A_1A_2A_3$ as bases, directly similar triangles $B_iA_jA_k$ are constructed interiorly giving the triangle $B = B_1B_2B_3$. Show that, if B has the maximum area when the sense of rotation of its vertices is opposite to that for A, the triangles $B_iA_jA_k$ must be isosceles with $\cot\alpha \cot V = 3$, where α is the base angle and V is the Brocard angle for A. Determine the form of the triangle B giving the maximum.

Solution by the Proposer. Let $\mathbf{t}_i$ and $\mathbf{u}_i$ be the vectors of A_i and B_i with the origin at the centroid G of the A triangle whose vertices we assume to have the positive sense of rotation $\alpha = \angle A_jA_iB_k$, $0 < \alpha < \pi$, and $A_iB_k/A_iA_j = p$, then

$$\mathbf{u}_1 = \mathbf{t}_2 + p\cos\alpha(\mathbf{t}_3 - \mathbf{t}_2) + p\sin\alpha\mathbf{k}\times(\mathbf{t}_3 - \mathbf{t}_2), \tag{1}$$

where $\mathbf{k}$ is a unit vector such that $\mathbf{t}_i$, $\mathbf{t}_j$, $\mathbf{k}$ has the positive sense of rotation, and two other equations are obtained from (1) by cyclic permutations of the subscripts. From these three equations it follows at once that G is also the centroid of the B triangle. Let S_a and S_b denote the area of the A and B triangles; then a long vector computation gives

$$\begin{aligned} S_b/S_a &= 3p^2 - 3p\cos\alpha + 1 - p\sin\alpha\cot V \\ &= 3(p\cos\alpha - \tfrac{1}{2})^2 + 3\left(p\sin\alpha - \frac{\cot V}{6}\right)^2 - \frac{1}{12}(\cot^2 V - 3). \end{aligned} \tag{2}$$

Since $\cot^2 V \geqq 3$, the minimum of the expression on the right is given by $2p\cos\alpha = 1$, $6p\sin\alpha = \cot V$. Hence for the minimum the triangles $B_kA_iA_j$ are isosceles with the base angle α such that $\cot\alpha\cot V = 3$.

Let b_i denote the length of side B_jB_k; then another rather long computation gives

$$\left(\frac{b_i}{GA_i}\right)^2 = \frac{\cot^2 V - 3}{4}. \tag{3}$$

Thus the minimizing B triangle is inversely similar to the triangle GA_1A_3', where A_3G is extended to A_3' so that $GA_3' = A_3G$. The area S_b in (2) is negative because of the sense of rotation of the vertices; hence a minimum rather than a maximum is desired.

Equilateral Hyperbola and Circle

4085 [1943, 330]. *Proposed by V. Thébault, San Sebastián, Spain*

Given an equilateral hyperbola (H) and a circle (O) passing through the center ω of (H), show that the necessary and sufficient condition for the existence of an infinite number of triangles inscribed in (H) and circumscribing the circle is that the center O of the circle lies on (H). Consider the envelope of the sides of these triangles.

Solution by Howard Eves, Syracuse University. Take O as origin of rectangular coordinates, and let the coordinate axes be parallel to the asymptotes of (H). Suppose ω has coordinates (k, h). Then the equations of (O) and (H) may be respectively taken as

$$(1) \qquad x^2 + y^2 - (h^2 + k^2) = 0, \qquad xy - kx - hy + c = 0.$$

The necessary and sufficient condition that an infinity of triangles circumscribe a conic S and be inscribed in a conic S' is, in terms of the customary invariant symbols (see art. 376 of Salmon's *A Treatise on Conic Sections*),

$$(2) \qquad \Theta^2 - 4\Delta\Theta' = 0.$$

Taking S and S' to be (O) and (H) above we find, by substituting in (1),

$$(3) \qquad c^2 + 4(h^2 + k^2)(- h^2/4 - k^2/4 + h^2/4 + k^2/4) = 0,$$

whence we must have $c=0$, or (H) must pass through O.

Let R, S, T, U be the points where (O) cuts the coordinate axes, and let V, W be the points where (O) cuts (H). Then the envelope of the sides of the triangles inscribed in (H) and circumscribed to (O) is the circle (O) minus the six points R, S, T, U, V, W. For these points the triangle degenerates.

Editorial Note. Another proof will be given. If there exists one triangle inscribed in (H) which circumscribes (O), then it is known that we can obtain another such triangle ABC by taking the vertex A as any point on (H) not also on (O) and the two tangents from A to (O) meeting (H) again in B and C; then BC must also be tangent to (O). Now consider the case of A at a point at infinity on (H), say in the direction of the asymptote parallel to the x axis; then in the notation of the above solution the equation of BC is

$$(hk - c)(y + h) - k^2(x - k) = 0.$$

In order for this line to be tangent to (O) the length of the perpendicular from O to it must be $(h^2+k^2)^{1/2}$, and this gives $k^2c^2=0$. It is shown below that k is not zero and hence $c=0$. Conversely, if O is on (H), the above equations show that there is a degenerate triangle of the above kind, and therefore an infinite number of the desired triangles. It will be seen that the theorem is true if we interchange the roles of the centers ω and O.

It is simpler to show synthetically that k is not zero; and in doing this it is interesting to make the whole proof synthetic. With the above notation for the degenerate triangle ABC, let T and T' be the points of contact with (O) of the parallels AB and AC. At least one of the two, say AC, meets (H) in a finite point C, and CB touches (O) in T''. Hence B must also be a finite point; and in the above proof, if k were zero, we must have $T\equiv\omega$, and this leads to the contradiction that $B\equiv A$. Thus (O) is the inscribed or an escribed circle of ABC, and therefore it is tangent to the nine point circle of ABC which is here the straight line through A', the midpoint of BC, and $\overline{C}$, $\overline{B}$, the feet of the parallel altitudes on AB, AC. Hence $A'\overline{C}$ is tangent to (O) at ω and ω is the symmetric

of T'' with respect to OA'. It is easily seen that $A'B\overline{C}$ is isosceles; and, if $T\omega$ and the asymptotes meet OB in W, U, V, that $WB\overline{C}$ and $WU\omega$ are also isosceles. Hence W is the common midpoint of OB and VU, and O must lie on (H).

Conversely, if O is on (H) and ω on (O), let AT and AT' be tangent to (O) at T and T' where as before A is at infinity on (H), let AT meet (H) again in B, and let the tangent from B to (O) touch it at T'' and meet the line AT' in C. We are to prove that C is on (H), and it will then follow that there are infinitely many of the desired triangles ABC where A may be any point on (H). If the asymptotes meet OB in U and V, where U is on ωA, we know that OB and VU have a common midpoint W, and that WT passes through ω. Let A' be the midpoint of BC and $\overline{C}$, $\overline{B}$, $\overline{W}$ be the symmetrics of C, B, W with respect to OA'. Then from symmetry with respect to OA' and the concurrence of $A'C$, $\overline{W}T'$, (O) in T'', the lines $A'\overline{C}$, WT must be concurrent on (O) in the symmetric of T'', and $A'\overline{C}$ must be the tangent at this point of concurrence. Hence this latter point must be ω, the symmetric of T''; and, if ωU meets BC in S, symmetry shows that $\overline{C}\omega = BS = T''C$. Thus C must lie on (H) and the proof is complete.

It can be shown synthetically that the isogonal conjugate of OT'' with respect to this special triangle ABC is (H), and from this follows that OT'' is the tangent to (H) at O. If we denote the tangents to (H) at O, B, C by t_o, t_b, t_c, then t_o and t_b meet on the line $W\omega T$, t_b and t_c meet on $A'\omega\overline{C}$, and t_c on t_o on $T'\omega E$, where E is the midpoint of OC.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to B. W. Jones, White Hall, Cornell University, Ithaca, New York.

Professor W. D. Cairns represented the Association at the inauguration of Howard F. Lowry as president of the College of Wooster on October 21, 1944.

Professor L. E. Dix represented the Association at the inauguration of Dr. Homer L. Dodge as president of Norwich University on October 9, 1944.

Professor C. A. Murray of West Texas State Teachers College represented the Association at the inauguration of Dr. W. M. Whyburn as president of Texas Technological College on September 30, 1944.

Assistant Professor G. M. Ewing of the University of Missouri, on leave at the Naval Ordnance Laboratory in Washington, D. C., has been promoted to an associate professorship.

Professor Orrin Frink, Jr., of Pennsylvania State College is on leave in the research division of the Aircraft Radio Laboratory at Wright Field, Dayton, Ohio.

Assistant Professor L. P. Hutchison of The Citadel has been appointed to an instructorship at Pennsylvania State College.

D. F. Johnson of Mercersburg Academy has accepted a position at Belmont Hill School, Massachusetts.

Ernest Johnston of the University of Minnesota has been appointed Associate Mechanical Engineer at the Naval Ordnance Laboratory in Washington, D. C.

P. S. Jones of the University of Michigan has been appointed to an instructorship in the University School of Ohio State University.

Professor S. H. Kimball of the University of Maine has been appointed head of the department of mathematics and astronomy.

H. W. Linscheid of the University of Nebraska has been appointed to an assistant professorship at Eastern New Mexico College.

Dr. A. T. Lonseth of Iowa State College has been appointed to an assistant professorship at Northwestern University.

Visiting Professor Szolem Mandelbrojt of The Rice Institute has been granted leave of absence to serve on a mission in London under General Koenig.

Professor C. A. Messick of Park College has been appointed acting chairman of the department during the absence of Professor H. E. Crull, who is in the United States Navy.

Professor I. F. Neff, head of the department of mathematics at Drake Unisity, has retired with the title of professor emeritus.

Professor J. B. Rosenbach of the Carnegie Institute of Technology has been named secretary of the division of humanistic and social studies.

Assistant Professor C. T. Ruddick of Alliance College has been appointed acting treasurer and acting associate professor of philosophy at Lake Erie College.

Professor Mary E. Sinclair, who has retired as head of the department at Oberlin College, has been appointed visiting professor at Berea College.

Assistant Professor B. M. Stewart of Denison University has been appointed to an assistant professorship at Michigan State College.

Dr. G. B. Thomas, Jr., of Massachusetts Institute of Technology has been promoted to an assistant professorship.

WAR INFORMATION

EDITED BY C. V. NEWSOM

Send news reports upon the utilization of mathematicians or mathematics in war activities to C. V. Newsom, Oberlin College, Oberlin, Ohio.

THE MATHEMATICAL TABLES PROJECT

The Mathematical Tables Project was set up by the Work Projects Administration in January, 1938, under the sponsorship of the National Bureau of Standards. In March, 1943, the sponsoring agency took over its operation with the support of the Office of Scientific Research and Development, the determination of the program being vested in the Applied Mathematics Panel of the National Defense Research Committee. Under the new auspices, the major emphasis is being placed on computations directly related to the war effort, but it is also planned to devote any available time to the important task of finishing various tables which were under way when the Work Projects Administration was discontinued.

Since its founding, the Project has done an amazing amount of work, employing at times a huge corps of computers and computing machines, working in two shifts. Professor A. N. Lowan has been the director of the Program. An effort has been made to select for tabulation functions of fundamental importance in pure and applied mathematics. In the selection and choice of the most useful range and interval of the argument, there has been a continual exchange of views with outstanding mathematicians, physicists, and engineers, both here and abroad.

Twenty large volumes of tables computed by the Mathematical Tables Project have already been published. Also, five shorter tables have appeared in various scientific journals; these are now available in pamphlet form. Two more volumes, namely, a table of associated Legendre functions and a table of arc sin x, are now in the process of being reproduced by the Columbia University Press. Moreover, in addition to present computations related to problems of war research, some time is being devoted to bringing to completion several other large volumes of spherical Bessel functions, Bessel functions of fractional order, Bessel functions of the second kind for complex arguments, exponential integrals for complex arguments, and Chebyschev polynomials. A considerable amount of work has also been done on Jacobi elliptic functions for real and imaginary arguments.

Tables now available are listed below. All except the last four titles may be obtained from the National Bureau of Standards; the last four volumes have been published by the Columbia University Press.

1. Tables of the exponential function e^x.
2. Tables of circular and hyperbolic sines and cosines for radian arguments.
3. Tables of sines and cosines for radian arguments.

4. Tables of sine, cosine, and exponential integrals. (2 volumes)
5. Tables of natural logarithms. (4 volumes)
6. Tables of probability functions. (2 volumes)
7. Tables of the moments of inertia and section moduli of ordinary angles, channels, and bulb angles with certain plate combinations.
8. Table of sine and cosine integrals for arguments from 10 to 100.
9. The hypergeometric and Legendre functions with applications to integral equations of potential theory.
10. Table of arc tan x.
11. Miscellaneous physical tables: Planck's radiation functions; and electronic functions.
12. Table of the zeros of the Legendre polynomials of order 1–16 and the weight coefficients for Gauss' mechanical quadrature formula.
13. Table of the confluent hypergeometric function and its first derivative.
14. Table of integrals $\int_0^x J_0(t)dt$ and $\int_0^x Y_0(t)dt$.
15. Table of $Ji_0(x)=\int_x^\infty [J_0(t)/t]dt$ and related functions.
16. Table of coefficients in numerical integration formulae.
17. Table of reciprocals of the integers from 100,000 through 200,009.
18. Table of the Bessel functions $J_0(z)$ and $J_1(z)$ for complex arguments.
19. Table of circular and hyperbolic tangents and cotangents for radian arguments.
20. Tables of Lagrangian interpolation coefficients.

NEW ARMED FORCES INSTITUTE CATALOGUE

The Army and the Navy have recently released a joint catalogue of the U. S. Armed Forces Institute. The new catalogue describes ways by which self-teaching texts may be effectively used by those in service, and outlines steps which the individual should take to procure an evaluation of his educational experience in terms of school and college credit. The Institute, with its nine overseas branches, is, as stated on the cover of the catalogue, "The Army-Navy School with the World Campus." The magnitude of the endeavor and the nature of the work being done deserve the attention of educators generally. The number of persons taking correspondence courses through the USAFI is already in excess of a quarter of a million, of whom over 50,000 are enrolled in university extension correspondence courses approved by the Institute. An average of more than 2000 persons in the Armed Forces are enrolling daily.

The Institute program consists of four major divisions, namely, Institute correspondence courses, university extension correspondence courses, self-teaching courses, and off-duty classes. The correspondence courses are of the traditional type. The self-teaching courses are especially for those students who prefer to study a subject entirely on their own, or who cannot be reached promptly and regularly by mail. In such a case, the Institute furnishes a textbook "which makes it possible to learn a subject without a teacher." Off-duty classes "are taught with specially prepared textbooks that include tests, drills,

and questions and answers, which follow the same method a teacher would use in a formal class. This makes it possible for a group of men to study a subject without a skilled instructor. (The men) appoint a leader or instructor, and get to work."

One fee, $2.00, enrolls any Army enlisted man or woman for Institute self-teaching and correspondence courses. The same fee applies to all personnel of the Navy, Marine Corps, and Coast Guard. So long as the student does acceptable work, he may continue to enroll for additional Institute self-teaching and correspondence courses without further charge. A commissioned or warrant officer of the Army may enroll for any Institute correspondence course by paying the cost of the course. The fee may be ascertained by writing to the United States Armed Forces Institute, Madison 3, Wisconsin, or to any oversea branch, and will cover the costs of texts, lesson service, and end-of-course test. The Government shares the cost of a university extension correspondence course with enlisted personnel of the Army and with all personnel of the Navy, Marine Corps, and Coast Guard; the Government will pay half the text and tuition fee for each course taken up to the amount of $20.00.

The accreditation service of the USAFI is explained by the new catalogue to service men in the following paragraph.

"The Armed Forces Institute does not itself grant high-school or college credit for courses offered to military personnel. The granting of school or college credit is the responsibility of the educational institution in which you may want to use such credit toward a diploma or a degree. The Institute will, however, help you to get school or college credit for any educational work you may do while you are in the service by submitting to the educational institution you designate, a detailed record of your educational achievement. All the major associations of high schools and colleges in the United States, and hundreds of individual institutions, have voted that special arrangements should be made for returning service personnel who want academic credit for their experience in the service. The special arrangements which have been recommended consist of a plan by which men and women who have been in the service will not be required to prove that they have attended specified courses for so many hours (the usual basis for granting credit), but will be given credit for what they have learned, no matter when or how they learned it. To make these arrangements effective, the Armed Forces Institute will give you special tests designed to measure what you have learned in the service. In addition, the Institute will prepare for you a record of the Institute and college or university extension courses you have studied, the service schools you have attended, and the service jobs you have had. On your request, it will forward a report on your examinations and a copy of your educational record to the school or college of your choice, and will obtain for you a statement of the amount of academic credit which will be granted you. If, therefore, you wish to obtain school or college credit for your work, write to the Institute headquarters or to the nearest Institute branch for an application for accreditation service."

Twenty-seven courses in mathematics are now offered by the USAFI. Only eleven of these courses, however, are regarded by the Institute as being upon the college level; they are plane trigonometry, college algebra, plane analytic geometry, descriptive geometry, spherical trigonometry, differential calculus, integral calculus, solid analytic geometry, differential equations, and engineering mathematics (parts I and II). All of the self-teaching texts in mathematics which have been developed up to the present time are in elementary or secondary mathematics.

DISPOSAL OF SURPLUS WAR PROPERTY

Many persons recall opportunities after the last war to secure surplus or used war material at a very reasonable cost. In view of the highly technical nature of the present conflict, it has seemed reasonable to assume that educational institutions might find it possible to obtain many desirable items of equipment at the conclusion of the war.

Important legislation on the disposal of surplus war property has been before Congress for some weeks. On August 22, the House passed the Colmer bill (H.R. 5125) which would set up a Surplus War Property Administration under the direction of one person, the administrator. He would determine "such policies governing prices and other terms and conditions of disposal . . . as he deems necessary to effectuate the objectives and policies of this Act." Moreover, it is provided that non-salable material may be donated to "any agency or institution supported by the Federal Government or any state or local government, or to any non-profit educational or charitable organization . . . ." Among the purposes of the act is the specification "To afford public, governmental, educational, . . . institutions . . . an opportunity to fulfill their legitimate needs."

Two bills on surplus war property have appeared in the Senate, one offered by Senators Stewart, Murray, and Taft (S. 2065), and the other by Senator Johnson (S. 2045). On August 23, a substitute version of S.2065 was reported out. This bill provides for a Surplus Property Board of eight members instead of a single administrator. Parts of Section 12 affecting educational institutions are quoted below.

"The Board may prescribe regulations for the disposition of surplus property to States, and political subdivisions thereof, including municipalities, and to tax-supported and non-profit institutions, as follows:

"(a) Surplus property that is appropriate for school, classroom or other educational use may be transferred to the Federal Security Agency for donation to the States and their political subdivisions and tax-supported educational institutions, and, . . . with the approval of the Board, to other nonprofit educational institutions which have been held exempt from taxation under section 101 (6) of the Internal Revenue Code.

"(b) Surplus medical supplies and equipment may be transferred to the Federal Security Administration for donation to the States and their political subdivisions and to tax-supported medical institutions, and, within rules and

regulations to be prescribed by the Federal Security Administrator, with the approval of the Board, to hospitals or other similar institutions not operated for profit which have been held exempt from taxation under Section 101 (6) of the Internal Revenue Code, and to the American Red Cross.

"(c) Any surplus property may be sold or leased to States, political subdivisions thereof, including municipalities, tax-supported institutions, and non-profit charitable, medical and educational institutions which have been held exempt from taxation under Section 101 (6) of the Internal Revenue Code, at discounts not to exceed 50 per centum of the sale or lease market value thereof, as the case may be, or 50 per centum of the highest price offered by any private purchaser or lessee, whichever is lower."

It is also specified in Section 12 that non-salable material may be donated to institutions. The Surplus Property Board exercises some control over property disposed of under Section 12 for a period of two years, especially "with respect to the maintenance of the property, (and) its continued use for the general purpose for which it was acquired . . . ."

MATHEMATICIANS NEEDED IN FEDERAL SERVICE

The Civil Service Commission has announced an examination for mathematicians to fill positions in various places throughout the country. At present there are needs at Langley Field in Virginia, in Aberdeen, Maryland, a few vacancies in Cleveland, Ohio, and some in Washington, D. C. Mathematicians are needed to apply their technical training to the solution of problems in ballistics, aerodynamics, vibration, and other scientific problems. The salaries range from $2,433 to $4,428 a year, including overtime pay.

In general, applicants for the $2,433 positions must have had at least 3 years of technical experience in mathematics requiring the applications of the principles of algebra, trigonometry, analytical geometry, and calculus. For the higher-grade positions, applicants must show additional experience of a progressively higher level.

College study which included courses in higher mathematics may be substituted for the prescribed experience on the basis of 1 year of academic study for each 9 months of experience. Graduate study in mathematics may be substituted year for year for the experience prescribed up to a maximum of three years.

No written test will be given and there are no age limits. Appointments to Federal positions are made in accordance with War Manpower Commission policies and employment stabilization programs.

Full information and application forms may be secured at first- and second-class post offices, from the Commission's regional offices, or direct from the U. S. Civil Service Commission, Washington 25, D. C.

THE MATHEMATICAL ASSOCIATION OF AMERICA

THE TWENTY-SEVENTH SUMMER MEETING OF THE MATHEMATICAL ASSOCIATION

The twenty-seventh summer meeting of the Mathematical Association of America was held at Wellesley College, Wellesley, Massachusetts, on Saturday, August 12, 1944, in conjunction with the summer meeting and colloquium of the American Mathematical Society and the meeting of the Institute of Mathematical Statistics. Two hundred and seventy-two persons were in attendance at the meetings, including the following one hundred and forty-one members of the Association.

C. R. Adams, Brown University
Louise Adams, High Point College
R. P. Agnew, Cornell University
H. E. Arnold, Wesleyan University
K. J. Arnold, Columbia University Statistical Research Group
L. A. Aroian, Hunter College

D. H. Ballou, Middlebury College
Helen P. Beard, Newcomb College
R. F. Belding, Vermont Academy
Garrett Birkhoff, Harvard University
G. D. Birkhoff, Harvard University
R. P. Boas, Jr., Harvard University
J. G. Bowker, Middlebury College
H. W. Brinkmann, Swarthmore College
E. T. Browne, University of North Carolina
R. E. Bruce, Boston University
R. C. Buck, Harvard University
Sister Leonarda Burke, Regis College, Massachusetts

W. D. Cairns, Oberlin College
R. H. Cameron, Massachusetts Institute of Technology
B. H. Camp, Wesleyan University
Mildred E. Carlen, Brown University
W. B. Carver, Cornell University
W. F. Cheney, Jr., University of Connecticut
Mary D. Clement, Wells College
Nancy Cole, Connecticut College
J. B. Coleman, New York, N. Y.
T. F. Cope, Queens College
Lennie P. Copeland, Wellesley College
Richard Courant, New York University
J. H. Curtiss, Bureau of Ships, Navy Department

Marguerite D. Darkow, Hunter College
F. F. Decker, Syracuse University
C. E. Dimick, U. S. Coast Guard Academy
Arnold Dresden, Swarthmore College
William H. Durfee, Yale University
P. S. Dwyer, University of Michigan

J. E. Eaton, Queens College
H. S. Everett, University of Chicago

W. H. Fagerstrom, College of the City of New York
Will Feller, Brown University
W. C. Foreman, U.S.N.R.
R. M. Foster, Polytechnic Institute of Brooklyn
J. S. Frame, Michigan State College
Philip Franklin, Massachusetts Institute of Technology

C. A. Garabedian, Wheaton College
R. E. Gaskell, Brown University
H. M. Gehman, University of Buffalo
B. H. Gere, U. S. Naval Academy
Michael Goldberg, Bureau of Ordnance, Navy Department
W. H. Gottschalk, University of Pennsylvania
V. G. Grove, Michigan State College

P. R. Halmos, Syracuse University
G. E. Hay, Brown University
Olive C. Hazlett, University of Illinois
G. A. Hedlund, University of Virginia
M. R. Hestenes, University of Chicago
T. H. Hildebrandt, University of Michigan
Einar Hille, Yale University
T. R. Hollcroft, Wells College

M. GWENETH HUMPHREYS, Newcomb College
C. C. HURD, U. S. Coast Guard Academy

R. N. JOHANSON, Boston University
B. W. JONES, Cornell University

G. K. KALISCH, Cornell University
WILFRED KAPLAN, Brown University
J. L. KELLEY, Aberdeen Proving Ground
L. M. KELLY, U. S. Coast Guard Academy
S. H. KIMBALL, University of Maine
J. R. KLINE, University of Pennsylvania
MAURICE KRAITCHIK, New School for Social Research

MARY E. LADUE, Barnard College
A. L. LANCKTON, U. S. Coast Guard Academy
J. A. LARRIVEE, U. S. Naval Observatory
JOSEPH LEHNER, Kellex Corporation

C. C. MACDUFFEE, University of Wisconsin
SAUNDERS MAC LANE, Harvard University
INGO MADDAUS, JR., Massachusetts Institute of Technology
JOHN MANDEL, Brooklyn, New York
R. J. MARCOU, Boston College
MORRIS MARDEN, University of Wisconsin
W. T. MARTIN, Syracuse University
V. O. MCBRIEN, College of the Holy Cross
N. H. MCCOY, Smith College
EDITH A. MCDOUGLE, University of Delaware
REV. P. H. MCGRATH, St. Peter's College
E. J. MCSHANE, Aberdeen Proving Ground
A. E. MEDER, JR., New Jersey College for Women
E. B. MODE, Boston University
C. N. MOORE, University of Cincinnati
R. K. MORLEY, Worcester Polytechnic Institute

C. A. NELSON, New Jersey College for Women
P. B. NORMAN, Columbia University

C. O. OAKLEY, Haverford College
RUFUS OLDENBURGER, Illinois Institute of Technology

GORDON PALL, McGill University
A. E. PITCHER, Aberdeen Proving Ground
A. L. PUTNAM, Yale University

J. F. RANDOLPH, Oberlin College
W. R. RANSOM, Tufts College
C. H. RAWLINS, JR., U. S. Naval Academy
MINA S. REES, Hunter College
C. F. REHBERG, New York University
C. E. RHODES, Union College
HARRIS RICE, Worcester Polytechnic Institute
R. G. D. RICHARDSON, Brown University
H. A. ROBINSON, U. S. Military Academy
SELBY ROBINSON, College of the City of New York
R. E. ROOT, U. S. Naval Academy
P. C. ROSENBLOOM, Brown University
LULU L. RUNGE, University of Nebraska
HELEN G. RUSSELL, Wellesley College

RAPHAEL SALEM, Massachusetts Institute of Technology
HENRY SCHEFFÉ, Syracuse University
W. A. SHEWHART, Bell Telephone Laboratories
M. F. SMILEY, U. S. Naval Academy
C. V. L. SMITH, U. S. Naval Academy
ANDREW SOBCZYK, Massachusetts Institute of Technology
E. R. STABLER, Hofstra College
MARION E. STARK, Wellesley College
E. P. STARKE, Rutgers University
H. W. STEINHAUS, Equitable Life Assurance Society
RUTH W. STOKES, Winthrop College
M. H. STONE, Harvard University
J. S. STUBBE, University of Cincinnati
OTTO SZASZ, University of Cincinnati

J. D. TAMARKIN, Brown University
MARIAN M. TORREY, Goucher College
J. I. TRACEY, Yale University
A. W. TUCKER, Princeton University
J. W. TUKEY, Princeton University

H. S. VANDIVER, University of Texas
ANDREW VAZSONYI, Harvard University

G. L. WALKER, Cornell University
R. M. WALTER, New Jersey College for Women
A. H. WHEELER, Clark University
J. H. WHITE, Summit, New Jersey
D. V. WIDDER, Harvard University
H. A. WOOD, Chance Vought Aircraft

MABEL M. YOUNG, Wellesley College

S. D. ZELDIN, Massachusetts Institute of Technology

Dormitory rooms were available for members of the organizations and their families in Tower Court, and excellent meals were served in the dining room of this same building. A reception for the visitors was given by the Administration of Wellesley College on Saturday evening in the Recreation Building; and on Sunday evening Professor Barnett of the Wellesley Music Faculty gave a piano recital in Billings Hall.

The joint dinner for the three organizations was held on Sunday evening at 6:30. Professor D. V. Widder of Harvard University acted as toastmaster, and he introduced Captain Mildred H. McAfee, President of Wellesley College and head of the WAVE organization, who gave a most happy address of welcome. President M. H. Stone of the Mathematical Society was introduced and spoke of the contribution which American mathematicians had made toward the prosecution of the war. Professor J. S. Frame then presented resolutions expressing the thanks and appreciation of the visiting organizations to President McAfee and the administration of Wellesley College, to Miss Copeland, chairman of the Department of Mathematics, to Miss Stark, chairman of the Committee on Arrangements, and to the other members of her committee, to Professor Barnett of the Music Department, and to all others who had given of their time and effort to make the meetings so thoroughly enjoyable. The resolutions were adopted by a rising vote.

The American Mathematical Society held sessions on Sunday and Monday. The twenty-sixth Colloquium consisted of four lectures on "Selected topics in the theory of semi-groups" given by Professor Einar Hille of Yale University. On Sunday afternoon, by invitation, Professor C.C. MacDuffee of the University of Wisconsin gave a lecture "On the composition of algebraic forms of higher degree."

The Institute of Mathematical Statistics held sessions on Sunday morning and afternoon in addition to the joint session with the Association on Saturday evening.

The program committee for the Association consisted of Professors J. I. Tracey, chairman; H. M. Gehman and J. J. Gergen. The sessions were held Saturday afternoon and evening.

First Session of the Association

1. "The stability of dynamical systems" by Dr. H. W. Bode, Director of Mathematical Research, Bell Telephone Laboratories.

2. "Some problems of a young instructor" (by one who has been), Professor Marion E. Stark, Wellesley College.

3. "Some aspects of the theory of functions of several complex variables" by Professor W. T. Martin, Syracuse University.

Joint Session of the Association with the Institute of Mathematical Statistics

Symposium on "Potential opportunities for statisticians and the teaching of statistics." Dr. W. A. Shewhart, President of the Institute, opened the discussion, followed by Professor Harold Hotelling. Professor Milton de Silva Rodriques of Sao Paulo University, Brazil, and Dr. Vaclav Myslivec, Czechoslovak Delegate to the United Nations Interim Commission on Food and Agriculture, were introduced and spoke briefly.

Meetings of the Board of Governors

Eight members of the Board were present at the meetings held Sunday morning and afternoon.

The following twenty-nine persons were elected to membership on applications duly certified:

Rev. H. W. Ball, A.M. (Boston Coll.) Missionary, St. George's Coll., Kingston, Jamaica, B.W.I.

W. E. Beeman, M.S. (N. Tex. State T.C.) Instr., North Texas State Teachers Coll., Denton, Tex.

John Bodnar, M.S. in M.E. (Illinois Inst. of Tech.) Instr., Math. and Physics, Jr. Coll. of Connecticut, Bridgeport, Conn.; Test Engr., Chance Vought Aircraft, Stratford, Conn.

A. V. Boswell, A.M. (Northwestern) Asso. Prof., Tennessee A. and I. State Coll., Nashville, Tenn.

C. H. Brown, Ph.D. (Kansas) Asso. Prof., Central Missouri State Teachers Coll., Warrensburg, Mo.

F. M. Carpenter, A.M. (Illinois) Instr., Missouri School of Mines, Rolla, Mo.

H. S. Christian, Jr. Student, Illinois Inst. of Tech., Chicago, Ill.

R. L. Duncan, A.B. (Tulsa) Instr., Univ. of Tulsa Evening Coll.; Div. Titleman, Carter Oil Co., Tulsa, Okla.

Anselm Fisher, A.M. (New York Univ.) Instr., Macalester Coll., St. Paul, Minn.

Sister Catherine Dorothea Fox, Ph.D. (Boston Coll.) President, Trinity Coll., Washington, D. C.

Aletha C. Gaddis, A.M. (Columbia) Instr., Carleton Coll., Northfield, Minn.

J. H. Giese, Ph.D. (Princeton) Aerodynamicist, Bell Aircraft Corp., Buffalo, N. Y.

F. G. Graff, A.M. (Pittsburgh) Major, CAC. Instr., U. S. Military Acad., West Point, N. Y.

D. B. Houghton, A.M. (Michigan) Instr., Princeton Univ., Princeton, N. J.

C. C. Hurd, Ph.D. (Illinois) Lt., U.S.N.R. U. S. Coast Guard Acad., New London, Conn.

P. A. Lagerstrom, Ph.D. (Princeton) Research mathematician, Bell Aircraft, Niagara Falls, N. Y.

A. L. Lanckton, A.M. (Duke) Lt., U.S.N.R. Instr., U. S. Coast Guard Acad., New London, Conn.

Mrs. Elizabeth C. Lukacs. Instr., Berea Coll., Berea, Ky.

F. L. Miksa. Switchman, Illinois Bell Telephone Co., Aurora, Ill.

E. P. Miles, Jr., Master's (Duke) Lt. (SC), U.S.N.R. Supply Officer.

Dewey Moore. Student, Berea Coll., Berea, Ky.

J. V. Pennington, Ph.D. (Rice) Vice Pres., Reed Roller Bit Co., Houston, Tex.

Valentina Potor, A.M. (Indiana) Instr., Berea Coll., Berea, Ky.

Elizabeth M. Ragland, A.M. (Kentucky) Teacher, Jr. High School, Lexington, Ky.

M. M. RESNIKOFF, M.S. (Chicago) Prof., State Teachers Coll., Minot, N. D.
P. C. ROSENBLOOM, Ph.D. (Stanford) Instr., Brown Univ., Providence, R. I.
J. A. TIERNEY, A.M. (Columbia) Instr., Norwich Univ., Northfield, Vt.
ANDREW VAZSONYI, Ph.D. (Budapest), M.S. (Harvard) Teaching Fellow, Grad. School of Engineering, Harvard Univ., Cambridge, Mass.
SISTER M. CLAUDIA ZELLER, Ph.D. (Michigan) Instr., Coll. of St. Francis, Joliet, Ill.

The Board received and placed on file the Report of the American Mathematical Society and the Mathematical Association of America to the Rockefeller Foundation on the activities of the War Policy Committee.

The Board voted in favor of suitable amendments to the by-laws to abolish institutional memberships in the Association, these amendments to be submitted to the Association at the next annual meeting.

It was voted to make an annual contribution of $350.00 per year for the three years 1945–47 to the support of Mathematical Reviews.

It was voted to contribute $200.00 for the calendar year 1944 to the support of the National Mathematics Magazine.

The question of financial help to Sections in securing outside speakers for Sectional meetings was discussed, and it was voted that such assistance be given under certain circumstances and in restricted amounts. The Secretary was instructed to inform all Section Secretaries with regard to the exact terms of the arrangement.

The President was authorized to appoint a committee to study the question of Regional Governors and their election and to report recommendations to the Board.

The Board passed a resolution giving the Finance Committee power to buy and sell securities for the Association's invested funds.

The Finance Committee reported that an audit of the accounts of the Association for the year 1943 had been made by Mr. Taylert of the Treasurer's office of Cornell University. The audit was accepted and approved by vote of the Board.

The resignation of Professor R. E. Langer as chairman of the Committee on Slaught Memorial Papers was presented and accepted by the Board with an expression of appreciation of the time and work that Professor Langer had given to this project. On nomination of President Cairns, Professor N. H. McCoy was appointed chairman of this committee, and the President was authorized to appoint one or more additional members of the committee.

On nomination of the Executive Committee the Board elected the following Nominating Committee: W. D. Cairns, J. S. Frame, and H. M. Gehman, chairman.

W. B. CARVER, *Secretary-Treasurer*

THE ANNUAL MEETING OF THE KANSAS SECTION

The thirteenth annual meeting of the Kansas Section of the Mathematical Association of America was held on Saturday, April 15, 1944, at Washburn University in Topeka, Kansas. This was a joint meeting with the Kansas Association of Teachers of Mathematics. Professor Paul Eberhart, Chairman of the Section, presided at the morning session, and Mr. Herbert Bishop of Manhattan High School presided at the afternoon session.

The attendance was eighty, including the following twenty-seven members of the Association: R. W. Babcock, Wealthy Babcock, Florence Black, Lucy T. Dougherty, Paul Eberhart, W. H. Garrett, Edison Greer, Emma Hyde, W. C. Janes, H. E. Jordan, C. F. Lewis, Anna Marm, C. T. McCormick, Thirza A. Mossman, Sister Jeanette Obrist, O. J. Peterson, P. S. Pretz, C. B. Read, D. H. Richert, G. W. Smith, E. B. Stouffer, W. T. Stratton, Sister M. Helen Sullivan, Gilbert Ulmer, E. B. Wedel, J. J. Wheeler, A. E. White.

At the business meeting the following officers were elected for the next year: Chairman, Edison Greer, Beech Aircraft Corporation; Vice-Chairman, C. T. McCormick, Fort Hays Kansas State College; Secretary, Anna Marm, Bethany College.

The following papers were presented:

1. *Algebraic functions*, by Professor D. H. Richert, Bethel College.

This paper was concerned with the properties of an algebraic function of a single variable. After such a function had been defined, the discussion centered around the relation of the definition to such concepts as the fundamental theorem of algebra, polynomials, algebraic numbers, and algebraic integers. A few examples were presented to illustrate the method by which it may be shown that any function expressible by means of radicals is algebraic.

2. *Mathematics in the navy training program*, by Professor G. W. Smith, University of Kansas.

In discussing the mathematics in the naval training programs at the University of Kansas, Professor Smith stated that each group of machinists' mates or electricians' mates took a twelve week course consisting of arithmetic, some elementary algebra, a little plane geometry, and a little trigonometry. He remarked that the mathematics taught in the V-12 program was substantially that included in the usual engineering curriculum.

3. *Mathematics in pre-radar training*, by Professor A. E. White, Kansas State College.

The pre-radar work at Kansas State College was organized by the department of electrical engineering in the fall of 1943, and continued for about eleven months. The purpose was to train men for radio operation and maintenance, and to prepare them for advanced radio work. Approximately three hundred and fifty men received this training. The work in mathematics consisted of an intensive review of elementary algebra (four hours per day for three days), after

which the trainees were divided into sections and assigned to regular class work in mathematics. The men devoted twelve hours per week for four weeks to algebra and trigonometry, and subsequently six hours per week for two weeks were spent in the study of the calculus.

4. *Mathematics in the army training programs*, by Dean E. B. Stouffer, University of Kansas.

The speaker gave a brief report on the experience of the mathematics department of the University of Kansas with the basic phase of the A.S.T.P. He stated that the care with which the students had been selected made it possible for them to carry a very heavy program of mathematics and science in a highly satisfactory manner.

5. *Mathematics in the manufacture of aircraft*, by Edison Greer, Beech Aircraft Corporation.

Mr. Greer remarked that higher mathematics plays an important part in many phases of the manufacture of aircraft, and that there seem to be few branches of mathematics which do not have a place somewhere in the industry. Numerous details were cited in support of these statements. Analytic geometry and descriptive geometry were said to be particularly useful. It was also remarked that much use is now being made of projective geometry in the construction of curves of the second degree.

6. *Mathematics in the army air force program*, by Professor O. J. Peterson, Kansas State Teachers College, Emporia.

Professor Peterson discussed the mathematical content and the objectives of the army air force college training program. He commented upon the effectiveness of a system of periodic diagnostic testing with follow-up remedial instruction aimed at bringing aviation students to acceptable proficiency in the fundamental mathematical skills. Attention was called to the possible benefits in applying similar technique more generally in civilian courses in colleges and secondary schools.

7. *Placement tests for air corps students*, by Professor C. B. Read, University of Wichita.

The speaker discussed the use of the Kansas Mathematics Test No. IV as a basis of sectioning students in the army air force program. The test, consisting of sixteen questions in simple arithmetic and twenty questions in algebra, was given to entering men. Results were used to divide the men into sections. This method of sectioning was found satisfactory, not only for mathematics classes, but also for other subjects.

8. *The Armed Forces Institute tests*, by Laura Z. Greene, Washburn University, introduced by the Secretary.

This paper dealt with the algebra, trigonometry, and analytic geometry tests published by the United States Armed Forces Institute. It was stated

that the tests have some weaknesses, but that in general they are well planned and will probably be useful either as course examinations or as placement tests for service men who return to college.

9. *Correlation of entrance test scores and term grades*, by Professor W. T. Stratton and Professor J. C. Peterson, Kansas State College.

The authors of this paper have been interested in securing a machine scorable type of examination in mathematics to replace the regular problem type of test, so that the results could be scored quickly and used for guidance and placement of freshmen. Their study showed a correlation of 0.74 between the machine scorable type and the problem type of test. The results of the study indicated that boys are more affected by the war situation than are the girls. The correlation of the first semester grades and the scores on the Iowa Physical Science Aptitude Test was 0.216 for boys and 0.678 for girls.

10. *Random jottings from an instructor's notebook*, by Professor C. B. Read, University of Wichita.

Professor Read described certain items which have been helpful in the presentation of material to students. Topics covered were of wide range, but in most cases had reference to the teaching of elementary mathematics. In particular, emphasis was placed upon the correct usage of terms, the recognition of alternative definitions, and the use of reasonable standards of rigor.

11. *Pre-induction courses in mathematics*, by Edna Austin, Topeka High School, introduced by the Chairman.

The speaker outlined the mathematics needed by men in the army, and the content of manuals prepared for the use of people in the military forces.

12. *The mathematics program in the high school at present and in the post-war period*, by T. J. LaRue, Junction City High School, introduced by the Secretary.

In this address the speaker stressed the need for a well-rounded, flexible mathematics program in the high schools. He said that such a program will result from following the recommendations of the M.A.A. and the National Council committees, and that those schools which have followed these recommendations have had little difficulty in meeting the needs of war time. It was stated that the principles which are stressed now will continue to be important after the war.

13. *Report of the committee on the improvement of instruction*, by Professor Gilbert Ulmer, University of Kansas.

This paper constituted the report of a committee appointed a year ago to study the problems confronting the mathematics teachers of Kansas. Professor Ulmer described certain problems which had been considered by the committee, and presented a recommendation that the Kansas State Department of Education be requested to establish a minimum requirement of one unit of mathe-

matics for graduation from accredited high schools of the state. The recommendation was adopted.

ANNA MARM, *Secretary*

THE APRIL MEETING OF THE OHIO SECTION

The twenty-ninth annual meeting of the Ohio Section of the Mathematical Association of America was held at the Ohio State University, Columbus, Ohio, on April 6, 1944. Sessions were held in the afternoon, at dinner, and in the evening. Professor Tibor Radó, Chairman of the Section, presided at these sessions.

Forty-five persons registered attendance, including the following thirty-four members of the Association: F. R. Bamforth, Grace M. Bareis, I. A. Barnett, H. M. Beatty, Henry Blumberg, C. T. Bumer, V. B. Caris, Nancy Cole, Rufus Crane, Wayne Dancer, B. C. Glover, Mary A. Goins, R. C. Hildner, Margaret E. Jones, L. C. Knight, A. C. Ladner, Edith J. McKissock, E. S. Manson, C. G. Maple, E. J. Mickle. C. C. Morris, Tibor Radó, J. F. Randolph, S. E. Rasor, R. B. Rice, L. D. Rodabaugh, K. C. Schraut, Mary E. Sinclair, B. M. Stewart, J. L. Synge, E. P. Vance, F. B. Wiley, C. O. Williamson, C. H. Yeaton.

The following officers were elected for the coming year: Chairman, J. B. Brandeberry, University of Toledo; Secretary-Treasurer, Rufus Crane, Ohio Wesleyan University; Member of the Executive Committee, H. M. Beatty, Ohio State University; Member of the Program Committee, H. M. MacNeille, Kenyon College. It is expected that the next meeting will be held at the Ohio State University on Thursday, April 5, 1945.

The following program was presented:

1. *On triangulation*, by Professor Tibor Radó, Ohio State University.

2. *Two rectangles in a quarter circle*, by Professor B. M. Stewart, Denison University.

In the first-quadrant quarter circle bounded by $x^2+y^2=1$, $x=0$, and $y=0$, consider the rectangle A_1 bounded by $x=0$, $x=\cos a$, $y=0$, $y=\sin a$, and the rectangle A_2 bounded by $x=0$, $x=\cos b$, $y=\sin a$, $y=\sin b$, subject to the restriction that $0<a<b<\pi/2$. The figure is related to a motor design problem arising in electrical engineering, where it is desired to maximize the area $A=A_1+A_2$ under the condition $A_1=A_2$. One feature of the problem is that the function $A=\sin 2a$ has at $a=\pi/4$ a maximum, discoverable by the elementary calculus, but not acceptable as a solution. The other feature is the variety of methods from the theory of equations and from elementary and advanced calculus which can be used to obtain the approximate solution $a=18°2'27''$ and $b=52°4'10''$.

3. *Demonstration of the new navy plotting board*, by Professor C. O. Williamson, College of Wooster.

Four problems were employed to demonstrate the rapidity with which the board enabled one to solve vector triangles in navigation.

4. *The life and work of Sir William Rowan Hamilton*, by Professor J. L. Synge, Ohio State University.

The speaker placed particular emphasis on Hamilton's contributions to the calculus of variations. The central idea which motivated Hamilton's early work (until his invention of quaternions) was the characteristic, or principal, function. By characterizing an optical or dynamical system by means of one such function, Hamilton placed himself in the category of men such as Descartes, Lagrange, and Laplace. For the major contribution of each of these consisted in a similar simple characterization of a general problem. The paper dealt also with Hamilton's life, his remarkable attainments as a child, his appointment to professorial rank while still an undergraduate, the many academic distinctions he received, and his habits of working.

5. *Development of college mathematics in Ohio in the past fifty years*, by Professor Mary E. Sinclair, Oberlin College.

6. *Future development of college mathematics in Ohio*, by Professor I. A. Barnett, University of Cincinnati.

The last two papers formed the subject of a round table discussion at the evening session.

RUFUS CRANE, *Secretary*

CALENDAR OF FUTURE MEETINGS

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

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ILLINOIS
INDIANA
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Baltimore, December 9, 1944
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA
NORTHERN CALIFORNIA, San Francisco, January 27, 1945
OHIO, Columbus, April 5, 1945
OKLAHOMA
PHILADELPHIA
ROCKY MOUNTAIN
SOUTHEASTERN
SOUTHERN CALIFORNIA, Los Angeles, March 10, 1945
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE
WISCONSIN, Milwaukee, May, 1945

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Edited by OTTO DUNKEL, Washington University, ORRIN FRINK, JR., Pennsylvania State College, and H. S. M. COXETER, University of Toronto.

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THE HARMONIC BOUNDARY VALUE PROBLEM FOR AN ELLIPSE OR AN ELLIPSOID

DUNHAM JACKSON, University of Minnesota

1. Introduction. The solution of Laplace's equation in the unit circle taking on given boundary values on the circumference can be represented by a series of the elementary harmonic functions $r^n \cos n\phi$, $r^n \sin n\phi$, in terms of the polar coordinates (r, ϕ), with coefficients determined by expanding the boundary values in a Fourier series.* The series is not without importance, even though the solution can be alternatively expressed in "closed" form by means of Poisson's integral. This paper† is concerned with the explicit details of a modification of the series by which it can be adapted to the solution of the corresponding problem for an arbitrary ellipse. The method is simple in principle and is such as to lend itself readily to actual calculation.

A similar transition is indicated from the solution of the boundary value problem for a sphere by means of Laplace series to a solution for an arbitrary ellipsoid; the formulas, necessarily much more complicated, are however not carried through to the same degree of explicitness.

The reader who is familiar with the theory of orthogonal curvilinear coordinates and Lamé series will recognize that there is room for further discussion as to the relation between that theory and the elementary approach followed here.

2. Solution of the boundary value problem for an ellipse in series of special harmonic functions. Let the ellipse be given by the parametric representation

$$(1) \qquad x = a \cos \theta, \qquad y = b \sin \theta, \qquad 0 \leqq \theta \leqq 2\pi, \qquad a > b.$$

This θ is of course not the polar angle in a polar coordinate system; the latter will enter into the discussion at a later stage. But it is easy to calculate the value of θ corresponding to an assigned point of the ellipse, or the coordinates corresponding to any assigned θ.

* See for example D. Jackson, Fourier Series and Orthogonal Polynomials, Carus Mathematical Monographs, No. 6, 1941, pp. 103–105.

† The paper is an "expository" one, in the sense that no single feature of it can lay claim to any appreciable degree of novelty. The facts were however in the present instance worked out by the writer during a period of restriction to the house by illness, and he does not know where to lay his hand on a connected account of them in the literature. The results of §4 can be obtained more compactly by use of the terminology of "confocal coordinates"; the formulation adopted here, on the other hand, avoids certain questions of detail associated with the introduction of those coordinates in general terms. The relation between §5 and the theory of Lamé series appears to be matter for further study.

For a comprehensive theory of developments in series of harmonic polynomials with which the problem treated here is associated in its general aspects see J. L. Walsh, The approximation of harmonic functions by harmonic polynomials and by harmonic rational functions, Bulletin of the American Mathematical Society, vol. 35, 1929, pp. 499–544; G. M. Merriman, On the expansion of harmonic functions in terms of normal-orthogonal harmonic polynomials, American Journal of Mathematics, vol. 53, 1931, pp. 589–596.

Let the boundary values to which a harmonic function is to be fitted be specified by a function $f(\theta)$. This function can be expanded in a Fourier series

$$\begin{aligned}(2)\qquad &\frac{a_0}{2} + a_1 \cos\theta + a_2 \cos 2\theta + \cdots \\ &+ b_1 \sin\theta + b_2 \sin 2\theta + \cdots\end{aligned}$$

which will converge uniformly to $f(\theta)$ if $f(\theta)$ is continuous and satisfies appropriate further hypotheses.

Each of the functions $\cos n\theta$, $\sin n\theta$ can be expressed as a polynomial of the nth degree in $\cos\theta$ and $\sin\theta$, hence in x/a and y/b, or, with different coefficients, as a polynomial of the nth degree in x and y, where x and y are associated with θ by the equations (1). That is to say, there is for each of the functions $\cos n\theta$, $\sin n\theta$ a polynomial of the nth degree in x and y which reduces on the ellipse to the cosine or sine of the multiple of θ in question. There are in fact infinitely many such polynomials in each case, for $n \geqq 2$, since values on the ellipse are unaffected by addition of any polynomial containing $(x/a)^2+(y/b)^2-1$ as a factor.

There are just $2n+1$ polynomials of the nth or lower degree which are linearly independent on the ellipse. For any polynomial can be reduced by use of the relation $y^2=b^2[1-(x/a)^2]$ to a form in which no term contains a higher power of y than the first; that is, any polynomial of the nth or lower degree can be expressed on the ellipse as a linear combination of the $2n+1$ monomials

$$1,\ x,\ y,\ x^2,\ xy,\ x^3,\ x^2y,\ \cdots,\ x^n,\ x^{n-1}y.$$

On the other hand, the constant 1 and the $2n$ harmonic polynomials forming the real and imaginary components of $(x+iy)^k$, $k=1, 2, \cdots, n$, are linearly independent on the ellipse, or on any closed curve in the plane. For if a linear combination of them with coefficients not all zero were to vanish on the curve it would be a harmonic function vanishing on the curve but not vanishing identically in its interior, in contradiction with the theorem of uniqueness for the boundary value problem. Consequently any polynomial of the nth degree can be expressed *on the ellipse* as a linear combination of these $2n+1$ harmonic polynomials. Let

$$u_0 = 1, \qquad u_1 = x, \qquad v_1 = y, \qquad u_2 = x^2 - y^2, \qquad v_2 = 2xy,$$

and in general let $(x+iy)^k = u_k(x, y)+iv_k(x, y)$, each of the functions u_k, v_k being a polynomial of the kth degree in the two variables together. In particular the polynomials representing $\cos n\theta$ and $\sin n\theta$ can be linearly expressed on the ellipse in terms of the u's and v's; there are uniquely determined linear combinations of u_0, u_1, v_1, $\cdots$, u_n, v_n which reduce on the ellipse to $\cos n\theta$ and $\sin n\theta$ respectively. Let these combinations, which are themselves harmonic polynomials in x and y, be denoted by $U_n(x, y)$ and $V_n(x, y)$. It is to be shown presently that the calculation of their coefficients can readily be carried through in detail.

The existence of a unique solution of Laplace's equation taking on the boundary values $f(\theta)$ being assumed as known, let $F(x, y)$ denote this solution. Let $s_n(\theta)$ denote the partial sum of the series (2) through terms of the nth order, and let

$$S_n(x, y) = \frac{a_0}{2} + a_1U_1(x, y) + a_2U_2(x, y) + \cdots + a_nU_n(x, y)$$
$$+ b_1V_1(x, y) + b_2V_2(x, y) + \cdots + b_nV_n(x, y).$$

Then $S_n(x, y)$ is a harmonic polynomial reducing on the ellipse to $s_n(\theta)$. If $f(\theta)$ is such that $s_n(\theta)$ converges uniformly toward $f(\theta)$, and if $\epsilon > 0$, then

$$| f(\theta) - s_n(\theta) | < \epsilon$$

for $0 \leqq \theta \leqq 2\pi$, when n is sufficiently large. As the difference $f(\theta) - s_n(\theta)$ represents the boundary values of the harmonic function $F(x, y) - S_n(x, y)$ on the ellipse, it follows from the maximum-minimum theorem that

$$| F(x, y) - S_n(x, y) | < \epsilon$$

throughout the interior of the ellipse, for the same values of n. When the U's and V's are known, the solution $F(x, y)$ of the boundary value problem has the series representation

$$F(x, y) = \frac{a_0}{2} + \sum_{k=1}^{\infty} [a_kU_k(x, y) + b_kV_k(x, y)],$$

the a's and b's being merely the ordinary Fourier coefficients of $f(\theta)$; and the convergence of the series throughout the interior of the ellipse is at least as rapid as that of the Fourier series for the boundary values.

By the least-square property of the Fourier series, $S_n(x, y)$ as an approximation to $F(x, y)$ is the harmonic polynomial of the nth degree at most such that the integral of the square of the error around the boundary is a minimum *for integration with respect to* θ. An approximation satisfying a different least-square criterion, for example that of minimizing the integral with respect to arc length (*cf.* Merriman, *loc. cit.*), could be obtained in theory by the use of orthogonal trigonometric sums with an appropriate weight function* in place of the simple monomials $\cos n\theta$ and $\sin n\theta$, but would not be so readily accessible to calculation.

3. Calculation of the special functions by the method of undetermined coefficients. The sequence of U's and V's begins with

$$U_0 = 1 = u_0, \qquad U_1 = x/a = u_1/a, \qquad V_1 = y/b = v_1/b,$$

which are equal to 1, $\cos\theta$, $\sin\theta$ on the ellipse. Also $V_2 = 2xy/(ab) = v_2/(ab)$ is the harmonic polynomial corresponding to $\sin 2\theta = 2 \sin\theta \cos\theta$. If the polynomial

* See D. Jackson, Orthogonal trigonometric sums, Annals of Mathematics, (2), vol. 34, 1933, pp. 799–814.

$(x^2/a^2)-(y^2/b^2)$, representing $\cos 2\theta=\cos^2\theta-\sin^2\theta$ on the curve, is expressed with undetermined coefficients as a linear combination of $u_2=x^2-y^2$ and $(x^2/a^2)+(y^2/b^2)$, it is found on evaluation of the coefficients that

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{1}{a^2+b^2}\left[2(x^2-y^2)+(b^2-a^2)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)\right],$$

which means for points on the ellipse, since $(x^2/a^2)+(y^2/b^2)=1$, that

$$(a^2+b^2)\cos 2\theta = 2u_2-(a^2-b^2),$$

that is to say, the polynomial U_2 is defined by the identity

$$(a^2+b^2)U_2 = 2u_2-(a^2-b^2)u_0.$$

By similar but progressively less simple calculations, which the reader need not repeat unless he cares to do so, inasmuch as an alternative method of verification is presently to be indicated, it is found that

$$\begin{aligned}(a^3+3ab^2)U_3 &= 4u_3-3(a^2-b^2)u_1,\\ (3a^2b+b^3)V_3 &= 4v_3-3(a^2-b^2)v_1,\\ (a^4+6a^2b^2+b^4)U_4 &= 8u_4-8(a^2-b^2)u_2+(a^2-b^2)^2u_0,\\ (4a^3b+4ab^3)V_4 &= 8v_4-8(a^2-b^2)v_2.\end{aligned}\tag{3}$$

In these identities a general law of formation begins to be recognizable empirically, especially if a symbol $v_0\equiv 0$ is introduced for the sake of uniformity, so that a term $(a^2-b^2)^2v_0$ can be added to the right-hand member of the last relation, while the one giving V_2 can be written in the form

$$2abV_2 = 2v_2-(a^2-b^2)v_0.$$

The parentheses by which U_n and V_n are multiplied on the left contain alternate terms of the binomial expansion of $(a+b)^n$, while the coefficients on the right, apart from the powers of a^2-b^2 in regular succession, are the coefficients of the polynomial expressing $\cos n\theta$ in terms of $\cos\theta$.

It is to be shown that this law of formation is indeed general, not by a formal proof in combinatorial terms, but by indication of a method of approach from which it will be apparent how such a proof could be constructed.

4. Calculation of the special functions by the algebra of complex quantities. Let the polar coordinates of an arbitrary point of the plane be (r,ϕ), connected with x and y by the equations

$$x=r\cos\phi,\qquad y=r\sin\phi,$$

so that on the ellipse

$$a\cos\theta=r\cos\phi,\qquad b\sin\theta=r\sin\phi.\tag{4}$$

The problem is to express $\cos n\theta$ and $\sin n\theta$ in terms of $r^k \cos k\phi$ and $r^k \sin k\phi$, $k=0, 1, \cdots, n$, by means of the last pair of relations; since

$$(x + iy)^k = r^k(\cos k\phi + i \sin k\phi),$$

the functions $r^k \cos k\phi$ and $r^k \sin k\phi$ are identical with the u_k and v_k of the earlier paragraphs. The resulting expressions will be linear combinations of the u's and v's which reduce to $\cos n\theta$ and $\sin n\theta$ when the point (r, ϕ) is on the ellipse. The equation of the ellipse no longer enters except as it is implicit in the parametric representation. It is to be borne in mind that a and b are constant, while θ, r, and ϕ are variable. The question as thus presented is merely one of algebraic and trigonometric manipulation based on the relations (4).

Let (real) quantities m and γ be defined so that

$$a = m \cosh \gamma, \qquad b = m \sinh \gamma, \tag{5}$$

that is, let

$$m = (a^2 - b^2)^{1/2}, \qquad \gamma = \tfrac{1}{2} \log \frac{a + b}{a - b};$$

when m and γ are thus determined,

$$me^{\gamma} = a + b, \qquad me^{-\gamma} = a - b,$$

and the relations (5) are fulfilled. Then

$$\begin{aligned} m^2 \cosh 2\gamma &= \tfrac{1}{2}m^2(e^{2\gamma} + e^{-2\gamma}) = \tfrac{1}{2}[(a + b)^2 + (a - b)^2] \\ &= a^2 + b^2, & m^2 \sinh 2\gamma &= 2ab, \\ m^3 \cosh 3\gamma &= a^3 + 3ab^2, & m^3 \sinh 3\gamma &= 3a^2b + b^3, \\ m^4 \cosh 4\gamma &= a^4 + 6a^2b^2 + b^4, & m^4 \sinh 4\gamma &= 4a^3b + 4ab^3, \end{aligned}$$

etc., so that the left-hand members of the identities involving the U's in (3) have on the ellipse the form $m^n \cosh n\gamma \cos n\theta$, while the left members involving the V's take the form $m^n \sinh n\gamma \sin n\theta$.

From (4) and (5),

$$m \cosh \gamma \cos \theta = a \cos \theta = r \cos \phi, \qquad m \sinh \gamma \sin \theta = r \sin \phi.$$

Hence

$$m(\cosh \gamma \cos \theta + i \sinh \gamma \sin \theta) = r(\cos \phi + i \sin \phi) = re^{i\phi}.$$

Let

$$b \cos \theta = r' \cos \psi, \qquad a \sin \theta = r' \sin \psi.$$

Then

$$m \sinh \gamma \cos \theta = r' \cos \psi, \qquad m \cosh \gamma \sin \theta = r' \sin \psi,$$

$$m(\sinh \gamma \cos \theta + i \cosh \gamma \sin \theta) = r'e^{i\psi}.$$

From these relations it follows that

$$
(6) \qquad \begin{aligned} re^{i\phi} + r'e^{i\psi} &= m(\cosh\gamma + \sinh\gamma)(\cos\theta + i\sin\theta) = me^{\gamma}e^{i\theta}, \\ re^{i\phi} - r'e^{i\psi} &= me^{-\gamma}e^{-i\theta}. \end{aligned}
$$

Consequently

$$
(7) \qquad m^n(e^{n\gamma}e^{ni\theta} + e^{-n\gamma}e^{-ni\theta}) = (re^{i\phi} + r'e^{i\psi})^n + (re^{i\phi} - r'e^{i\psi})^n.
$$

The real and pure imaginary parts of the left-hand member are respectively

$$
m^n(e^{n\gamma} + e^{-n\gamma})\cos n\theta = 2m^n \cosh n\gamma \cos n\theta
$$

and

$$
im^n(e^{n\gamma} - e^{-n\gamma})\sin n\theta = 2im^n \sinh n\gamma \sin n\theta,
$$

so that these quantities can be evaluated by separating real and imaginary components on the right of (7). On the other hand, multiplication of the two equations (6) by each other gives

$$
(8) \qquad r^2e^{2i\phi} - r'^2e^{2i\psi} = m^2.
$$

The rest of the calculation is probably easier to follow on a first reading if carried through, not for arbitrary n, but for a particular value of n small enough to permit writing the formulas at full length, and at the same time large enough to illustrate the essential points involved. With $n=6$, one-half the right-hand member of (7) is

$$
r^6e^{6i\phi} + 15r^4r'^2e^{i(4\phi+2\psi)} + 15r^2r'^4e^{i(2\phi+4\psi)} + r'^6e^{6i\psi},
$$

the terms with odd powers of r' in the binomial expansions cancelling each other. On replacement of $r'^2e^{2i\psi}$ by $r^2e^{2i\phi}-m^2$ from (8) this takes the form

$$
(9) \qquad r^6e^{6i\phi} + 15r^4e^{4i\phi}(r^2e^{2i\phi} - m^2) + 15r^2e^{2i\phi}(r^2e^{2i\phi} - m^2)^2 + (r^2e^{2i\phi} - m^2)^3,
$$

which becomes after expansion and collection of terms

$$
(10) \qquad A_6r^6e^{6i\phi} + A_4m^2r^4e^{4i\phi} + A_2m^4r^2e^{2i\phi} + A_0m^6
$$

with numerical coefficients $A_6, \cdots, A_0$. It is clear from the structure of the formulas that the coefficients are the same as those which are obtained if $\cos 6\theta$ as the real part of $(\cos\theta+i\sin\theta)^6$ is written in the form

$$
\begin{aligned} &\cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta - \sin^6\theta \\ &\qquad = \cos^6\theta + 15\cos^4\theta(\cos^2\theta - 1) + 15\cos^2\theta(\cos^2\theta - 1)^2 + (\cos^2\theta - 1)^3 \end{aligned}
$$

and rearranged as a polynomial in $\cos\theta$,

$$
\cos 6\theta = A_6\cos^6\theta + A_4\cos^4\theta + A_2\cos^2\theta + A_0.
$$

The expression (9) can in fact be represented by $m^6C_6(re^{i\phi}/m)$, if $C_n(\cos\theta)$ is the polynomial giving $\cos n\theta$ in terms of $\cos\theta$.

Separation of reals and pure imaginaries in (10) then gives

$$m^6 \cosh 6\gamma \cos 6\theta = (a^6 + 15a^4b^2 + 15a^2b^4 + b^6) \cos 6\theta$$
$$= A_6 r^6 \cos 6\phi + A_4 m^2 r^4 \cos 4\phi + A_2 m^4 r^2 \cos 2\phi + A_0 m^6,$$
$$m^6 \sinh 6\gamma \sin 6\theta = (6a^5b + 20a^3b^3 + 6ab^5) \sin 6\theta$$
$$= A_6 r^6 \sin 6\phi + A_4 m^2 r^4 \sin 4\phi + A_2 m^4 r^2 \sin 2\phi$$

for points (r, ϕ) on the ellipse, that is to say,

$$(a^6 + 15a^4b^2 + 15a^2b^4 + b^6) U_6 = A_6 u_6 + A_4(a^2 - b^2) u_4 + A_2(a^2 - b^2)^2 u_2 + A_0(a^2 - b^2)^3,$$
$$(6a^5b + 20a^3b^3 + 6ab^5) V_6 = A_6 v_6 + A_4(a^2 - b^2) v_4 + A_2(a^2 - b^2)^2 v_2.$$

It is clear that the work that has been done in the last two paragraphs for $n=6$ is applicable in principle for an arbitrary value of n, requiring merely a repetition of the same steps with an appropriate notation for the general binomial coefficients. In tabulating as many of the U's and V's as may be desired for computational purposes, the polynomials in a and b which appear in the left-hand members of the identities can of course be written down at once from the powers of $(a+b)^n$. The coefficients corresponding to the A's, identified by the fact that they serve to express $\cos n\theta$ in terms of powers of $\cos \theta$, can be obtained rapidly in succession from the recurrence relation

$$\cos (n + 1)\theta = 2 \cos \theta \cos n\theta - \cos (n - 1)\theta.$$

The u's and v's can be found as polynomials in x and y from the expansion of $(x+iy)^k$, or can be left in the form $r^k \cos k\phi$, $r^k \sin k\phi$.

5. The boundary value problem for an ellipsoid. The method outlined above can be adapted in its earlier stages to the solution of the harmonic boundary value problem for an ellipsoid, though the formulas do not appear ultimately to be similarly manageable.

Let the ellipsoid, with the rectangular equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \tag{11}$$

be represented parametrically by

$$x = a \cos \theta, \qquad y = b \sin \theta \cos \phi, \qquad z = c \sin \theta \sin \phi. \tag{12}$$

The assigned boundary values can be regarded as constituting a given function $f(\theta, \phi)$. Under suitable hypotheses $f(\theta, \phi)$ can be expanded in a uniformly convergent Laplace series in terms of the spherical harmonics

$$\cos n\phi \sin^n \theta P_m^{(n)}(\cos \theta), \qquad \sin n\phi \sin^n \theta P_m^{(n)}(\cos \theta), \tag{13}$$

$m=0, 1, 2, \cdots; n=0, 1, \cdots, m$, where $P_m^{(n)}(\cos \theta)$ denotes the nth derivative of the Legendre polynomial $P_m(\cos \theta)$ with respect to $\cos \theta$ as argument.*

* For the form of the series see for example the Carus Monograph cited above, pp. 118–126.

For $m=0, 1, 2$ the functions (13) are

$$1, \quad \cos\theta, \quad \cos\phi\sin\theta, \quad \sin\phi\sin\theta,$$
$$\tfrac{3}{2}\cos^2\theta - \tfrac{1}{2},\ 3\cos\phi\sin\theta\cos\theta,\ 3\sin\phi\sin\theta\cos\theta,\ 3\cos 2\phi\sin^2\theta,\ 3\sin 2\phi\sin^2\theta;$$

since $\sin n\phi=0$ for $n=0$ just $2m+1$ functions are defined for each m. It is apparent by inspection that the expressions listed can be written by means of (12) in the alternative form

$$(14)\qquad \begin{array}{lllll} 1, & x/a, & y/b, & z/c, & \\ \tfrac{3}{2}(x/a)^2 - \tfrac{1}{2}, & 3xy/(ab), & 3xz/(ac), & 3[(y/b)^2 - (z/c)^2], & 6yz/(bc). \end{array}$$

It can be shown* that each of the functions (13) is expressible as a polynomial of the mth degree in the three variables $\cos\theta$, $\sin\theta\cos\phi$, $\sin\theta\sin\phi$, and so by the virtue of (11) in x/a, y/b, z/c or in x, y, z.

By substitution of the value of z^2 from (11), any polynomial can be reduced as far as values on the ellipsoidal surface are concerned to a form in which no term contains a power of z above the first. There are just $2m+1$ monomials of the mth degree, and

$$1 + 3 + 5 + \cdots + (2m+1) = (m+1)^2$$

monomials of the mth or lower degree, which contain no power of z higher than the first. On the other hand,† there are just $2m+1$ linearly independent homogeneous harmonic polynomials of the mth degree, and $(m+1)^2$ independent homogeneous harmonic polynomials of the mth or lower degree, and these can be taken as representing in terms of x, y, z the functions

$$(15)\qquad \rho^m \cos n\omega \sin^n \psi P_m^{(n)}(\cos\psi), \qquad \rho^m \sin n\omega \sin^n \psi P_m^{(n)}(\cos\psi),$$

if (ρ, ψ, ω) are spherical coordinates of an arbitrary point of space, so that

$$x = \rho\cos\psi, \qquad y = \rho\sin\psi\cos\omega, \qquad z = \rho\sin\psi\sin\omega.$$

These harmonic polynomials are linearly independent not merely as functions of three variables but also as functions in two dimensions on the ellipsoidal surface, since a relation of linear dependence connecting them there would contradict the theorem of uniqueness for the boundary value problem.

Consequently any polynomial is expressible on the surface as a linear combination of harmonic polynomials, and this is true in particular of the polynomial representations of the functions (13). If the terms of the Laplace series for $f(\theta, \phi)$ are replaced by the corresponding harmonic polynomials, there is obtained a series of harmonic functions which converges uniformly to $f(\theta, \phi)$ on the surface, under suitable hypotheses, and so converges uniformly to the solu-

* See for example Carus Monograph, pp. 126–127; for the present purpose the factor ρ of the passage cited is to be set equal to 1, and x, y, z (from page 106 of the Monograph) are to be replaced by z, x, y.

† See Carus Monograph, pp. 126–129.

tion of the boundary value problem throughout the interior of the ellipsoid, by the same sort of application of the maximum-minimum theorem that was used in the next to the last paragraph of §2 for the ellipse.

The polynomials corresponding to (15) for $m=0$, 1, 2 are

$$\begin{array}{lccccc} & 1, & x, & y, & z, & \\ (16) & x^2-\tfrac{1}{2}y^2-\tfrac{1}{2}z^2, & 3xy, & 3xz, & 3(y^2-z^2), & 6yz. \end{array}$$

It can be readily verified that these satisfy Laplace's equation. Of the nine polynomials (14), all but in fifth and eighth are harmonic as they stand, being constant multiples of the corresponding members of the sequence (16). Each of the remaining two can be expressed with undetermined coefficients as a linear polynomial in $x^2-\frac{1}{2}y^2-\frac{1}{2}z^2$, $3(y^2-z^2)$, and $(x/a)^2+(y/b)^2+(z/c)^2$. By evaluation of the coefficients and substitution of the value 1 for $(x/a)^2+(y/b)^2+(z/c)^2$ it is found that on the ellipsoid

$$\tfrac{3}{2}(x/a)^2-\tfrac{1}{2}=(1/D)\left[A_1(x^2-\tfrac{1}{2}y^2-\tfrac{1}{2}z^2)+B_1(3y^2-3z^2)+C_1\right],$$
$$3\left[(y/b)^2-(z/c)^2\right]=(1/D)\left[A_2(x^2-\tfrac{1}{2}y^2-\tfrac{1}{2}z^2)+B_2(3y^2-3z^2)+C_2\right],$$

with

$$D=\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2},\qquad A_1=\frac{3}{2a^2}\left(\frac{1}{b^2}+\frac{1}{c^2}\right),\qquad B_1=\frac{1}{4a^2}\left(\frac{1}{c^2}-\frac{1}{b^2}\right),$$
$$C_1=\frac{1}{a^2}-\frac{1}{2b^2}-\frac{1}{2c^2},$$
$$A_2=\frac{3}{a^2}\left(\frac{1}{c^2}-\frac{1}{b^2}\right),\qquad B_2=\frac{1}{2a^2b^2}+\frac{1}{2a^2c^2}+\frac{2}{b^2c^2},\qquad C_2=3\left(\frac{1}{b^2}-\frac{1}{c^2}\right).$$

It is clear that the work would have to be continued much further to bring to light any general law of formation of the coefficients.

NOTE ON THE t-TEST

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1. Definition. Given a normal universe whose mean α is assumed to be known but whose standard deviation σ is unknown, the hypothesis that a given sample of $n'=n+1$ elements with mean $\bar{x}$ and variance $s^2=\sum(x-\bar{x})^2/n$ may be considered as a random sample of such a universe at a certain level of probability P is accepted or rejected according as

$$t=\frac{(\bar{x}-\alpha)\sqrt{n'}}{s} \tag{1}$$

has a value less or greater than that derived from

$$(2) \qquad P = \frac{2\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi n}\,\Gamma\left(\frac{n}{2}\right)} \int_t^\infty \frac{dt}{\left(1+\frac{t^2}{n}\right)^{(n+1)/2}}.$$

In case n is very large the value of t for $P=.05$ is 1.96; in case the sample has only $n'=2$ elements so that n has its minimum value 1, t is 12.7.

2. Distribution of t for samples of 2. For this case of $n=1$ it is sometimes practicable to derive the distribution of t in (1) for universes other than normal. We have indeed

$$(3) \qquad \bar{x} = \frac{x_1 + x_2}{2}, \qquad s = \frac{|x_1 - x_2|}{\sqrt{2}}.$$

If $f(x)$ is the postulated probability distribution of the underlying universe, the probability of the observed sample (x_1, x_2) is the integral

$$(4) \qquad \iint f(x_1)f(x_2)dx_1dx_2$$

extended over all parts of the x_1x_2-plane which contain points (x_1, x_2). As the distribution in the plane is symmetrical, with respect to x_1 and x_2, we may use the integral

$$2\iint f(x_1)f(x_2)dx_1dx_2$$

extended over all parts of the x_1x_2-plane containing points (x_1, x_2) with $x_1 \geqq x_2$ and we may take $s=(x_1-x_2)/\sqrt{2}$. We may then solve for

$$x_1 = \bar{x} + \tfrac{1}{2}\sqrt{2}\,s, \qquad x_2 = \bar{x} - \tfrac{1}{2}\sqrt{2}\,s, \qquad dx_1dx_2 = \sqrt{2}\,d\bar{x}ds$$

and consider the integral

$$(5) \qquad 2\sqrt{2}\iint f(\bar{x} + \tfrac{1}{2}\sqrt{2}\,s)f(\bar{x} - \tfrac{1}{2}\sqrt{2}\,s)d\bar{x}ds$$

over all possible values of $\bar{x}$ and s. If we eliminate $\bar{x}$ by introducing from (1)

$$(6) \qquad \bar{x} = \frac{st}{\sqrt{2}} + \alpha, \qquad d\bar{x} = \frac{s}{\sqrt{2}}dt$$

we have

$$(7) \qquad 2\iint f\left(\frac{st}{\sqrt{2}} + \alpha + \tfrac{1}{2}\sqrt{2}\,s\right)f\left(\frac{st}{\sqrt{2}} + \alpha - \tfrac{1}{2}\sqrt{2}\,s\right)sdsdt$$

extended over all effective parts of the ts-plane, and if the proper limits for s

in terms of t can be found and the integrations can be performed, the distribution of t may be obtained.

For the case of the J-type exponential distribution

$$f(x) = ae^{-ax}, \qquad x \geqq 0,$$

with a mean value $1/a$, the result is

$$\frac{2a^2}{e^2}\int\int e^{-\sqrt{2}ast}sdsdt. \tag{8}$$

The region of integration in the x_1x_2-plane is that below the line $x_1=x_2$, $\bar{x}$ may have any non-negative value, s ranges from $s=0$ on that line to $s=\sqrt{2}\bar{x}$ on the x_1-axis, and for this value of s we have

$$s = \frac{\sqrt{2}/a}{1-t} \quad \text{since} \quad \alpha = 1/a. \tag{9}$$

The region of integration in the $\bar{x}s$-plane is that under the line $s=\sqrt{2}\bar{x}$. As t is constant along the lines issuing from the point $(\bar{x}=1/a,\ s=0)$ in the $\bar{x}s$-plane, the range of s is from 0 to ∞ for $t>1$ whereas for $t<1$ the range is from 0 to the value of s given by (9). The integration must therefore lead to two different expressions, which are:

$$\frac{1}{e^2t^2} \quad \text{if} \quad t > 1 \quad \text{and} \quad \frac{1}{e^2t^2}\left(1 - \frac{1+t}{1-t}e^{-2t/(1-t)}\right) \quad \text{if} \quad t < 1, \tag{10}$$

the total range of t being from $-\infty$ to $+\infty$.

The two functions have the common value e^{-2} at $t=1$. The distribution has a mode at $t=.42$ where it takes the value $2.41/e^2=.326$. At $t=0$ the value is $2/e^2=.271$. The integral for $t<1$ which vanishes at $t=-\infty$ is

$$1 - \frac{1}{e^2t} + \frac{1}{e^2}\left(\frac{1}{t} - 1\right)e^{-2t/(1-t)}$$

from which it may be ascertained that the median is at $t=-.40$. For the normal distribution for $n=1$ the t distribution is

$$\frac{1}{\pi}\,\frac{1}{1+t^2}$$

which is symmetrical with the mode at $t=2$ of value .318. This curve falls off as $.318/t^2$ for large values of t, whether positive or negative, whereas the distribution (10) falls off as $.135/t^2$ for large positive and as $1.135/t^2$ for large negative t and is very skew. Neither distribution has a mean because the integrand falls off only as $1/t^2$ at infinity.

3. Critical ratios for $P=.05$. If we should automatically take $t=12.7$ from the tables for t there would not be a chance of .025 for $t>12.7$ or for $t>-12.7$;

the chance for $t>12.7$ would actually be about .01 and for $t<-12.7$ would be about .08, making a total of .09 instead of .05. What values of t would reject the hypothesis at the .05 level? That would depend on how one might define "significant" values of t for the skew curve. If we were to take those values beyond which the tail of the distribution contained an area of .025 we should find the limits at $t=5.4$ and $t=-44$. If, however, we should set the limits of t by the condition that a line parallel to the axis should cut the distribution (10) in two points such that the total area in the two tails beyond those points should be .05 the positive value of t would be about 10, the negative value about -30.

It is not to be supposed that from any sample of 2 elements (or from any really small sample) one could tell the type of the universe from which the sample was drawn; if known at all, it would have to be as a result of previous experience with large samples or on the basis of adequate a priori theory. Furthermore if the universe were not normal presumably some test of significance other than the distribution of t defined by (1) should be used. Yet, as many persons will very naturally apply the t-test quite automatically, it is worth while to show students by elementary examples that if the universe is not normal there may be considerable differences between the probabilities determined from the tables for t and those calculated from various distributions. The reader can readily satisfy himself that if the universe were rectangular instead of normal, the .05 level would be for $t=19$ instead of 12.7.

THE ABEL-DINI AND ALLIED THEOREMS

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1. These observations are in the nature of a postscript to two papers in this MONTHLY: (i) T. H. Hildebrandt's paper *Remarks on the Abel-Dini theorem*, vol. 49 (1942), pp. 441–445, and (ii) my paper *Remarks on some generalizations of Cauchy's condensation and integral tests*, vol. 48 (1941), pp. 180–185. For the sake of brevity these two papers will be referred to as **H** and **R** respectively in what follows. Theorem I of **H** and its further extensions are all included in the third alternative of Theorem 1a of **R**. Further, Theorem 1a of **R**, in common with the Abel-Dini Theorem, can be derived from de la Vallée-Poussin's theorem quoted at the outset of **R**. This derivation of the Abel-Dini theorem is exactly like the derivation of its analogue for convergent series from another theorem of de la Vallée-Poussin which runs:*

THEOREM 1. *If $m_n \downarrow 0$ and $F(x)\uparrow \infty$ as $x\to +0$, then $\sum(m_n-m_{n+1})F(m_n)$ converges with $\int_0 F(x)dx$ and $\sum(m_n-m_{n+1})F(m_{n+1})$ diverges with $\int_0 F(x)dx$.*

To deduce from this the analogue of the Abel-Dini theorem we first note that $\sum(m_n-m_{n+1})/m_{n+1}$ is divergent. The divergence of $\sum(m_n-m_{n+1})/m_n$ now follows since, either $2m_{n+1}>m_n$ for all sufficiently large n which means

* Cours d'Analyse, vol. 1, 1926, p. 399, Ex. 5.

$(m_n - m_{n+1})/m_n > (m_n - m_{n+1})/2m_{n+1}$ for all such n, or $2m_{n+1} \leqq m_n$ for an infinity of n which means $(m_n - m_{n+1})/m_n \geqq \frac{1}{2}$ for these n. Hence we have in any case the divergence of $\Sigma(m_n - m_{n+1})/m_n^p$ for $p=1$ and consequently for $p>1$ also. Thus we are led to the

COROLLARY. *If* $m_n \downarrow 0$, $\sum(m_n - m_{n+1})/m_n^p$ *converges or diverges according as* $p<1$ *or* $p \geqq 1$.

Theorems related to this very much as Theorems I, II of **H** are related to the Abel-Dini theorem appear as particular cases of the following deduction from Theorem 1 when $F(x) = 1/xl_1(1/x) \cdot l_2(1/x) \cdots [l_k(1/x)]^p$, where $l_k\xi = \log\log \cdots$ (k times)ξ.*

THEOREM 1a. *If, in Theorem* 1, *either* $\overline{\lim}\ (m_n^{-1} - m_{n-1}^{-1}) < \infty$, *or* $\underline{\lim}\ (m_n - m_{n+1})/(m_{n-1} - m_n) > 0$, *or* $\underline{\lim}\ m_n/m_{n-1} > 0$, *then* $\sum(m_n - m_{n+1})F(m_{n+1})$ *converges or diverges with* $\int_0 F(x)dx$.

2. The treatment accorded above to the Abel-Dini theorem and to its analogue support a point of view which is explained in **R** and which may be briefly expressed thus. Our familiar convergence theorems as well as canonical forms for the criteria of convergence can be shown to have their origin in Maclaurin's method of condensing a series by means of an integral. This point of view helps us to achieve a pedagogically satisfactory "systematization of the general theory of convergence." For instance it enables us to bring Kummer's test within the scope of such a systematization more naturally than attempts like Knopp's.† Also it dispenses with a special argument like Bromwich's to explain why a *divergent* series should figure prominently in this criterion for convergence.‡ Furthermore it makes it clear that Kummer's test is only a criterion 'of the second kind' which corresponds to Pringsheim's generalization of Cauchy's root-test§ regarded as a criterion 'of the first kind.' These points, implicit in **R**, will now be made explicit with reference to the following statement of Kummer's and Pringsheim's criteria.

$\mathcal{C}$. *If* $a_n > 0$, $d_n > 0$ ($n = 1, 2, \cdots$), *the latter are the terms of a divergent series and* $D_n = \sum_{\nu=1}^{n} d_\nu$, *then*

either (i)

$$\left(\frac{a_n}{d_n}\right)^{1/D_n} \leqq \theta < 1 \qquad \text{[\textit{PRINGSHEIM}]},$$

or (ii)

$$\frac{1}{d_n}\left(\frac{a_{n+1}/d_{n+1}}{a_n/d_n} - 1\right) \leqq -\rho < 0 \qquad \text{[\textit{KUMMER}]},$$

ensures the convergence of $\sum a_n$.

* For a discussion of some allied problems see my paper, On certain theorems of Pringsheim, Tôhoku Math. Journ. 43, 1937, pp. 122–126.

† Knopp, Infinite Series, §42, in particular, pp. 310–311.

‡ Bromwich, Infinite Series (ed. 2), p. 38.

§ Math. Annalen, vol. 35, 1890, p. 340; or Knopp, *loc. cit.*, p. 309.

These results are obtained with $F(x)=e^{-\rho x}$, (i) in Theorem 1 of **R**, (ii) in the first alternative of Theorem 1a of **R**. For, (i) implies that

$$\frac{a_n}{d_n} \leqq e^{-\rho D_n} \qquad (e^{-\rho} = \theta,\ \rho > 0)$$

(ii) has a similar implication since $0<\rho d_n<1$ and therefore

$$\frac{a_{n+1}/d_{n+1}}{a_n/d_n} \leqq 1 - \rho d_n < e^{-\rho d_n}$$

whence

$$\frac{a_n/d_n}{a_1/d_1} < e^{-\rho D_{n-1}} = e^{-\rho D_n + \rho d_n} < e^{-\rho D_n + 1}$$

or

$$\frac{a_n}{d_n} < K e^{-\rho D_n} \qquad \left(K = \frac{a_1}{d_1} e\right).$$

Theoretically $\mathcal{C}$ is incomplete without a statement such as $\mathcal{D}$ which sets forth a criterion for the divergence of $\sum a_n$ in terms of a *convergent* series.

$\mathcal{D}$. *If* $a_n>0$, $c_n>0$ $(n=1, 2, \cdots)$, *the latter are the terms of a convergent series and* $R_n=\sum_{\nu=n+1}^{\infty} c_\nu$, *then*

either (i) $$\left(\frac{a_n}{c_n}\right)^{R_n} \geqq \lambda > 1,$$

or (ii) $$\left(\frac{1}{R_n^{-1} - R_{n-1}^{-1}}\right)\left(1 - \frac{a_{n-1}/c_{n-1}}{a_n/c_n}\right) \geqq \rho > 0^*$$

ensures the divergence of Σa_n.

These results follow from Theorem 1 when we put $F(x)=e^{\rho/x}$, $m_n=R_n$.

3. A theorem, of much the same scope as Theorem I of **H**, and like it included in the third alternative of Theorem 1a of **R**, is:

THEOREM 2. *If* $D_n=\sum_{\nu=1}^{n} d_\nu$ $(d_\nu>0)\to\infty$ *and* $D_n/D_{n-1}=O(1)$ *as* $n\to\infty$, *then the series*

$$\sum \frac{d_n}{D_n^p} (\log D_n)^q, \dagger$$

$$\sum \frac{d_n}{D_n^p}\left(\frac{d_1}{D_1} + \frac{d_2}{D_2} + \cdots + \frac{d_n}{D_n}\right)^q,$$

are convergent for $p>1$, *or* $p=1$ *and* $q<-1$; *divergent otherwise.*

* The hypothesis in $\mathcal{D}$ (ii) is the analogue of the hypothesis in $\mathcal{C}$ (ii) in the form:

$$\left(\frac{1}{D_n - D_{n-1}}\right)\left(\frac{a_n/d_n}{a_{n-1}/d_{n-1}} - 1\right) \leqq -\rho < 0.$$

† The case $d_n=1$ is given in Bromwich, *loc. cit.*, p. 460, Ex. 22.

The case $p>1$ of the second series in this theorem admits of a considerable generalization in the direction of certain theorems of Hardy.* The generalization may be stated as follows.

THEOREM 3. *If* $\{d_n\}$ *is as in Theorem 2,* $\{a_n\}$ *non-negative and such that*

$$A_n = a_1d_1 + a_2d_2 + \cdots + a_nd_n \to \infty \quad as \quad n \to \infty,$$

then

$$\sum \frac{d_n}{D_n^p} A_n^q \leqq K \sum \frac{d_n}{D_n^p} (a_nD_n)^q,$$

where $p>1$, $q\geqq 1$, $K=K(p, q)$ *depends on* p *and* q *only, and the series on the right-hand side is convergent.*†

Proof. For $p>1$,

$$\phi_n = \frac{d_n}{D_n^p} + \frac{d_{n+1}}{D_{n+1}^p} + \cdots < \int_{D_{n-1}}^{\infty} \frac{dx}{x^p} = \frac{D_{n-1}^{1-p}}{p-1} = O(D_n^{1-p}).$$

Taking $A_0=0$, we have

$$\begin{aligned}
\sum_1^N \frac{d_n}{D_n^p} A_n^q &= \sum_1^N (\phi_n - \phi_{n+1}) A_n^q \leqq \sum_1^N \phi_n (A_n^q - A_{n-1}^q) \\
&= \sum_1^N O\left(\frac{1}{D_n^{p-1}}\right)(A_n^q - A_{n-1}^q) \\
&\leqq K \sum_1^N \frac{A_n^{q-1} a_n d_n}{D_n^{p-1}} \\
&\leqq K \left\{ \sum_1^N \frac{d_n}{D_n^q} (a_nD_n)^q \right\}^{1/q} \left\{ \sum_1^N \frac{d_n}{D_n^p} A_n^q \right\}^{(q-1)/q}
\end{aligned}$$

where we assume $q>1$ and use Hölder's inequality. The theorem for $q>1$ follows at once from this. When $q=1$ the argument remains valid without any appeal to Hölder's inequality.

That the theorem is not necessarily true for $p\leqq 1$ can be verified as follows. For $p\leqq 1$, $q>0$ the series on the left-hand side of the conclusion of Theorem 3 is divergent since A_n^q increases with n. The series on the right-hand side may be convergent as in the case $a_n = 1/D_n \log D_n$, $p=1$, $q>1$.

Theorem 3 becomes trivial when $\sum a_nd_n$ is convergent in which case it may be replaced by

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* See for instance Hardy, Littlewood and Polya, Inequalities, p. 247, Theorem 332.

† This result with $d_n=1$, is stated and proved by Hardy and Littlewood in Journal für Math., vol. 157, 1927, pp. 143–145.

THEOREM 4. *If* $\{d_n\}$ *is as in Theorem* 2, $\{a_n\}$ *non-negative and such that* $\sum a_n d_n$ *is convergent, and if*

$$A_n = a_n d_n + a_{n+1} d_{n+1} + \cdots ,$$

then, for $p<1$, $q\geqq 1$,

$$\sum_m^\infty \frac{d_n}{D_n^p} A_n^q \leqq K \sum_m^\infty \frac{d_n}{D_n^p} (a_n D_n)^q,$$

where $K=K(p, q)$, *and the series on the right-hand side is convergent.*

This theorem, like the previous, is established by an obvious generalization of the argument employed by Hardy and Littlewood, *loc. cit.*, to prove the case $d_n=1$.

We see that the theorem is not necessarily true for $p \nless 1$ from the example: $p=1$, $a_n=1/D_n(\log D_n)^{1+\delta}$, $0<\delta<1/q$.

It may be observed that Theorems 3 and 4 invite comparison with Theorems 1.1, 1.2, 2.1, 2.2 of E. T. Copson's *Note on series of positive terms* in the Journal of the London Mathematical Society, vol. 3 (1928) pp. 49–51.

DIFFERENCE EQUATIONS IN AVERAGE VALUE PROBLEMS

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1. Introduction. Although there have been some very interesting applications* of difference equations to probability problems, there seems to have been no organized attempt to apply them to average (or mean) value problems. In fact, there have been relatively few problems stated which require the average number of trials necessary to produce a certain number of successes or the average value of a stochastic or chance variable. These average value problems which have been discussed usually have been solved by infinite series. The method of difference equations is often easier and more direct than the use of series even in the simpler problems and becomes practically indispensable in some of the more complicated ones.

A discussion of difference equations and the *general* solution of a linear difference equation has been included in §2; these results are not new, but it appears that they are not well known and are of sufficient importance to warrant their inclusion. The reader may prefer to read the section only hastily, or omit it, before proceeding with the succeeding sections, and refer back as occasion demands.

Section 3 contains representative theorems on average value problems which are well-known; their solutions are obtained here from difference equations.

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* Uspensky, J. V., Introduction to Mathematical Probability, McGraw-Hill Book Company, Inc., New York & London, 1937, pp. 166–7.

Some new problems are discussed in §4, both for their intrinsic interest and as vehicles to illustrate methods of solving linear difference equations in one variable. One of these problems was proposed as an advanced problem in this MONTHLY. Although its solution was obtained from a difference equation in three or four ways it appears that direct methods of solving these equations are not familiar mathematical techniques. Uspensky gives many excellent applications of them to problems in probability, and his book contains a great deal of material not to be found in other books on the same subject in the English language.

The problem in (4.3) is unique in that Bayes' theorem is required to establish the difference equation which yields its solution.

2. Difference equations in one variable. An equation in x, $f(x)$, $f(x+1)$, $f(x+2), \cdots, f(x+n)$, where x is an independent variable, is called a difference equation* in the unknown function $f(x)$. A difference equation in which both $f(x)$ and $f(x+n)$ appear, but not $f(x+n+m)$, $m=1, 2, 3, \cdots$, is said to be of *order* n.

The function $\Delta f(x)=f(x+1)-f(x)$ is called the *difference* of $f(x)$, and the difference of $\Delta f(x)$, namely,

$$\Delta^2 f(x) = f(x+2) - 2f(x+1) + f(x)$$

is called the *second difference* of $f(x)$; successive differences are defined similarly. These equations can be solved for $f(x+1), f(x+2), \cdots, f(x+n)$ and substituted in equation (2.11) below to produce an equation of the form

$$A_n(x)\Delta^n f(x) + A_{n-1}(x)\Delta^{n-1}f(x) + \cdots + A_0(x)f(x) = \phi(x).$$

It was this notation which suggested the name *difference equation.*

2.1. Linear difference equations. If each $f(x+k)$, $0 \leqq k \leqq n$, appears linearly then the equation is called linear. We shall consider only linear difference equations in one unknown function. The general one can be written as

$$a_n(x)f(x+n) + a_{n-1}(x)f(x+n-1) + \cdots + a_0(x)f(x) = \phi(x), \tag{2.11}$$

where the coefficients $a_i(x)$ and $\phi(x)$ are known functions. In all the problems which we shall consider, these coefficient functions are rational. If $\phi(x)=0$, then equation (2.11) is called homogeneous. If $\phi \neq 0$, then the equation

$$a_n(x)f(x+n) + a_{n-1}(x)f(x+n-1) + \cdots + a_0(x)f(x) = 0 \tag{2.12}$$

is called the *associated homogeneous equation* of (2.11).

We shall consider first the solution of a homogeneous linear difference equation since the solutions of the non-homogeneous equation can be derived from those of its associated homogeneous equation.

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* Batchelder, P. M., An Introduction to Linear Difference Equations, Harvard University Press, 1927.

Boole, George, Treatise on the Calculus of Finite Differences, 1860.

If in any way we can obtain particular solutions $f_1(x)$, $f_2(x)$, $\cdots$, $f(x)$, of the homogeneous equation (2.12), then it follows by direct substitution that

$$f(x) = p_1(x)f_1(x) + p_2(x)f_2(x) + \cdots + p_n(x)f_n(x) \tag{2.13}$$

is also a solution, where the p's are arbitrary functions of period* one. In any problem where x assumes only the values $x=k+1$, $k+2$, $k+3$, $\cdots$, (where k is a constant) the difference equation will have a solution of the form

$$f(x) = c_1f_1(x) + c_2f_2(x) + \cdots + c_nf_n(x), \tag{2.14}$$

where the c's are constants. This follows since $p(x+1)\equiv p(x)$ has a constant value when x assumes any of the values $k+1$, $k+2$, $k+3$, $\cdots$.

The solutions (2.14) are not only solutions of equation (2.12), but are the only† solutions of the equation provided the particular solutions $f_1(x)$, $f_2(x)$, $\cdots$, $f_n(x)$ form a *fundamental system of solutions*, *i.e.*, are linearly independent. Then $f(x)$ is called the *general solution* of equation (2.12).

It is often possible to obtain a fundamental system of solutions of a linear homogeneous difference equation with constant coefficients, namely,

$$k_nf(x+n) + k_{n-1}f(x+n-1) + \cdots + k_0f(x) = 0,$$

of the form $f(x)\equiv\alpha^x$. When this substitution is made, the resulting *characteristic equation* of degree n,

$$k_n\alpha^n + k_{n-1}\alpha^{n-1} + \cdots + k_0 = 0 \tag{2.15}$$

determines n values, α_1, α_2, $\cdots$, α_n of α. Then the functions

$$\alpha_1^x, \alpha_2^x, \cdots, \alpha_n^x$$

constitute a fundamental system of solutions provided no two values of the α_i's are equal.

When exactly r roots of the characteristic equation are equal, the difference equation has solutions of the form $\alpha^x u(x)$, where $u(x)$ is a polynomial of degree $(r-1)$; that is, the r functions α_1^x, α_2^x, $\cdots$, α_r^x, where $\alpha_1=\alpha_2=\cdots=\alpha_r$, can be replaced by the r independent particular solutions

$$\alpha_1^x, x\alpha_1^x, \cdots, x^{r-1}\alpha_1^x. \tag{2.16}$$

The general solution of the non-homogeneous equation (2.11) can be obtained‡ from the general solution of its associated homogeneous equation pro-

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* *Definition. If $p(x+k)=p(x)$, then $p(x)$ is a periodic function of period k.*

The difference equation

$$p(x+1) - p(x) = 0$$

is satisfied by every function whose value at $x+1$ is equal to its value at x, that is, by every periodic function of period *one*. The simplest analytic functions of period *one* are constants.

† Batchelder, *loc. cit.*, pp. 6–10.

‡ Batchelder, *loc. cit.*, pp. 13–16.

vided some particular solution $F(x)$ of (2.11) can be obtained. Then, as in the case of a linear differential equation, the most general solution of (2.11) is the sum of the particular solution and the general solution of the associated homogeneous equation, namely,

$$f(x) = F(x) + c_1f_1(x) + c_2f_2(x) + \cdots + c_nf_n(x).$$

After the general solution of the difference equation has been found, sufficient initial or boundary conditions will be needed to impose upon the general solution to obtain the particular solution which satisfies the proposed problem. In some problems there will be sufficient obvious initial conditions, and in others they will have to be obtained in any way possible.

2.2. The solution of a particular difference equation of order one. The method just discussed will be applied to the solution of the difference equation

$$f(x) = f(x - 1) + k. \tag{2.21}$$

In practice it is often more convenient to replace the symbol $f(x)$ by N_x; equation (2.21) then becomes

$$N_x = N_{x-1} + k. \tag{2.22}$$

When we substitute $N_x = \alpha^x$ in the associated homogeneous equation, the value of α is determined to be $\alpha = 1$ and the general solution is $N_x = c_1$. In order to find a particular solution of (2.22) let $N_x = Ax + B$. Then we must have

$$Ax + B \equiv Ax - A + B + k.$$

Then $A = k$ and since the value of B is arbitrary, let $B = 0$. The general solution of (2.22) is then

$$N_x = c_1 + kx.$$

This difference equation is established in sections (3.3)–(3.6) and in each case a known boundary condition is that $N_0 = 0$. When this condition is imposed it follows that $c_1 = 0$ and the solution is

$$N_x = kx. \tag{2.23}$$

The procedures outlined in this section will be illustrated and utilized in §§3, 4.

3. Familiar average value problems and their solutions by difference equations. The following average value problems can be found in texts by Coolidge,* Uspensky, and other authors. With the exception of a few,† this list is complete. Different writers have obtained the solutions of these problems; often the solutions have been obtained by series and sometimes by applications of propositions (3.1) and (3.2), which we shall not prove.

* Coolidge, J. L., An Introduction to Mathematical Probability, Oxford, 1925, p. 28.

† See Uspensky, *loc. cit.*, Chapter IX, for example.

(3.1) *Proposition 1. The mathematical expectation (the average value) of the sum of several variables is equal to the sum of their expectations.*

(3.2) *Proposition 2. The mathematical expectation of the product xy of two independent variables x and y is equal to the product of their expectations.*

We shall illustrate the application of difference equations to the solution of several of these same problems. The method is essentially the same for all of them and avoids the necessity of devising ingeneous methods for the problems individually.

(3.3) *Proposition 3. If the probability of success on an individual trial is p, the average number of trials necessary to produce r successes is r/p.*

Solution: Let $f(r)=N_r$ be the average number of trials required to produce exactly r successes. Then after the first trial the number of successes yet to be obtained will be either $(r-1)$ or r with probabilities p and q, respectively, and we have the following difference equation:

$$N_r = p(1 + N_{r-1}) + q(1 + N_r), \quad \text{or}$$
$$N_r = N_{r-1} + 1/p.$$

Since the average number of trials to produce zero successes is equal to zero, then $N_0=0$ and from (2.23)

$$N_r = r/p.$$

(3.4) *Proposition 4. The average of the number of successes in n trials with constant probability p is np.*

Solution: Let N_n be the average number of successes in n trials. After the first trial, the total number of successes will be either: (a) one more than the number of successes in the remaining $(n-1)$ trials provided the first trial was a success, or (b) just the number of successes in the remaining $(n-1)$ trials provided the first trial was a failure. Then,

$$N_n = p(1 + N_{n-1}) + qN_{n-1},$$

or,

$$N_n = N_{n-1} + p.$$

Again, $N_0=0$ and from (2.23) it follows that

$$N_n = np.$$

(3.5) *Proposition 5. The average value of the discrepancy $d=r-np$ from the most likely number of successes in n trials of an event (where p is the probability of success on each trial) is zero.*

Solution: The discrepancy $d=r-np$ in n trials is equal to the sum of the discrepancies d_1 and d_2 in the first n_1 and last n_2 trials as is evident from the following equation:

$$d = r - np = (r_1 + r_2) - (n_1 + n_2)p = d_1 + d_2.$$

Let N_x be the average value of the discrepancy in x trials. Then by arbitrarily selecting $n_1=1$ and $n_2=n-1$, it follows from (3.1) that

$$N_n = N_1 + N_{n-1}.$$

But

$$N_1 = p(1-p) + q(0-p) = 0.$$

Then

$$N_n = N_{n-1} \quad \text{and since} \quad N_1 = 0, \quad \text{the solution is}$$
$$N_n = 0.$$

(3.6) *Proposition 6. The average value of the square of the discrepancy is npq.*

Solution: The square of the discrepancy d in n trials is not equal to the sum of the squares of the discrepancies d_1, d_2, but it does satisfy the following relation;

(3.61) $$d^2 = d_1^2 + d_2^2 + 2d_1d_2.$$

Let N_x be the average value of the square of the discrepancy in x trials, and let $n_1=1$ and $n_2=n-1$. It follows from (3.61), (3.1) and (3.2) that;

$$N_n = N_1 + N_{n-1}.$$

Since

$$N_1 = p(1-p)^2 + q(-p)^2 = pq,$$

then

$$N_n = N_{n-1} + pq.$$

Again, since $N_0=0$, we have from (2.23) that

$$N_n = npq.$$

The solutions by difference equations of these two preceding theorems seem to be more direct than any of the orthodox methods which appear in books on statistics and probability.

4. Some new mean value problems. Although difference equations have not been used previously to solve the problems of the preceding section, solutions of the following ones become either impossible or very difficult without recourse to them.

(4.1) *Proposition 7.* Mr. A has x coins and Mr. B has y. They match coins until one player has won all the coins. What is the average number of tosses required to end the game?*

Solution: Let N_x be the average number of tosses required to end the game from the time Mr. A has x coins and Mr. B has y coins. Since after the first toss,

* This MONTHLY, Vol. 47, No. 5, pp. 324–5.

Mr. A will have either $x-1$ or $x+1$ coins with equal probabilities, the following difference equation can be deduced:

$$(4.11)\qquad \begin{aligned} N_x &= (1 + N_{x-1})/2 + (1 + N_{x+1})/2, \\ 2N_x &= N_{x-1} + N_{x+1} + 2. \end{aligned}$$

This difference equation of order 2, with the obvious boundary conditions $N_0 = N_{x+y} = 0$, are sufficient to solve the problem.

As outlined in (2.1), the general procedure to find particular solutions of the associated homogeneous equation is to make the substitution

$$N_x = \alpha^x.$$

Then α must satisfy the characteristic equation

$$2\alpha = 1 + \alpha^2,$$

and $\alpha = 1, 1$. Since the roots are equal it follows from (2.16) that two particular solutions are 1 and x, and the general solution of the associated homogeneous equation is

$$N_x = A + Bx.$$

A particular solution of the non-homogeneous equation can often be found by substituting in it a polynominal with unknown coefficients of appropriate degree. If we substitute $N_x = Rx^2 + Sx + T$ in (4.11), it follows that

$$N_x = -x^2$$

is a particular solution of (4.11). Then the general solution of (4.11) is

$$N_x = -x^2 + Bx + A.$$

The obvious boundary conditions, $N_0 = 0$ and $N_{x+y} = 0$ are sufficient to determine A and B, and the solution is

$$(4.12)\qquad N_x = -x^2 + (x + y)x = xy.$$

Besides illustrating a method of solving a difference equation which is frequently applicable, the result itself is rather unexpected and interesting. It is unexpected to find that, when $x = 1$, $y = 20$, the average number of tosses is as large as 20 when only *one* toss is required in about half the trials.

Although the notation N_x may suggest that player A is in some way singled out for attention, this is not the case, and N_x is the average number of tosses required to end the game by eliminating either player.

(4.2) *Proposition 8*. When the probabilities in the preceding proposition are equal to p and $q (p \neq q)$ on each individual trial, a solution becomes more difficult and the form more complicated. In this case, the difference equation which corresponds to (4.11) is the following:

$$(4.21)\qquad N_x = pN_{x+1} + qN_{x-1} + 1.$$

The characteristic equation of the associated homogeneous equation is

$$\alpha = p\alpha^2 + q,$$

and the roots are

$$\alpha = 1, \qquad q/p.$$

Equation (4.21) has a particular solution of the form $N_x = bx$ where b is determined by the equation $bx = pb(x+1) + qb(x-1) + 1$,
whence

$$b = (q - p)^{-1}.$$

Then the general solution of (4.21) is

$$N_x = R + S(q/p)^x + x(q - p)^{-1}.$$

Again it is obvious that $N_0 = N_{x+y} = 0$, and these conditions are sufficient to determine the following solution of the problem:

$$N_x = \frac{xq^{x+y} - (x + y)q^x p^y + yp^{x+y}}{(p - q)(p^{x+y} - q^{x+y})}. \tag{4.22}$$

Since *Proposition 8* reduces to *Proposition 7* when $p = q$, we should expect, that solution (4.22) would reduce to (4.12) when $p = q$. The function (4.22) is then indeterminate and of the form 0/0; it is not difficult to show that its limiting value as $p \to q \to \frac{1}{2}$ is $N_x = xy$.

(4.3) *Proposition 9*. In the matching game of (4.1), sometimes it will be player A who is eliminated and sometimes it will be player B. *What is the average number of tosses A_x required to eliminate player A?*

The a priori probabilities $\bar{p}_x$ and $\bar{p}_y$ that A (who has x coins) or B (who has y coins) will be eliminated* are:

$$\bar{p}_x = y/(x + y), \qquad \bar{p}_y = x/(x + y).$$

The following difference equation can be established on the basis of Bayes' theorem:

$$A_x = \frac{1}{2}\left[\frac{y+1}{x+y}(1 + A_{x-1}) + \frac{y-1}{x+y}(1 + A_{x+1})\right] \div \frac{y}{x+y}, \quad \text{or} \tag{4.31}$$

$$(y + 1)A_{x-1} = 2yA_x - (y - 1)A_{x+1} - 2y.$$

The characteristic equation of the associated homogeneous equation is

$$(y + 1) = 2y\alpha - (y - 1)\alpha^2;$$

and,

$$\alpha = 1, \qquad (y + 1)/(y - 1).$$

Then $A_x = C$ is a solution of the homogeneous equation, but $D(y+1)^x/(y-1)^x$

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* Whitworth, W. A., Choice and Chance, G. E. Stechert & Co., New York, Reprint of Fifth Edition, p. 231.

cannot be expected to be a solution, nor is it, since $(y+1)/(y-1)$ is not a constant. It can be shown, however, that x/y is a second solution.

Then the general solution of the homogeneous equation is

$$A_x = C + D\left(\frac{x}{y}\right)^x.$$

A particular solution of (4.31) can be obtained by substituting $A_x = Rx^2 + Sxy + Ty^2$, and is

$$A_x = x(x + 2y)/3.$$

The general solution of (4.31) is then

$$A_x = C + D\left(\frac{x}{y}\right)^x + x(x + 2y)/3.$$

One obvious boundary condition is $A_0 = 0$. In this problem, A_{x+y} is meaningless but $A_1 = 1$ when $x = y = 1$, since exactly one toss is then required to eliminate either A or B. These two boundary conditions are sufficient to determine that $C = D = 0$, and the solution of the problem is

$$A_x = x(x + 2y)/3.$$

5. Conclusion.* Difference equations can be applied to many problems in summation of series, determination of coefficients in a series solution of a differential equation, in replacing the method of mathematical induction, and in other ways. Application of them is often more direct than the standard procedures.

DISCUSSIONS AND NOTES

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The Department of Discussions and Notes is open to all forms of activity in college mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

REVERSION OF SERIES WITH APPLICATIONS

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It has been the experience of the author that men doing research work in applied mathematics are frequently baffled by a difficulty that could be resolved by the method of reversion of series. It is his conviction that every student who is trained for engineering or other scientific work should be taught

* It is my pleasure to express here an appreciation of the interest and suggestions of H. M. Gehman in connection with this paper.

this subject at some time in his courses in mathematics. Texts on algebra usually present this subject in connection with the theory of undetermined coefficients but it will be presented here as a topic in differential calculus. This approach has certain advantages over the algebraic method. Applications will be given to show how the method may be used.

Let $y=f(x)$ be such that $f(0)=0$ and $f(x)$ is expansible in a series in powers of x; then by Maclaurin's theorem.

$$y = px + qx^2/2! + rx^3/3! + sx^4/4! + tx^5/5! + \cdots, \qquad (p \neq 0), \tag{1}$$

in which p, q, r, s, and t stand respectively for the values of the first, second, third, fourth, and fifth derivatives of y with respect to x, each evaluated for $x=0$.

If this series be reverted we may write

$$x = Py + Qy^2/2! + Ry^3/3! + Sy^4/4! + Ty^5/5! + \cdots \tag{2}$$

in which P, Q, R, S, and T stand respectively for the first, second, third, fourth, and fifth derivatives of x with respect to y evaluated for $x=0$ for which value y is, also, zero.

From the relation $dy/dx=1/(dx/dy)$, we find by successive differentiations the following relations

$$\begin{aligned} &P = 1/p, \qquad Q = -q/p^3, \qquad R = (3q^2 - pr)/p^5. \\ &S = (10pqr - 15q^3 - p^2s)/p^7, \\ &T = (15p^2qs + 10p^2r^2 - 105pq^2r + 105q^4 - p^3t)/p^9. \end{aligned} \tag{3}$$

These relations determine the values of the coefficients of the powers of y in the reverted series (2).

One advantage of using the calculus approach is that $y=f(x)$ does not have to be expanded into a series in order to obtain the reverted series giving the value of x in a power series in y as it does in the method of undetermined coefficients.

APPLICATIONS

1. Revert the series

$$y = x + x^2 + x^3/2 + x^4/6 + x^5/24 + \cdots$$

and find the value of x that satisfies $10xe^x=1$. From the successive derivatives of y with respect to x, evaluated for $x=0$, we find the values $p=1$, $q=2$, $r=3$, $s=4$, $t=5$; and by means of equations (3) $P=1$, $Q=-2$, $R=9$, $S=-64$, $T=625$; so that by equation (2)

$$x = y - y^2 + 3y^3/2 - 8y^4/3 + 125y^5/24 \cdots$$

which is the required reverted series. Since the given series is the expansion of xe^x, the value of y that makes x satisfy $10xe^x=1$ is 0.1. This value put into the reverted series gives $x=0.09128$.

2. Compute the root of $x^3+10x^2+8x-120=0$ that lies between 2 and 3. In order to have the reverted series converge as rapidly as possible we reduce the roots by 3 rather than by 2 and get the equation $x^3+19x^2+95x+21=0$. If now we let

$$y = x^3 + 19x^2 + 95x$$

so that $y=0$ for $x=0$, the equation will be satisfied when $y=-21$. From the assumed value of y we find

$$p = 95, \qquad q = 38, \qquad r = 6, \qquad s = 0;$$

whence

$$P = 1/95, \qquad Q = -38/(95)^3, \qquad R = 3762/(95)^5, \qquad S = -606480/(95)^7.$$

These values give by equation (2)

$$x = y/95 - (1/5)(y/95)^2 + (33/475)(y/95)^3 - (14/475)(y/95)^4 \cdots$$

and for $y=-21$, $x=-0.23165$; so that the corresponding root of the original equation is $3-0.23165=2.76835$.

3. Solve the equation

$$x + e^x + \sin x + \tan x = 2.$$

Let

$$y = x + e^x + \sin x + \tan x - 1,$$

so that $y=0$ for $x=0$; then the given equation will be satisfied when $y=1$. From the assumed value of y we find

$$p = 4, \qquad q = 1, \qquad r = 2, \qquad s = 1, \qquad t = 18;$$

whence

$$P = 1/4, \quad Q = -1/64, \quad R = -5/1024, \quad S = 49/(4)^7, \quad T = -1007/(4)^9.$$

When these values are substituted into equation (2) we get

$$x = y/4 - 2(y/16)^2 - (10/3)(y/16)^3 + (49/6)(y/16)^4 - (1007/30)(y/16)^5 \cdots .$$

When y is set equal to 1 in this, we find $x=0.24147$ which is the required root.

THE ADJOINT OF A LINEAR DIFFERENTIAL EQUATION

L. H. McFarlan, University of Washington

Let the expression $L(y)$ be defined by

$$L(y) = a_0y^{(n)} + a_1y^{(n-1)} + \cdots + a_ny, \tag{1}$$

where the a's are functions of x. Associated with such an expression is another form,

$$M(z) = b_0z^{(n)} + b_1z^{(n-1)} + \cdots + b_nz, \tag{2}$$

known as the adjoint of $L(y)$. The b's are functions of x which are expressible in terms of the a's and certain of their derivatives, which are assumed to exist. The expressions $L(y)$ and $M(z)$ satisfy a relation of the form

$$zL(y) - yM(z) = dR/dx \tag{3}$$

where R is a bilinear form with coefficients expressible in terms of the a's and certain of their derivatives and is linear in each of the two sets of functions $(y, y', \cdots . y^{(n-1)})$ and $(z, z', \cdots, z^{(n-1)})$. The differential equation $M(z)=0$ is said to be the adjoint differential equation of $L(y)=0$ and the relation is a symmetrical one.

A simple method of obtaining the form $M(z)$ from $L(y)$ is given by using one of the forms of the Euler equation of the calculus of variations. The left hand member of the expression (3) will be designated by ψ. It will be assumed that the function $z(x)$ has been assigned and that the form $M(z)$ is determined but is, so far, unknown. Then ψ may be considered as a function of the set of variables $(x, y, y', \cdots, y^{(n)})$ and is an exact differential of the expression R. Hence if one were to consider the problem of minimizing the integral

$$I = \int_{x_0}^{x_1} \psi dx \tag{4}$$

the values of I would depend only upon the sets of initial values of $(x, y, y', \cdots, y^{(n-1)})$ at x_0 and x_1 and not on the particular form of the function $y(x)$. A necessary and sufficient condition for this to be the case is that the Euler differential equation for such an integral

$$\psi_y - \psi'_{y'} + \psi''_{y''} + \cdots + (-1)^n \psi^{(n)}_{y^{(n)}} = 0 \tag{5}$$

vanishes identically.

For the particular form of ψ as defined above, the expression (5) is seen to be

$$a_n z - M(z) - (a_{n-1}z)' + (a_{n-2}z)'' + \cdots + (-1)^n (a_0 z)^{(n)} = 0, \tag{6}$$

which immediately gives the expression

$$M(z) = (-1)^n (a_0 z)^{(n)} + (-1)^{n-1} (a_1 z)^{(n-1)} + \cdots + a_n z.$$

AN ACKNOWLEDGEMENT

H. V. Craig, University of Texas

With regard to my recent comment in the June–July MONTHLY concerning the development of the formulas for $D \sin x$ and $D \cos x$ by vector methods, I learn from Professor Louis Brand that he presented essentially the same observation in his book *Vectorial Mechanics* (1930), example 1, page 178. The basic facts are so well known and the argument so obvious I suspected that the method had been recorded by some of the earlier workers in quaternions and vector analysis. A time consuming, although obviously insufficient, search of the literature was fruitless. I regret that I failed to find Professor Brand's example.

CLUBS AND ALLIED ACTIVITIES

EDITED BY J. S. FRAME

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to J. S. Frame, Michigan State College, East Lansing, Michigan.

CLUB REPORTS 1943–44

Mathematics Club, Oberlin College

Semi-monthly meetings were held at which members of the club presented papers:

Why study mathematics, by Professor C. H. Yeaton

Geometric constructions with paper folding, by Rodney Hood

Geometric constructions with compass only, by Sarah-Lou Lotz

Maps, by Alan Chaney

The game of Nim, by Elizabeth Frazier

Simple diophantine equations, by Norman Weinstein

Vectors, by George Ritchie

Finite geometries, by Dora Sherman: a talk based on the article *Four finite geometries*, by H. F. MacNeish, this MONTHLY, Jan. 42, p. 15

What is a proof?, by Thomas Hargrove

Distances between points of a Cantor set, by Professor J. F. Randolph

A new distance function, by Mary Kinsman

Aerial navigation constructions, by Robert Graves

Repeating decimals, by Elizabeth MacKay

Algebraic equations, by Artha-Jean Burington

The major project of the club for the year was the writing and adoption of a constitution.

In addition to teas at the beginning of each meeting, the club sponsored a Christmas party, at which Professor Vance and Professor Rodabaugh debated the subject

Resolved that Pi(e) is more delectable than mathematical.

The club also sponsored a party honoring Mrs. C. H. Yeaton, who retired from the faculty this year. The last meeting of the year was a banquet in honor of Professor Mary E. Sinclair, head of the mathematics department, who also retires this year, after thirty-seven years of service to the college. The club is looking forward to its fiftieth anniversary next year.

Officers for the year were: President, Dora Sherman; Vice-President, George Ritchie; Secretary-Treasurer, Ruth Cheney; Program Chairman, Rodney Hood; Social Chairman, Mary Kinsman; Faculty Advisers, Professor Vance and Professor Rodabaugh. Officers elected for the year 1944–45 are: President, Ruth Cheney; Vice-President and Program Chairman, Rodney Hood; Secretary-Treasurer, Mary Kinsman; Social Chairman, Sarah-Lou Lotz; Publicity Chairman, Norman Weinstein; Faculty Adviser, Professor Vance.

Kappa Mu Epsilon, Mount St. Scholastica College

The theme for the year was *Mathematics as it is studied and taught in other countries*. Each monthly meeting was given over to a particular country or group of countries as follows:

FRANCE:

Sur les probabilités, LaPlace, translated by Ann Hughes
La géometrie, Descartes, treated by Patricia Warwick
Sophie Germain, by Mary Lou Maloney
Comparative study of mathematical textbooks, by Virginia Meyers.

SPANISH-SPEAKING COUNTRIES (open meeting with guest speakers):

Rural education and personal teaching experience in Mexico, by Mary Rachel Robleda from Mexico City
Student's view of Mexican education, by Fanny Mary Dorantes from Mexico City
Report on the nature of South American mathematical periodicals, by Mary Catherine Growney
Translation of a letter from Bernardo Baidaff, editor of *Boletin Mathematico*, by Katherine Zeller
Status of mathematics in the University of the Philippines, by Virginia Harrison.

BRITISH ISLES:

Brief history of the development of mathematics in England, by Margaret Mary Wolters.
London Royal Society, by Mary Frances Herrman
Newton-Leibnitz controversy, by Doris Beckman
Invariant twins—Sylvester and Cayley, by Mary Ann Wirtz.

GERMANY:

Life and work of Kepler, by Mary Jane Fox
Mathematical contributions of Gauss, by Mary Davis
Translation of a German work on the origin of counting, by Katherine Molloy.

The major social events were the Christmas party and the formal buffet supper following the initiation of new members. Noteworthy activities of the year were the U. S. War Stamp Drive and Mathematical Olympics. This latter was arranged in bazaar form featuring intellectual concessions such as *Mathematical Fortune-telling*; identifying pictures of mathematicians by clever clues; demonstration of Marchant Electric Calculator. The purpose of the festival was to determine the eligibility of prospective members. The program closed with the presentation of *Snow White and the Seven Dwarfs*.

For five consecutive years the honor of valedictorian has gone to a member of the Kansas Gamma Chapter—this year to Miss Virginia Meyers of Evanston, Illinois, chapter president, who also merited election to Kappa Gamma Pi, National Honorary Society. Kansas Gamma was also honored by the publication

in the spring issue of the *Pentagon*, of the research paper of one of its alumnae members, Muriel Thomas.

Officers-elect for the year 1944–45 are: President, Mary Lou Maloney; Vice-President, Mary Catherine Growney; Secretary, Katherine Zeller; Treasurer, Ann Hughes; Secretary Descartes and Faculty Sponsor, Sister Helen Sullivan, O.S.B.

Women's College Mathematics Club, University of Delaware

Meetings of the Club are always preceded by an informal tea, which is served to all who are interested in Mathematics on the campus. The five papers presented were:

Modern developments of some ancient ideas, by Dr. C. J. Rees

Foundations of mathematics, by Dr. Murray Mannos

An introduction to topology, by Dr. G. L. Walker

Must we use the base ten? by Dr. G. C. Webber

Opportunities for women with statistical training in market research, by Miss Elizabeth Watters. In this most interesting talk the opportunities were most vividly and clearly pointed out, and statistical procedure was explained and was used in illustrating work of this nature.

The Mathematics Club makes an annual contribution to the library of at least one book. The selection for the past year was *Aesthetic Measure*, by G. D. Birkhoff. Officers of the Club were: President, Agnes Wright; Vice-President, B. Catharine Carrick; Corresponding Secretary, Astrid Delitzsch; Treasurer, Jean Dukek; Chairman of Teas, Mary Edith Boyce.

Mathematics Club, Hunter College

Topics and speakers for the year 1943–44 were as follows:

Paths of heavenly bodies, by Mary Dolciani

Cryptography, by Marie Johnson

Origin of mathematical symbols, by Catherine Porcheddu

Excluded areas of algebraic curves, by Theo Gelbfarb

Asymptotes, by Melitta Lowy

The nine point circle, by Carol Podell

A few mathematical facts, by Augusta Schurrer and Gladys Heinlein

Pythagorean triplets, by Ilse Novak

Two theorems, by Professor Jewell Hughes Bushey

Pictorial topology, by Professor Saunders Mac Lane of Harvard University—an open meeting attended by about 175 members and guests.

The Club's two social events were the *Get Acquainted Tea* and the *Mad Hatter Tea Party*, financed by two successful cake sales. Officers for the year were: President, Theresa Danielson; Vice-President, Carol Podell; Secretary, Theo Gelbfarb; Treasurer, Mary Dolciani; Publicity Manager, Leila Rubashkin; Faculty Advisers, Mrs. Helen Kutman, Professor Marguerite Darkow.

Pi Mu Epsilon, Hunter College

The general topic considered for discussion by the Beta Chapter of *Pi Mu Epsilon* during the year 1943–44 was *Statistical theory and practice*. The fifteen papers presented were:

Probability functions, by Constance Heiden

Moment generating functions, by Mary Demitrack

Applications of the moment generating function to the normal probability function, by Lydia Blumenthal

The approach to normality of the Bernouilli and Poisson distributions, by Melitta Lowy

Semivariants, by Mildred Finger

Semivariants of the normal probability function, Poisson probability function, and the Type III probability function, by Gertrude Crosby

Semivariants and the distribution of sample means, by Marie Johnson

A method for the evaluation of a sum, by Gladys Rappaport

The moments of the sum or difference of two dependent variables, by Mae Perlstein

Some applications of the moments of the sum or difference of two variates, by Nelly Szabo

Fiducial probability and the Bernouilli distribution, by Regina Benbosset

Fiducial limits applied to the Poisson distribution, by Grace Leight

The theory of estimation, by Marie Lombardi

The method of maximum likelihood, by Miriam Josephson

The moments of a product, by Felicitas Reich.

Twenty-seven new members were initiated during the year. Prizes for best papers during the first and second semesters were copies of *Differential and Integral Calculus*, *v.*2, by Richard Courant, and *Mathematical Statistics*, by S. S. Wilkes, presented respectively to Marie Johnson and Lydia Blumenthal.

Officers for the current year were: Director, Dr. Leo Aroian; President, Pauline Palacek; Recording Secretary, Mary Dolciani; Corresponding Secretary, Gladys Heinlein; Treasurer, Melitta Lowy.

Mathematics Club, Case School of Applied Science

The following talks were presented at the meetings of the Club:

Supersonics, by J. T. Leiss

Some unintegrable integrals, by Professor P. E. Guenther: a most successful talk held at the home of Dean T. M. Focke

Mathematics of the electron, by Bernard Cohen

Geometry of the complex plane, by G. S. Springer

The density distribution of the stars, by Professor S. W. McCuskey of the Physics Department.

Officers of the Mathematics Club for the past season were: President, Bernard Cohen; Vice-President, George Springer; Secretary, Herbert Rutemiller. The Faculty Adviser was Professor Max Morris.

PROBLEMS AND SOLUTIONS

ELEMENTARY PROBLEMS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND H. S. M. COXETER

Send all communications concerning Elementary Problems and Solutions to H. S. M. Coxeter, 24 Strathearn Boulevard, Toronto 10, Canada.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 646. *Proposed by Orrin Frink, Jr., Xenia, Ohio*

Prove that any two conjugate planes through a secant of a sphere meet the sphere in orthogonal circles.

E 647. *Proposed by V. Thébault, San Sebastián, Spain*

Let (m_1, m_2, m_3, m_4) be the barycentric coordinates of a point G with respect to a regular tetrahedron $A_1A_2A_3A_4$ of edge a (so that G is the centroid of masses m_1, m_2, m_3, m_4 at A_1, A_2, A_3, A_4). Obtain an expression for the distance A_4G.

E 648. *Proposed by Mary L. Boas and R. P. Boas, Jr., Tufts College and Harvard University*

Show that, when $x=2\cos\pi/(n+1)$, the n-rowed determinant

$$\begin{vmatrix} x & 1 & 0 & 0\cdots 0 & 0 \\ 1 & x & 1 & 0\cdots 0 & 0 \\ 0 & 1 & x & 1\cdots 0 & 0 \\ 0 & 0 & 1 & x\cdots 0 & 0 \\ \cdot & \cdot & \cdot & \cdot\;\cdot\;\cdot\;\cdot & \cdot \\ 0 & 0 & 0 & 0\cdots x & 1 \\ 0 & 0 & 0 & 0\cdots 1 & x \end{vmatrix}$$

has the value zero.

E 649. *Proposed by L. A. Santaló, Rosario, Argentine Republic*

A set of parallel line segments will be called "linear" if all of them can be cut by one straight line. Show by an example that an infinite set of parallel segments in one plane may have the property that every subset of three is linear while the whole set is not linear. (The segments are "open": not including their end points.)

E 650. *Proposed by Lloyd Dulmage, University of Manitoba*

If we first arrange n letters in a row, in a definite order, and then arrange below these letters

(*a*) a second row containing the same n letters so that no letter is repeated in any column, the number of possible arrays is ${}_2K_n$;

(*b*) a second row containing p of the letters of the first row together with $n-p$ other letters, so that no letter is repeated in any column, then the number of arrays is ${}_2K_{n,p}$;

(*c*) a second row containing a definite p of the letters of the first row (and $n-p$ empty spaces) so that exactly some q of these p letters do not appear below any of the p chosen letters (but the remaining $p-q$ letters appear below some $p-q$ of the p chosen letters), no letter being repeated in any column, then the number of arrays is ${}_2K_{n,p,q}$;

(*d*) two rows (second and third) each containing the same n letters, so that no letter is repeated in any column, then the number of arrays is ${}_3K_n$.

Show that the functions so defined satisfy the following relations:

$$(1)\qquad {}_2K_{n,p} = \sum_{r=0}^{p} (-1)^r \binom{p}{r} (n-r)!,$$

$$(2)\qquad {}_2K_{n,p,q} = \binom{p}{q}\binom{n-p}{q}\, {}_2K_{p,p-q},$$

$$(3)\qquad {}_2K_n = \sum_{q=0}^{\min(p,n-p)} {}_2K_{n,p,q}\cdot {}_2K_{n-p,n-p-q}$$

(for each value of p from 0 to n),

$$(4)\qquad {}_3K_n = \sum_{p=0}^{n} \sum_{q=0}^{\min(p,n-p)} (-1)^p \binom{n}{p}\, {}_2K_{n,p,q}({}_2K_{n-p,n-p-q})^2.$$

SOLUTIONS

Arc and Area of an Epicycloid

E 600 [1943, 634]. *Proposed by J. H. Butchart, Grinnell College*

If the radii of the fixed and rolling circles are a and b respectively, the length of one arch of an epicycloid is $8(a+b)b/a$, and the area bounded by one arch and the fixed circle is

$$\pi(3a^2 + 8ab + 4b^2)b^2/a(a + 2b).$$

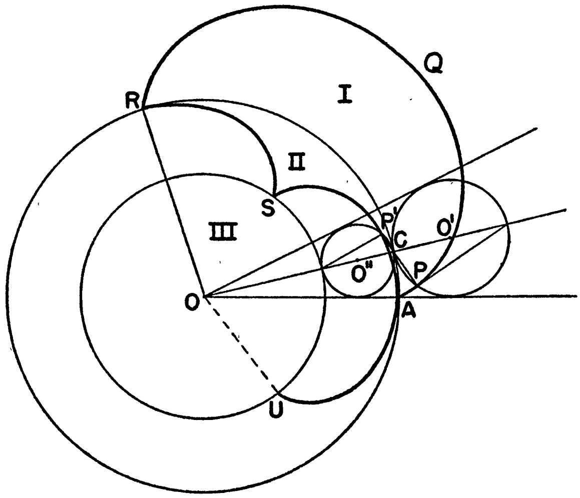
Corresponding formulas for the hypocycloid are obtained by changing the sign of b. Prove these results synthetically.

Solution by the Proposer. The formulas will be proved for the epicycloid $APQR$ generated by rolling a circle (O', b) on a circle (O, a). Modifications for the hypocycloid are obvious. Consider a circle $(O'', ab/(a+2b))$ tangent to (O') and homothetic to it with respect to the center O, and let it roll on an inner circle centered at O. As the point P, fixed on (O'), describes the epicycloid, the point P', fixed on (O''), describes the evolute. This evolute, an epicycloid whose arch subtends the angle $2\pi b/a$, is clearly similar to the original epicycloid, the

ratio of similitude being $a:(a+2b)$. Half an arch of the evolute, being equal to SQ, is

$$\frac{2ab}{a+2b}+2b=\frac{4ab+ab^2}{a+2b}\cdot$$

Hence the entire arch of the original epicycloid is $8(a+b)b/a$.



To determine the area denoted by I, we find three linear relations connecting I, II, and III. Since III is equivalent to $OUAS$, we get

$$(\mathrm{I}+\mathrm{II}+\mathrm{III})/\mathrm{III}=(a+2b)^2/a^2.$$

Also

$$\mathrm{II}+\mathrm{III}=\pi ab.$$

A third relation is found by noting that $P'C/CP=a/(a+2b)$. An element of area of the region II+I can consist, to within infinitesimals of higher order, of an infinitesimal triangle formed by neighboring tangents to the evolute with base on the epicycloid. The arc of the fixed circle separating II from I can be replaced by a line parallel to the base of this triangle, dividing the sides in the ratio $a:(a+2b)$. The difference which this change makes is an infinitesimal of higher order than the small triangle itself. From this argument we see that

$$\mathrm{II}/(\mathrm{II}+\mathrm{I})=a^2/4(a+b)^2.$$

These three relations can be written as follows:

$$a^2\mathrm{I}+a^2\mathrm{II}-(4ab+4b^2)\mathrm{III}=0,$$

$$\mathrm{II}+\mathrm{III}=\pi ab,$$

$$a^2\mathrm{I}-(3a^2+8ab+4b^2)\mathrm{II}=0.$$

Solving for I, we get the announced result.

Incidentally, the same diagram gives the radius of curvature very directly as

$$\frac{4(a+b)b}{a+2b}\sin\frac{\phi}{2},$$

where ϕ is the angle $OO'P$.

A Second Meigs Hall Problem

E 611 [1944, 162]. *Proposed by D. F. Barrow, University of Georgia*

Suppose n students be standing an examination in a row of seats with an aisle at either end. If they finish in random order find the average number of disturbances caused by students passing over one another. (If a student passes over 3 others, he causes 3 disturbances. Assume that they go out so as to cause the smallest number of disturbances. Cf. E 531 [1943, 202 and 513].)

Solution by C. M. Feller, Providence, R. I. Suppose that, at a certain moment, there are k students left. If $k=2j$, the expected value of the number of disturbances caused by the first leaving student is

$$\frac{2}{k}\{0+1+2+\cdots+(j-1)\}=\frac{j-1}{2}=\frac{k}{4}-\frac{1}{2};$$

if $k=2j+1$, the expected value becomes

$$\frac{2}{k}\left\{0+1+2+\cdots+(j-1)+\frac{j}{2}\right\}=\frac{j^2}{k}=\frac{k}{4}-\frac{1}{2}+\frac{1}{4k}\cdot$$

Summing over all k from 1 to n, we obtain the required average

$$\frac{n(n-3)}{8}+\frac{1}{4}\left(1+\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{m}\right),$$

where m is the greatest odd number not exceeding n.

Also solved by D. H. Browne, Irving Kaplansky, and the proposer. The proposer remarks that, in the case of several rows of students, we can find the average number of disturbances for each row and add the results. For instance, five rows of six students make an average of $13\frac{1}{6}$ disturbances, while six rows of five make only $9\frac{4}{5}$.

A Product of Four Consecutive Integers

E 612 [1944, 162]. *Proposed by V. Thébault, San Sebastián, Spain*

Show that $1110\cdot1111\cdot1112\cdot1113=1235431^2-1$ for any radix greater than 5. Generalize this result.

I. *Solution by E. D. Schell, Arlington, Virginia.* If 1_n is understood to mean a sequence of n digits 1 in the scale of r, let

$$R=r^n+r^{n-1}+\cdots+r+1\tfrac{1}{2}=1_{n+1}+\tfrac{1}{2}.$$

Then

$$R^2 = 123 \cdots n(n+1)n \cdots 321 + 1_{n+1} + \tfrac{1}{4}$$
$$= 123 \cdots n(n+2)(n+1)n \cdots 32 + \tfrac{1}{4}$$

and

$$1_n0 \cdot 1_{n+1} \cdot 1_n2 \cdot 1_n3 = (R - \tfrac{3}{2})(R - \tfrac{1}{2})(R + \tfrac{1}{2})(R + \tfrac{3}{2})$$
$$= R^4 - \tfrac{5}{2}R^2 + \tfrac{9}{16} = (R^2 - \tfrac{5}{4})^2 - 1$$
$$= \{123 \cdots n(n+2)(n+1)n \cdots 31\}^2 - 1$$

for any radix greater than $n+2$. The special case in the problem is when $n=3$.

II. *Solution by E. P. Starke, Rutgers University.* This is a special case of the algebraic identity

$$a(a+b)(a+2b)(a+3b) = (a^2 + 3ab + b^2)^2 - b^4,$$

in which we take $a = r^3 + r^2 + r$ (where r is the radix) and $b = 1$, so that

$$a^2 + 3ab + b^2 = r^6 + 2r^5 + 3r^4 + 5r^3 + 4r^2 + 3r + 1.$$

Since no coefficient exceeds 5, the identity is significant for any radix greater than 5. Other examples may be made up by taking other expressions for a; *e.g.*,

$$10100 \cdot 10101 \cdot 10102 \cdot 10103 = 102040301^2 - 1 \quad \text{for} \quad r > 4,$$
$$12 \cdot 13 \cdot 14 \cdot 15 = (17\epsilon)^2 - 1 \qquad \text{for} \quad r > \epsilon = 11.$$

Also solved by H. N. Carleton, W. R. Talbot, J. A. Tierney and the proposer.

A Criterion for Unequal Cevians

E 613 [1944, 162]. *Proposed by L. M. Kelly, U. S. Coast Guard Academy*

Can a triangle have two equal symmedians without being isosceles?

Solution by the Proposer.

LEMMA 1. *If the internal cevians AD and BE of the triangle ABC are such that $\angle BAD > \angle ABE$ and $\angle CAD > \angle CBE$, then $BE > AD$.*

Proof. Select a point F on CD so that $\angle FAD = \angle CBE$. Let BE meet AD at H, and AF at G. Since triangle FAB has a greater angle at A than at B, $FB > FA$. Since triangles FAD and FBG are similar, it follows that $AD < BG < BE$.

REMARK. If two cevians satisfy the conditions of this lemma, so do their isogonal conjugates.

LEMMA 2. *Two medians satisfy the conditions of Lemma* 1 *unless the triangle is isosceles.*

Proof. Let AD and BE be medians, and suppose that $BC > AC$. From the familiar formula for the median in terms of the sides it follows that $AD < BE$, whence $HB > HA$, and $\angle BAD > \angle ABE$. If CAD is obtuse the remaining condition is obviously satisfied. If not,

$$\frac{\sin \measuredangle CAD}{\sin C} = \frac{CD}{AD} = \frac{BC}{2AD}, \qquad \frac{\sin \measuredangle CBE}{\sin C} = \frac{CE}{BE} = \frac{AC}{2BE};$$

so

$$\frac{\sin \measuredangle CAD}{\sin \measuredangle CBE} = \frac{BC\cdot BE}{AC\cdot AD} > 1,$$

and

$$\measuredangle CAD > \measuredangle CBE,$$

as required.

From the two lemmas and the remark it follows that a triangle must be isosceles if two of its symmedians are equal. The famous Steiner-Lehmus theorem is another simple consequence of Lemma 1. Other special cases are of interest.

Also solved (by direct computation of the symmedians) by W. B. Carver, Howard Eves, R. A. Johnson, and Alan Wayne.

A Reducible Polynomial

E 614 [1944, 162]. *Proposed by Joseph Rosenbaum, Bloomfield, Conn.*

Prove that the polynomial $(x+y)^n-x^n-y^n$ is divisible by x^2+xy+y^2 when $n\equiv 5 \pmod 6$, and by $(x^2+xy+y^2)^2$ when $n\equiv 1 \pmod 6$.

Solution by C. D. Olds, Purdue University. Write

$$Q = (x+y)^n - x^n - y^n$$

and

$$p = x^2 + xy + y^2 = (x - \omega y)(x - \omega^2 y),$$

the three cube roots of unity being 1, ω, ω^2. From elementary algebra we know that $1+\omega^n+\omega^{2n}=0$ provided n is not a multiple of 3. Now, p divides Q if Q vanishes when $x=\omega y$ (for then it will vanish again for the conjugate value $x=\omega^2 y$); and p^2 divides Q if $\partial Q/\partial x$ also vanishes when $x=\omega y$.

When $x=\omega y$, we have

$$Q = y^n\{(1+\omega)^n - \omega^n - 1\} = y^n\{(-\omega^2)^n - \omega^n - 1\}$$

and

$$\begin{aligned}\partial Q/\partial x &= n(x+y)^{n-1} - nx^{n-1} = ny^{n-1}\{(1+\omega)^{n-1} - \omega^{n-1}\}\\ &= ny^{n-1}\{(-\omega^2)^{n-1} - \omega^{n-1}\}.\end{aligned}$$

The former vanishes when n is odd and not a multiple of 3; the latter when $n-1$ is even and a multiple of 3, *i.e.*, when $n\equiv 1 \pmod 6$.

This theorem was first proved by Cauchy in 1840 (*Oeuvres*, vol. 4, pp. 499–504). When n is a prime, it can be used to discuss the case $x^p+y^p+z^p=0$ of Fermat's Last Theorem. Numerous references to this problem, together with many applications and generalizations, will be found in Dickson, *History of the Theory of Numbers*, vol. 2, chap. 26.

Also solved by Murray Barbour, F. M. Carpenter and W. W. Gandy (together), Irving Kaplansky, E. D. Schell, Murray Spiegel, E. P. Starke, and the proposer.

The Generalized Water-fetching Puzzle

E 451 [1942, 125–127]. *Proposed by W. E. Buker, Pittsburgh Public Schools*

There are three containers, having capacities of a, b, c quarts, where $a>b>c$ (positive integers). With the largest container full and the others empty, it is desired to divide the liquid into two equal portions, using these containers and no others. For what values of a, b, c is a solution possible?

Partial solution by William Scott, Fort Monroe, Va. In the editorial note to the previous solution it was shown that a sufficient condition for the occurrence of a solution with $a<b+c-2$ is that $\frac{1}{2}a\equiv c \pmod{r}$, where $r\equiv \pm c \pmod{b-c}$ and $0<r<\frac{1}{2}(b-c)-1$. The editor asked for an example to show that this sufficient condition is not also necessary. Here is such an example:

$$a = 18, \qquad b = 17, \qquad c = 4.$$

The sequence of s's is 0, 17, 13, 9, $\cdots$. But $r=4$, and $\frac{1}{2}a$ is not a multiple of 4.

The Three Ladders

E 616 [1944, 231]. *Proposed by J. S. Cromelin, Chicago*

A 60′ ladder and a 77′ ladder rest with their lower ends against a building on the east side of a street, their upper ends against the opposite building on the west wide. A third ladder rests with its lower end against the building on the west side, its upper end against the building on the east side. It crosses the first ladder at a height of 17′, the second at a height of 19′. What is the length of the third ladder, and how wide is the street (to the nearest inch)?

Solution by Murray Barbour, Michigan State College. Let the width of the street be x feet, and let the ladders cross at horizontal distances a and b feet from the west side of the street. Then, by similar right triangles,

$$\frac{x-a}{17} = \frac{x}{\sqrt{60^2-x^2}}, \qquad \frac{x-b}{19} = \frac{x}{\sqrt{77^2-x^2}}, \qquad \frac{a}{17} = \frac{b}{19}.$$

Solving the first two equations for a, b, and substituting in the third, we obtain

$$(60^2 - x^2)^{-1/2} - (77^2 - x^2)^{-1/2} = 2/323.$$

There is no advantage in rationalizing this equation, for the resultant equation would be quartic in x^2. By successive interpolation we soon find $x=36.235\cdots$. Thus, to the nearest inch, the width is 36′ 3″.

The length of the third ladder (in feet) can be expressed as either

$$\frac{x\sqrt{17^2+a^2}}{a} \qquad \text{or} \qquad \frac{x\sqrt{19^2+b^2}}{b}.$$

Using the above value for x, either formula gives the length 44′ 10″.

Also solved by D. H. Browne, H. N. Carleton, William Douglas, Howard Eves, L. S. Shively, and E. C. Varnum.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known text-books or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4141. *Proposed by H. S. M. Coxeter, University of Toronto*

Prove synthetically that the four lines of 4018 [1943, 125] are (in general) four tangents to a twisted cubic.

4142. *Proposed by G. Pólya, Stanford University*

Find a sequence of real numbers $a_1, a_2, a_3, \cdots$ so that $\sum_1^\infty a_n$ converges, $\sum_1^\infty a_n^3$ diverges, $\sum_1^\infty a_n^5$ converges. More generally, let C be an arbitrarily given (finite or infinite) class of positive integers. There exists a sequence of real numbers $a_1, a_2, \cdots, a_n, \cdots$ adapted to C so that, for $l=1, 2, 3, \cdots$, the series

$$a_1^{2l-1} + a_2^{2l-1} + \cdots + a_n^{2l-1} + \cdots$$

converges or diverges according as l does or does not belong to C.

4143. *Proposed by P. Erdös, Purdue University*

Let $p_1 < p_2 < \cdots < p_n < \cdots$ be the consecutive primes. Prove that

$$\frac{p_n!}{p_n(p_n+1)\cdots(p_{n+1}-1)}$$

is always an integer except when $p_n = 3$.

4144. *Proposed by V. Thébault, San Sebastián, Spain*

For what base is a number of eight digits of the form *ababcdcd* the square of a four digit number *mnmn*, given that *a b c d m n* is a permutation of six consecutive digits?

SOLUTIONS

Affine Geometry

4087 [1943, 391]. *Proposed by Betty Dick and B. M. Stewart, Michigan State College*

Let P be a plane polygon with vertices $A_1, A_2, \cdots, A_n$, and consider $A_{n+k} = A_k$. Determine points $B_1, B_2, \cdots, B_n$ such that B_i is on the side A_iA_{i+1} with $A_iA_{i+1} = k \cdot A_iB_i$, where k is a fixed real number. Let the lines A_iB_{i+1} and

$A_{i+1}B_{i+2}$ intersect in the point C_i, thus determining the polygon Q with the vertices $C_1, C_2, \cdots, C_n$. Let $R(k)$ be the ratio of the area of Q to the area of P, wherein not to restrict the type of polygon we use Klein's definition of area, *Elementary Mathematics from an Advanced Standpoint, Geometry*, p. 10. (1) Show that k and $R(k)$ are invariants under affine transformations. (2) As a corollary to (1) show that for any triangle we have $R(k)=(k-2)^2/(k^2-k+1)$. (3) For any parallelogram we have $R(k)=(k-1)^2/(k^2+1)$. (4) Show that $R(k)$ does not have the same value for all quadrilaterals.

This problem is a generalization of the so-called Problem of Steinhaus which asserts for any triangle $R(3)=1/7$. An early reference, suggestive of this problem, is problem 276 in *Mathesis*.

Solution by Howard Eves, Syracuse University. (1) This part follows from the facts that the ratio of two segments on a line is an absolute invariant, and the area of a polygon is a relative invariant, with respect to the group of affine transformations.

(2) Since any three non-collinear finite points may be carried affinely into any other three non-collinear finite points it follows that any triangle is affinely equivalent to an equilateral triangle. But it is easy to show that, for an equilateral triangle,

$$R(k) = (k - 2)^2/(k^2 - k + 1).$$

By (1) this relation is then true for all triangles.

(3) As in (2), and since parallel lines carry affinely into parallel lines, it follows that any parallelogram is affinely equivalent to a square. It is easy to show that, for a square,

$$R(k) = (k - 1)^2/(k^2 + 1).$$

By (1) this relation is then true for all parallelograms.

(4) To show that $R(k)$ does not have the same value for all quadrilaterals, consider a quadrilateral, S, three of whose vertices are the vertices of an equilateral triangle, and whose fourth vertex, A_4, is near one of the other vertices, say A_3. Then it is readily seen from a figure that, as $A_4 \to A_3$, $R(2) \to 1/6$. But for a parallelogram $R(2)=1/5$.

Solved also by J. H. Butchart and the proposers.

Editorial Note. The solution by the proposers is similar to the above. For (2) the triangle $(0, 0)$, $(k, 0)$, $(0, k)$ is used; for (3) the square with three vertices as above and the fourth (k, k) is used; and for (4) a trapezoid with one base twice the length of the other gives $R(2)=156/775$, whereas for a parallelogram $R(2)=1/5$. For (2) Butchart used vectors for a triangle $A_1A_2A_3$ and eliminated the vector for A_3 from the vector equations for B_2 and B_3 and thus obtained $A_2C_1/C_1B_3=k/(k-1)^2$, $A_2C_2/C_2B_3=k(k-1)$. Additional combinations give the desired result. For (3) the square with side k is used, then Q is a square with side $C_1C_2=k(k-1)/(k^2+1)^{1/2}$ and this gives the desired result for $R(k)$.

Two Special Radical Centers

4088 [1943, 392]. *Proposed by V. Thébault, San Sebastián, Spain*

If three circles $A(\rho)$, $B(\rho)$, $C(\rho)$ with the same radius ρ are described in the triangle ABC, and then the circles with centers A, B, C orthogonal respectively to $C(\rho)$, $A(\rho)$, $B(\rho)$; these three circles have the same radical center M_1 whatever the value of ρ. The same is true of three circles with centers A, B, C orthogonal respectively to $B(\rho)$, $C(\rho)$, $A(\rho)$, the radical center being M_2. If O is the circumcenter of ABC, show also that: (1) The triangles ABC and OM_1M_2 have the same centroid; (2) The straight line M_1M_2 is perpendicular to the straight line through the centroid and the Lemoine point; (3) if M_1' and M_2' are the symmetrics of M_1 and M_2 with respect to O, then M_1M_1' and M_2M_2' are parallel to the Euler line.

Solution by Howard Eves, Syracuse University. We shall use rectangular coordinates with the circumcenter O as origin, and shall vectorize our algebra. Let a, b, c be the sides of the triangle, G the centroid. H the orthocenter, and L the Lemoine point. Denote the coordinates of A, B, C, M_1, M_2, G, H, L by the vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, $\mathbf{m}_1$, $\mathbf{m}_2$, $\mathbf{g}$, $\mathbf{h}$, $\mathbf{l}$ respectively.

The equation of the radical axis through M_1 and perpendicular to BC is

$$(\mathbf{x}-\mathbf{w})\cdot(\mathbf{x}-\mathbf{w})-(\mathbf{x}-\mathbf{v})\cdot(\mathbf{x}-\mathbf{v})=(a^2-\rho^2)-(c^2-\rho^2)=a^2-c^2,$$

or, since $\mathbf{w}\cdot\mathbf{w}=\mathbf{v}\cdot\mathbf{v}$

$$2\mathbf{x}\cdot(\mathbf{v}-\mathbf{w})=a^2-c^2. \tag{1}$$

Similarly, the equation of the radical axis through M_1 and perpendicular to CA is

$$2\mathbf{x}\cdot(\mathbf{w}-\mathbf{u})=b^2-a^2. \tag{2}$$

We shall solve this pair of equations by use of the matrix operator ϵ defined as follows. The vector $\mathbf{u}$ has the components u_1, u_2 and may be written as the matrix (u_1, u_2), then

$$\epsilon=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},\qquad \mathbf{u}\epsilon=(u_1, u_2)\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}=(-u_2, u_1)$$

and hence $\mathbf{u}$ and $\mathbf{u}\epsilon$ are perpendicular and equal in length. We have also

$$(\mathbf{u}\epsilon)\cdot(\mathbf{u}\epsilon)=\mathbf{u}\cdot\mathbf{u};\qquad (\mathbf{u}\epsilon)\cdot\mathbf{u}=0;\qquad (\mathbf{u}\epsilon)\cdot\mathbf{v}=\begin{vmatrix}u_1 & u_2\\ v_1 & v_2\end{vmatrix}=|\mathbf{u},\mathbf{v}|.$$

The solution of the pair of equations $\mathbf{x}\cdot\mathbf{u}=r$, $\mathbf{x}\cdot\mathbf{v}=s$ is

$$\mathbf{x}=\frac{s\mathbf{u}-r\mathbf{v}}{|\mathbf{u},\mathbf{v}|}\epsilon$$

which is easily verified by substitution.

Thus the solution of (1) and (2) is

$$(3) \qquad \mathbf{m}_1 = \frac{\mathbf{u}(a^2 - c^2) + \mathbf{v}(b^2 - a^2) + \mathbf{w}(c^2 - b^2)}{4\Delta}\epsilon,$$

where Δ is the area of ABC. Similarly, we find

$$(4) \qquad \mathbf{m}_2 = \frac{\mathbf{u}(b^2 - a^2) + \mathbf{v}(c^2 - b^2) + \mathbf{w}(a^2 - c^2)}{4\Delta}\epsilon.$$

These results are independent of ρ.

Now it is geometrically evident that H lies on the radical axis of the circles $B(c)$ and $C(b)$. Thus the equation of the altitude on side BC is

$$(\mathbf{x} - \mathbf{w})\cdot(\mathbf{x} - \mathbf{w}) - (\mathbf{x} - \mathbf{v})\cdot(\mathbf{x} - \mathbf{v}) = b^2 - c^2, \quad \text{or} \quad 2\mathbf{x}\cdot(\mathbf{v} - \mathbf{w}) = b^2 - c^2.$$

Similarly the equation of the altitude on side CA is

$$2\mathbf{x}\cdot(\mathbf{w} - \mathbf{u}) = c^2 - a^2.$$

Solving simultaneously we get

$$\mathbf{h} = \frac{\mathbf{u}(b^2 - c^2) + \mathbf{v}(c^2 - a^2) + \mathbf{w}(a^2 - b^2)}{4\Delta}\epsilon.$$

Also we know that

$$\mathbf{l} = \frac{a^2\mathbf{u} + b^2\mathbf{v} + c^2\mathbf{w}}{a^2 + b^2 + c^2}, \qquad \mathbf{g} = \frac{\mathbf{u} + \mathbf{v} + \mathbf{w}}{3}.$$

From (3), (4), (5) we see that $\mathbf{m}_1 + \mathbf{m}_2 = \mathbf{h}$. But, since O, G, H are collinear and $OH = 3OG$, it follows that $\mathbf{h} = 3\mathbf{g}$. Hence $\mathbf{m}_1 + \mathbf{m}_2 = 3\mathbf{g}$ and G is the centroid of M_1M_2O.

We also see that the vectors $\mathbf{m}_1 - \mathbf{m}_2$ and $(\mathbf{l} - \mathbf{g})\epsilon$ are proportional, whence it follows that M_1M_2 is perpendicular to LG. Finally, since M_1M_2' and M_2M_1' are parallel to GO, it follows that each is parallel to the Euler line. Incidentally the nine point center is the midpoint of M_1M_2, and M_1HM_2O is a parallelogram.

Solved also by the proposer.

Editorial Note. The proposer used the theorem on the difference of the powers of the vertex A with respect to the circles with centers at B and C orthogonal to $A(\rho)$ and $B(\rho)$ respectively, and considered the two cases of radical axes. For the remaining part the triangle $A_1B_1C_1$ was introduced formed by the perpendiculars to BC at C, to CA at A, and to AB at B, so that O is the Lemoine point for $A_1B_1C_1$. The barycentric coordinates of M_1, M_2, and their associates with respect to $A_1B_1C_1$ and $A_2B_2C_2$, the symmetric of $A_1B_1C_1$ with respect to O, were used to complete the proof.

In the solution above the vector equations (1) and (2) suggest the following form for their solution

$$\mathbf{m}_i = A_i(\mathbf{v} + \mathbf{w}) + B_i(\mathbf{w} + \mathbf{u}), \qquad i = 1, 2,$$

where the scalars A_i, B_i are to be determined by the pair of equations which $\mathbf{m}_i$ satisfy. An examination of the figure enables us to write down immediately six scalar products, the remaining four of which are cyclic permutations of

$$(\mathbf{w}+\mathbf{u})\cdot(\mathbf{u}-\mathbf{v}) = (\mathbf{w}+\mathbf{u})\cdot(\mathbf{w}-\mathbf{v}) = ac\cos\beta,$$

where α, β, γ are the angles opposite the sides a, b, c. We then find that

$$2A_1bc\cos\alpha = b^2-a^2, \qquad 2B_1ca\cos\beta = c^2-a^2$$
$$2A_2bc\cos\alpha = c^2-b^2, \qquad 2B_2ca\cos\beta = a^2-b^2.$$

We have also $(\mathbf{h}-\mathbf{u})\cdot(\mathbf{v}-\mathbf{w})=(\mathbf{h}-\mathbf{v})\cdot(\mathbf{w}-\mathbf{u})=0$, and these give after trigonometric reductions

$$2\mathbf{h}\cdot(\mathbf{v}-\mathbf{w}) = b^2-c^2, \qquad 2\mathbf{h}\cdot(\mathbf{w}-\mathbf{u}) = c^2-a^2.$$

Since $\mathbf{m}_1+\mathbf{m}_2$ satisfy the same two equations, we have $\mathbf{h}=\mathbf{m}_1+\mathbf{m}_2$.

Finally we have after reductions

$$2abc\cos\alpha\cos\beta(\mathbf{m}_1-\mathbf{m}_2) = a\cos\beta(2b^2-c^2-a^2)(\mathbf{v}+\mathbf{w}) - b\cos\alpha(2a^2-b^2-c^2)(\mathbf{u}+\mathbf{w}),$$
$$3(a^2+b^2+c^2)(\mathbf{1}-\mathbf{g}) = (2b^2-c^2-a^2)(\mathbf{v}-\mathbf{w}) + (2a^2-b^2-c^2)(\mathbf{u}-\mathbf{w});$$

and it is easily verified that they give $(\mathbf{m}_1-\mathbf{m}_2)\cdot(\mathbf{1}-\mathbf{g})=0$. It should be noted in the above that, if the triangle has a right angle, we may assume that it is γ; also the case of the equilateral triangle is exceptional. The three vectors satisfy the relation

$$a\cos\alpha\mathbf{u} + b\cos\beta\mathbf{v} + c\cos\gamma\mathbf{w} = 0.$$

We may also use as an operator the unit vector $\mathbf{k}$ normal to the horizontal plane of the figure and directed vertically upward. The vector $\mathbf{k}\times\mathbf{u}$ is the vector $\mathbf{u}$ turned through the positive right angle without altering its length; $\mathbf{u}$, $\mathbf{k}\times\mathbf{u}$, $\mathbf{k}$ forming a rectangular system in positive order of rotation. We then have $[\mathbf{k}\times(\mathbf{v}-\mathbf{w})]\cdot(\mathbf{w}-\mathbf{u})=[\mathbf{k}\times(\mathbf{u}-\mathbf{w})]\cdot(\mathbf{v}-\mathbf{w})=2\Delta$; and the solution of the system (1), (2) is

$$4\Delta\mathbf{m}_1 = \mathbf{k}\times[(b^2-a^2)(\mathbf{v}-\mathbf{w})+(c^2-a^2)(\mathbf{w}-\mathbf{u})]$$

and Eves' proof may be applied with this operator.

The Monge Point

4090 [1943, 456]. *Proposed by N. A. Court, University of Oklahoma*

The polar plane, with respect to the "quasipolar" sphere of a tetrahedron (T), of a point on the circumsphere of (T) trisects the segment joining the Monge point of (T) to the diametric opposite of the given point on the circumsphere.

Note. The "quasipolar" sphere (Q) of a tetrahedron (T) has for its center the Monge point M of (T) and for the square of its radius one third of the power of M for the circumsphere (O) of (T). See *Bull. Am. Math. So.* vol. 48, 1942, p. 583.

Solution by the Proposer. Let L be the given point on the sphere (O), and P the second point of intersection of (O) with the line LM. The traces L' of the polar plane λ of L for the quasi-polar sphere (Q) on LMP is the inverse of L for (Q), hence $ML\cdot ML'$ is equal to the square of the radius of (Q). On the other hand $ML\cdot MP$ is the power of M for the sphere (O), hence, taking into account the definition of (Q), we have

$$ML\cdot MP = 3ML\cdot ML', \quad \text{or} \quad MP = 3ML'.$$

Let L'' be the diametric opposite of L on (O). The plane λ is perpendicular to the line LML', hence λ cuts the plane LPL'' along a line perpendicular to LML' at L' and therefore parallel to the line PL''. Hence the proposition.

Remark. This is an extension to the general tetrahedron of a known property of the orthocentric tetrahedron. See the proposer's *Modern Pure Solid Geometry*, pp. 266, 267, art. 814.

Editorial Note. This theorem can be extended to Euclidean space of n dimensions, $n\geqq 2$, where we suppose that the Monge point M of the simplex S is not on its circumsphere (C). Taking M as the origin of vectors, we have the following equations

$$(M):\qquad \mathbf{x}^2 - m = 0; \qquad\qquad (C):\qquad \mathbf{x}^2 - 2\mathbf{c}\cdot\mathbf{x} + nm = 0;$$

where $\mathbf{c}$ is the vector of C, the circumcenter of S, $m\neq 0$ by the above assumption, and (M) is the "quasi polar" sphere of S. The above equations are given in the solution of 4049 [1943, 578]. Let P_1 on (C) have the vector $\mathbf{x}_1$, then P_1', its diametric opposite on (C), has the vector $2\mathbf{c}-\mathbf{x}_1$, and the polar plane of P_1 with respect to (M) has the equation $\mathbf{x}_1\cdot\mathbf{x}-m=0$. Let this polar plane and the straight line MP_1' meet in Q; then Q has the vector $t(2\mathbf{c}-x_1)$, where $t(2\mathbf{c}\cdot\mathbf{x}_1-\mathbf{x}_1^2)-m=0$, and from the equation of (C) we get $(tn-1)m=0$. From this follows that $MQ=MP_1'/n$, which is the desired result. It is obvious from the equation of (C) that the power of M with respect to (C) is nm.

Three Related Series

4091 [1943, 457]. *Proposed by Morgan Ward, California Institute of Technology*

Given the three series

$$z - \frac{z^5}{2\cdot4\cdot5} + \frac{z^9}{2\cdot4\cdot6\cdot8\cdot9} - \frac{z^{13}}{2\cdot4\cdot6\cdot8\cdot10\cdot12\cdot13} + \cdots,$$

$$\frac{z^3}{2\cdot3} - \frac{z^7}{2\cdot4\cdot6\cdot7} + \frac{z^{11}}{2\cdot4\cdot6\cdot8\cdot10\cdot11} - \frac{z^{15}}{2\cdot4\cdot6\cdot8\cdot10\cdot12\cdot14\cdot15} + \cdots,$$

$$\frac{z^2}{1\cdot2} - \frac{z^6}{1\cdot3\cdot5\cdot6} + \frac{z^{10}}{1\cdot3\cdot5\cdot7\cdot9\cdot10} - \frac{z^{14}}{1\cdot3\cdot5\cdot7\cdot9\cdot11\cdot13\cdot14} + \cdots,$$

prove that the sum of the squares of the first two series is double the third series.

Solution by B. P. Gill, City College, N. Y. Denote the three series in the order given by f, g, h. Then term by term differentiation, which is clearly permissible, yields, if accents denote derivatives with respect to z, (1) $f''=-zg'$, $g''=zf'$. From these follow (2) $f'f''+g'g''=0$, whence $f'^2+g'^2=C$. Putting $z=0$ gives $C=1$, so that (3) $f'^2+g'^2=1$. Equations (2) and (3) also follow from the fact, which we shall not need, that $f'=\cos(z^2/2)$, $g'=\sin(z^2/2)$. Now differentiation of (1) and substitution show that f and g are both solutions of the differential equation $L[w]=zw'''-w''+z^3w'=0$, that is, $L[f]=L[g]=0$.

Applying a similar argument to the two functions h and

$$k\equiv 1-\frac{z^4}{1\cdot 3\cdot 4}+\frac{z^8}{1\cdot 3\cdot 5\cdot 7\cdot 8}-\frac{z^{12}}{1\cdot 3\cdot 5\cdot 7\cdot 9\cdot 11\cdot 12}+\cdots,$$

we get $h''=1+zk'$, $k''=-zh'$, $L[h]=-1$. But if we write $y=f^2+g^2$, straightforward computation using (2) and (3) gives $L[y]=2(fL[f]+gL[g]-1)=-2$. Hence $2h$ and y are both solutions of $L[w]=-2$, while also $y(0)=2h(0)=0$, $y'(0)=2h'(0)=0$, $y''(0)=2h''(0)=2$. Hence, because of the uniqueness of the solution of the differential equation under these last initial conditions, $y=2h$, which is what was to be proved.

Solved also by A. B. Farnell, F. A. Ficken, Elijah Swift, H. E. Vaughan and the proposer.

Reciprocal Simplexes

4079 [1943, 264]. *Proposed by S. Beatty, Univ. of Toronto*

If two simplexes, in projective n-space, are reciprocal with respect to a quadric, so that each vertex of one is the pole of a bounding hyperplane of the other, then the $n+1$ lines joining corresponding vertices are associated,* in the sense that every $(n-2)$-space which meets n of them meets the remaining one also. Reciprocally, the $n+1$ of the $(n-2)$-spaces of intersection of corresponding hyperplanes are such that every line which meets n of them meets the remaining one also.

Solution by J. A. Todd, Cambridge University, and H. S. M. Coxeter, University of Toronto. Let one simplex be the simplex of reference $X_1X_2\cdots X_{n+1}$, and $\sum\sum A_{i,j}X_iX_j=0$ be the tangential equation of the quadric, so that the other simplex is $Y_1Y_2\cdots Y_{n+1}$ where (*e.g.*) Y_1, being the pole of $x_1=0$, is $(A_{1,1}, A_{1,2},\cdots,A_{1,n+1})$. A variable point of the line X_1Y_1 is $(\lambda_1, A_{1,2},\cdots,A_{1,n+1})$, where λ_1 is a parameter. The condition for n such points, one on each of the lines X_1Y_1, X_2Y_2, $\cdots$, X_nY_n, to lie in an $(n-2)$-space, is the same as the condition for the first $n-1$ points and the whole line X_nY_n to lie in a hyperplane, namely

* For example, when $n=3$, the four lines are generators of a regulus. The case where $n=2$ is Hesse's theorem.

$$\begin{vmatrix} \lambda_1 & A_{1,2} & \cdots A_{1,n-1} & A_{1,n} & A_{1,n+1} \\ A_{2,1} & \lambda_2 & \cdots A_{2,n-1} & A_{2,n} & A_{2,n+1} \\ \cdot & \cdot & \cdot\;\cdot\;\cdot & \cdot & \cdot \\ A_{n-1,1} & A_{n-1,2} & \cdots \lambda_{n-1} & A_{n-1,n} & A_{n-1,n+1} \\ A_{n,1} & A_{n,2} & \cdots A_{n,n-1} & A_{n,n} & A_{n,n+1} \\ 0 & 0 & \cdots 0 & 1 & 0 \end{vmatrix} = 0, \quad \text{or}$$

$$\begin{vmatrix} \lambda_1 & A_{1,2} & \cdots A_{1,n-1} & A_{1,n+1} \\ A_{2,1} & \lambda_2 & \cdots A_{2,n-1} & A_{2,n+1} \\ \cdot & \cdot & \cdot\;\cdot\;\cdot & \cdot \\ A_{n-1,1} & A_{n-1,2} & \cdots \lambda_{n-1} & A_{n-1,n+1} \\ A_{n,1} & A_{n,2} & \cdots A_{n,n-1} & A_{n,n+1} \end{vmatrix} = 0.$$

This is a relation between the $n-1$ parameters $\lambda_1, \lambda_2, \cdots, \lambda_{n-1}$ (any $n-2$ of which may be chosen arbitrarily). But, since $A_{i,j}=A_{j,i}$, the relation is symmetrical between the suffixes n and $n+1$. Thus every $(n-2)$-space which meets the lines $X_1Y_1, X_2Y_2, \cdots, X_nY_n$ meets $X_{n+1}Y_{n+1}$ too.

Editorial Note. After the appearance of the problem in print, Coxeter reported that it had been pointed out to him that the three dimensional case is treated in a pure geometrical manner in H. F. Baker's *Principles of Geometry*, vol. 3, p. 41, Ex. 7. He stated also that Baker's proof is readily extensible to higher space by induction. A later communication reported that an analytical treatment of the general case, different from the above solution, is given by H. F. Baker, *Polarities for the nodes of a Segre cubic primal in space of four dimensions*, Proc. Camb. Phil. Soc. 32, 1936, p. 510.

Excenter in Hyperbolic Geometry

4082 [1943, 329]. *Proposed by H. S. M. Coxeter, University of Toronto*

In hyperbolic geometry, obtain the condition

$$\cos \tfrac{1}{2}A < \sin \tfrac{1}{2}B + \sin \tfrac{1}{2}C$$

for the triangle ABC to have a proper excircle beyond BC.

Solution by the Proposer. Let E be the excenter, if there is one, so that BE and CE are the external bisectors of the angles B and C. From the triangle ABC, with $BC=a$, we have

$$\sin B \sin C \cosh a = \cos B \cos C + \cos A.$$

Similarly, from the triangle EBC,

$$\cos \tfrac{1}{2}B \cos \tfrac{1}{2}C \cosh a = \sin \tfrac{1}{2}B \sin \tfrac{1}{2}C + \cos E.$$

Eliminating a, we have

$$4 \sin \tfrac{1}{2}B \sin \tfrac{1}{2}C(\sin \tfrac{1}{2}B \sin \tfrac{1}{2}C + \cos E) = \cos B \cos C + \cos A$$
$$= 2 \cos^2 \tfrac{1}{2}A - 2 \sin^2 \tfrac{1}{2}B - 2 \sin^2 \tfrac{1}{2}C + 4 \sin^2 \tfrac{1}{2}B \sin^2 \tfrac{1}{2}C,$$

whence

$$\cos^2 \tfrac{1}{2}A = \sin^2 \tfrac{1}{2}B + \sin^2 \tfrac{1}{2}C + 2 \sin \tfrac{1}{2}B \sin \tfrac{1}{2}C \cos E.$$

The condition for E to be an ordinary (or accessible) point is $\cos E<1$. This is a much simpler form of the condition than that given on p. 239 of my *Non-Euclidean Geometry*, 1942.

Tangents to Kiepert's Hyperbola

4089. [1943, 456]. *Proposed by J. H. Butchart, Grinnell College*

The tangent to Kiepert's hyperbola at a vertex of the triangle is the harmonic conjugate of the symmedian with respect to the altitude and median from that vertex. It meets the corresponding side of the medial triangle at a point on the tangent to the hyperbola at the centroid.

Editorial Note. The proposer gave the following remark:

To establish these, I use the similar isosceles triangles constructed on the sides of the given triangle together with the definition of a conic by projectively related pencils of rays. The second part is a consequence of the properties of the inscribed quadrangle.

We shall add some details which may aid in following the proof. In the solu- of problems 3882, 3883 [1940, 401–404] it is stated on p. 404, that if on the sides of triangle ABC directly similar isosceles triangles P_aBC, P_bCA, P_cAB are constructed, all directed interiorly, or all exteriorly, the straight lines AP_a, BP_b, CP_c are concurrent in a point P; and, as the base angle of the similar triangles varies, P describes a conic κ through A, B, C, G, H, the last two points being the centroid and orthocenter of ABC. Indications of the proof are given by means of projective pencils of rays with centers at two vertices. The conic κ is Kiepert's hyperbola, the isogonal conjugate of the circumdiameter of ABC through the symmedian point K. In that proof the case where the isosceles triangles are interiorly directed with the base angle A, the angle opposite BC, is considered, and this gives the point P at the vertex A. Denote by K_a' the point P_a for this base angle, then $K_a'A$ meets κ in only one point A, whereas for a base angle slightly different from A, the line P_aA meets κ in A and also in a second point P. Hence $K_a'A$ is the tangent at A to κ. Now consider the isosceles triangle K_aCB symmetric to $K_a'BC$ with respect to A_0 the midpoint of BC; then the pencil of rays AK_a', AA_0, AK_a, AH is harmonic, since AH and $K_a'K_a$ are parallel. It is known that AK_a passes through K; this proves the first part.

The second theorem is true for any conic through A, B, C, G; this conic must be a hyperbola since G is inside triangle ABC. Consider the degenerate hexagon $AA\ B\ GG\ C$, where AA, GG denote the tangents t_a, t_g to the conic at A, G. The three points (t_a, t_g), $(AB, GC)=C_0$, $(BG, CA)=B_0$ are collinear, and this proves

a more general theorem: A necessary and sufficient condition for a conic through A, B, C to be a rectangular hyperbola is that it passes through H. For such a conic we prove in the same way that the tangents at A and H meet on B_hC_h, the corresponding side of the pedal triangle for H with respect to ABC.

Ruled Quadrics

4094 [1943, 516]. *Proposed by N. A. Court, University of Oklahoma.* A. If a line moves so that the segment intercepted on it by two fixed skew lines subtends a right angle at a fixed point in space, the line generates a ruled quadric (This proposition is due to E. Bobillier, 1797–1832. It was proved analytically in *Nouvelles Annales de Mathématiques*, 1862, pp. 318, 320 and 379, 380).

B. The segment intercepted by two elements of the same system of a ruled quadric on a variable element of the complementary system subtends a right angle at each of two fixed points in space.

Solution by James Jenkins, Student, University of Toronto. A. Let a, b be the two fixed skew lines and P the fixed point. Let AB be the moving line, with A on a and B on b. The flat pencil of lines PA is projectively related to an axial pencil of perpendicular planes through P (with axis perpendicular to Pa). If the planes Pa, Pb are not perpendicular, the latter will meet this axial pencil in a related flat pencil of lines PB; so $A\overline{\wedge}B$, and AB traces out a regulus. If the planes Pa, Pb are perpendicular, we have instead two flat pencils of lines AB, one joining a particular point on a to every point on b, and the other vice versa.

B. Let a, b be two lines of a regulus. Take any three lines of the associated regulus (or complementary system), and using as diameters the segments cut off on these by a and b, describe three spheres. In general, two of these spheres will meet in a circle which will meet the third in two points. Each of the three segments then subtends a right angle at either of these points. But (by A) all lines giving segments which subtend right angles at these two points belong to a regulus. Since three lines suffice to determine a regulus, these are just the lines of the given regulus. Of course, the two common points of the three spheres may be real, coincident, or conjugate imaginary.

The words "in general" here mean "provided the midpoints of the three segments are not collinear," or "provided the ruled quadric is a hyperboloid." If, on the other hand, the ruled quadric is a paraboloid, then the three spheres have collinear centers and consequently have either no common points or a common circle. In the former case there is no point from which the segments all subtend right angles; in the latter there are infinitely many. Since the points at infinity on a and b form one such segment, the latter case arises only when a and b are "perpendicular."

Solved also by the proposer.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to B. W. Jones, White Hall, Cornell University, Ithaca, New York.

Professor Solomon Lefschetz of Princeton University has been elected an honorary member of the Mexican Mathematical Society.

Associate Professor Reinhold Baer of the University of Illinois has been promoted to a professorship.

Dr. T. A. Bancroft of Iowa State College has been promoted to an assistant professorship.

Dr. F. C. Biesele of the University of Utah has been promoted to an assistant professorship.

Dr. E. E. Blanche of Curtiss-Wright Corporation has been appointed principal statistician and chief of operations for the Foreign Economic Administration in Washington, D. C.

Associate Professor N. R. Bryan of the University of Maine has been promoted to a professorship.

Dr. R. E. Byrne has been appointed to an assistant professorship at the University of California at Los Angeles.

Dr. R. H. Cole of the University of Western Ontario has been promoted to an assistant professorship.

Assistant Professor Mary E. Decherd of the University of Texas has retired.

Assistant Professor H. L. Dorwart of Washington and Jefferson College has been promoted to an associate professorship.

Professor W. S. Erickson of the State Teachers College at Minot, North Dakota, has been appointed to a fellowship in mechanics at Brown University.

Associate Professor W. M. Ewing of Lehigh University has been appointed to an associate professorship at Columbia University.

Dean T. M. Focke of the Case School of Applied Science has been retired with the title emeritus.

Dr. Hans Fried of Swarthmore College has been appointed lecturer.

E. S. Grable of the University of Richmond has been promoted to an assistant professorship.

Dr. Mary C. Graustein has been appointed to an assistant professorship at Tufts College.

Associate Professor E. A. Hedberg of Baylor University, Waco, Texas, has been appointed visiting associate professor in electrical communications at Massachusetts Institute of Technology.

Dr. Olaf Helmer of the College of the City of New York has been appointed lecturer at the New School for Social Research, New York City.

Assistant Professor J. T. Hurt of the Agricultural and Mechanical College of Texas has been promoted to an associate professorship.

Associate Professor A. E. Johns of McMaster University has been promoted to a professorship.

Dr. R. E. Johnson has been appointed to an assistant professorship at Mount Holyoke College.

Assistant Professor Roberta F. Johnson of Wilson College, Chambersburg, Pennsylvania, has been promoted to an associate professorship.

Assistant Professor H. L. Krall of Pennsylvania State College has been promoted to an associate professorship.

Dr. P. E. Lewis of Oklahoma Agricultural and Mechanical College has been promoted to an assistant professorship.

Dr. R. R. R. Luckey of Houghton College has been promoted to an associate professorship.

Professor E. L. Mackie of the University of North Carolina has been appointed Dean of Men.

Dr. Leonard McFadden of Virginia Polytechnic Institute has been promoted to an associate professorship.

Dr. E. J. Mickle of Ohio State University has been promoted to an assistant professorship.

Assistant Professor R. H. Moorman of Tennessee Polytechnic Institute has been granted leave of absence to serve as acting associate professor at the College of Charleston, South Carolina.

Dr. C. G. A. Nordling of the University of Connecticut has been promoted to an assistant professorship.

Miss Elizabeth A. Oliphant of Texas College of Arts and Industries, Kingsville, Texas, has been promoted to an assistant professorship.

Assistant Professor T. S. Peterson of the University of Oregon has been promoted to an associate professorship.

Reverend J. H. Raymond of St. Martin's College, Lacey, Washington, has been promoted to an assistant professorship.

Dr. I. F. Ritter of New York University has been promoted to an assistant professorship.

Associate Professor J. S. Rosen of Eastern New Mexico College has been appointed to an assistant professorship at the University of Kansas City.

Dr. E. H. Rothe has been appointed to an assistant professorship at the University of Michigan.

Assistant Professor P. A. Samuelson of Massachusetts Institute of Technology has been promoted to an associate professorship.

Assistant Professor Edith R. Schneckenburger of Michigan State Normal College, Ypsilanti, Michigan, has been promoted to an associate professorship.

Dr. Abraham Schwartz of Pennsylvania State College has been promoted to an assistant professorship.

R. E. Smith of Allegheny College has been granted a year's leave of absence to teach engineering and mathematics at Brown University.

Associate Professor D. V. Steed of the University of Southern California has been promoted to a professorship.

Assistant Professor A. E. Taylor of the University of California at Los Angeles has been promoted to an associate professorship.

Professor E. H. Taylor of Eastern Illinois State Teachers College has retired.

Assistant Professor Abraham Wald of Columbia University has been promoted to an associate professorship.

Assistant Professor S. E. Warschawski of Washington University has been promoted to an associate professorship.

Associate Professor Antoni Zygmund of Mount Holyoke College has been promoted to a professorship.

The following appointments to instructorships have been announced:
Connecticut College: Dr. Josephine M. Mitchell
Michigan State College: Dr. Margaret Mauch
Smith College: Aida Kalish
University of California: Dr. S. A. Schaaf
University of Chicago: Dr. R. M. Martin
University of Michigan: Dr. A. A. Grau

Professor G. D. Birkhoff of Harvard University died November 12, 1944. He was a charter member of the Mathematical Association.

Professor J. A. Shohat of the University of Pennsylvania died October 8, 1944.

WAR INFORMATION

EDITED BY C. V. NEWSOM

Send news reports upon the utilization of mathematicians or mathematics in war activities to C. V. Newsom, Oberlin College, Oberlin, Ohio.

GENERAL EDUCATION FOR THE ARMED FORCES

Ten basic objectives and twelve specific courses for general education for members of the armed forces are outlined in a report which a committee of the American Council on Education has recently completed at the request of the United States Armed Forces Institute. Dean T. R. McConnell of the University of Minnesota was chairman of the committee. The suggested instructional materials, planned primarily for the period following the end of hostilities, are now being prepared by the Institute for use in correspondence study and group instruction. According to Dr. George F. Zook, president of the Council, "The proposed program should serve as an effective bridge between military activities and the return of men and women to civilian life." The courses have been developed for men and women who would normally register for work at the upper senior high school or junior college level. However, it is assumed that persons who have served in the armed forces will be more mature than the usual high school or junior college student, so the courses have been prepared for an adult point of view.

The report has recently been made available for civilian use under the title, "A Design for General Education." Dr. Zook said, "The committee had not proceeded far with its work before it discovered that civilian educational institutions—particularly secondary schools, junior colleges, and colleges—would be as much interested in this report as the armed forces. These institutions recognize that the postwar educational programs for service personnel will probably demand new curricular patterns. Many schools and colleges are already studying their programs to get ready for this new responsibility. In addition, teachers and administrators throughout the country are reconsidering their provisions for general education." So it is hoped the study will assist institutions in serving returning service personnel who have enrolled in courses similar to those described.

The committee defines a general education as "the type of education which the majority of our people must have if they are to be good citizens, parents, and workers." Ten fundamental objectives are then formulated for such a program. Objective VI, of interest to mathematicians, specifies that a "general education should lead the individual as a citizen in a free society . . . to use scientific methods in the solution of his problems, ..." An explanatory footnote emphasizes that mathematical skills involved in this objective are usually obtained before the student reaches the level of general education considered in the outline; if such skills are not present, it is suggested that the student review courses offered

at the elementary or high school level. "Computational skill beyond the level of high school algebra may be developed as part of a student's program of specialization but is not included as an essential element in his general education."

The courses proposed by the report are as follows: personal and community health; oral and written communications; problems of social adjustment; marriage and family adjustment; development of American thought and institutions; problems of American life; America in international affairs; science—biological and physical; literature—American life and ideals in literature readings; form and function of art in society; music in relation to human experience; philosophy and religion—the meaning and value of life; and vocational orientation. Extensive bibliographies are provided for each course, except for the one on science.

Some mathematics is included as one of the "units" in the outline of the course in physical science. The following syllabus of the unit attempts to combine the general discussion of the unit and its topical outline, as they appear in the report.

1. Numbers. The evolution of the real number system; and some mention of imaginary numbers.
2. Algebra. Importance of "generalized numbers"; the function concept; some review of algebraic operations. "The convenience of algebra" is to be emphasized.
3. Geometry. The abstract nature of geometry; the use of symbolism in making proofs.
4. Postulational Structure of Mathematics. Deductive reasoning as applied in mathematics; Euclidean and non-Euclidean geometries; noncommutative algebras; "mathematical truth."
5. Trigonometry. Definitions of trigonometric functions and illustrations of their use; graphs.
6. Analytic Geometry. "Graphing of a few equations."
7. Differential Calculus. The derivative presented as a time-rate.
8. Integral Calculus. "Qualitative description of the methods in geometric language only."

EDUCATIONAL PROGRAM FOR THE ARMY OF OCCUPATION

The following bulletin was released by the War Department on September 29, 1944.

"Soldiers serving in the Army of Occupation or awaiting shipment home after the defeat of Germany will have an opportunity to further their education or receive practical training to prepare for civilian jobs under an extensive program provided by the Army.

"To the extent that the military mission of a unit will allow, parts of the duty day that are now devoted to strictly military training will be utilized for instruction in academic or vocational subjects, or supervised athletics and recreation. Troops will have free choice as to which phases of the program they follow,

but will be encouraged to enroll in activities that have a bearing on their individual postwar plans and ambitions.

"The program will be applicable in all inactive theaters of operation when the military situation in each permits, but will be especially useful during the anticipated period between the defeat of Germany and the fall of Japan, when large numbers of troops will remain in Europe for a considerable length of time. While many will be sent to the Pacific, many others will stay for occupation duties, and some will be sent home.

"Since troops have been sent abroad at an ever-increasing rate for almost three years, the return of any of the men will require some time, and will depend upon the availability of shipping. A substantial part of the inevitable waiting period can be used profitably in one or more of the educational activities.

"Academic curricula will range in level from the sixth grade through second year college, and will include courses in the liberal arts and in the scientific and pre-professional fields. Many of the courses are designed to aid soldiers who plan to continue their educations after leaving the Army. Facilities of inactive educational institutions may be used for study centers, and opportunities may be given to attend courses at foreign colleges or universities. The nature of the subjects taught at each school will be determined by the preferences of the soldiers.

"Soldiers whose ability to read and write is below the fifth grade standard will be encouraged to enroll in special classes.

"Officers and enlisted men who meet the qualifications will be designated—without regard to rank—to teach the courses. Instructors' outlines and textbooks have been prepared by leading educators, and all teachers will undergo a brief training period.

"In addition to academic subjects, courses in mechanical and technical subjects will be given, and practical training in trades and vocations will be provided for men who plan to go directly into civilian employment after discharge. Specialist personnel will teach the classes, and equipment of the technical services such as Signal Corps, Quartermaster Corps and Ordnance Department will be used for practical work.

"No one will be delayed in returning to the United States by participation in the program. When a soldier receives shipping orders he will pack up and leave immediately, regardless of the stage of any course he might be taking.

"The material for each course is divided into units of work, each unit requiring 20 hours of classroom study. Insofar as possible, each unit is made independent of subsequent units, so that if a student stops in the middle of a course to go home, he still will have benefited from the instruction.

"Schools were operated for servicemen overseas after the World War, when more than 9,000 soldiers attended an Army university at Beaune, France, and several thousand more took courses at British and French universities. With a far larger number of Army personnel overseas during this war, the scope of the present program will be infinitely greater.

"During the present war, more than 800,000 servicemen, both in the United States and overseas, have taken correspondence and self-teaching courses in their off-duty hours from the United States Armed Forces Institute, an agency sponsored jointly by the Army and Navy. Textbooks and educational advice is provided, as well as aid in obtaining academic credit in schools and colleges. When hostilities cease in each of the war theaters, the educational activities of the Institute will be carried on during part of the duty-time of the servicemen.

"The entire program will, of course, be subject to considerations of military necessity. Theater commanders will decide which units under their command may substitute educational activities for portions of their military training programs."

REGISTRATION IN USAFI COURSES

Bulletin No. 72 of "Higher Education and National Defense," issued by the American Council on Education, reveals the trend of registrations in the correspondence courses sponsored by the United States Armed Forces Institute. During the months of April, May, and June, 1944, a total of 22,463 men in the Army enrolled for correspondence courses. Of this number, nearly one-fourth (5,118) registered for work in mathematics; moreover, 1,722 men wanted courses more advanced than elementary arithmetic and algebra. Other courses which proved to be popular, with the enrollment in each, were bookkeeping and accounting (3,167), engines and boilers (1,593), auto mechanics (1,505), radio and communication (1,497), and English grammar (1,112).

For the same period of time, a total of 11,444 men from the Navy signed for correspondence courses. Of this number, 3,368 men registered for work in mathematics; moreover, 985 wanted work more advanced than elementary arithmetic and algebra. Other courses popular with men from the Navy were bookkeeping and accounting (985), engines and boilers (882), refrigeration and air conditioning (712), English grammar (712), and American history (701).

During the same three months, a total of 10,740 men from both the Army and the Navy enrolled through the Institute for correspondence courses sponsored by colleges and universities. The four most popular classifications of courses were social studies (2,027), English (2,024), mathematics (1,951), and business (1,435).

The American Council on Education warns that caution should be used in the interpretation of this data. For one thing, a basic motive of men in their selection of USAFI correspondence courses is the "improvement of the individual's military effectiveness and to procure advance in rating."

THE FUTURE OF THE NAVY V-12 PROGRAM

An announcement by the Navy Department on October 15, 1944, revealed that it is not planned to enroll any new trainees in any of the Navy college training programs for the term beginning March 1, 1945. The announcement emphasized, however, that students already in training "will continue the train-

ing as scheduled." The decision to eliminate new enrollments for that period was based upon revised estimates of future officer requirements.

At the time of the announcement, approximately 69,000 men were enrolled in the V-12 Program. This figure, however, is expected to drop to about 50,000 on November 1, when many of the men first registered in the Program will have completed the prescribed number of terms. Based upon an estimate of the number of men in the various classes of the Program, it appears that the total enrollment may drop to 30,000 men on March 1.

It should be distinctly understood that the Navy has given no indication at the present time that it is planned to discontinue the V-12 Program at an early date. The future of the Program will undoubtedly depend upon military exigencies.

ENGINEERING EDUCATION AFTER THE WAR

The following is an excerpt from the report of the Committee on Engineering Education after the War, sponsored by the Society for the Promotion of Engineering Education, as summarized by Dean H. P. Hammond, Chairman, before the annual meeting of the Society on June 24, 1944.

"The committee calls attention to two recent trends in engineering and industry for which provision should be made in educational programs. One of these trends is the rapid increase in the application of the engineering method to the management and operation of industry as is evidenced by the phenomenal success of graduates in these activities. The other is the greatly increased need for engineers equipped to practice at high scientific and creative level. The remark was made recently by an engineer having unusual knowledge of technical developments during the war, that if engineers are not to be equipped to work on new developments at high creative level, physicists are likely to take over the designing of the devices which will use the scientific principles they are discovering—as indeed they are already doing in some instances.

"In order to provide for the satisfaction of the needs incident to these trends, the committee suggests, for consideration, a plan of curricular differentiation in the fourth year, through which three options would be offered within each major professional curriculum:

(1) Continuation of the present type of 4-year program essentially as a terminal curriculum, but with modifications advocated by the committee, for a majority of students.

(2) An alternative fourth year emphasizing subjects dealing with the management of construction and production enterprises.

(3) A fourth year intended to prepare for additional years of advanced study by strengthening the student's command and extending his knowledge of basic sciences and mathematics, and by introducing him to the methods of advanced study. This fourth year and the year or years of graduate study to follow would be planned as a unit rather than as two stages marked by the usual differences of undergraduate and postgraduate programs."

THE MATHEMATICAL ASSOCIATION OF AMERICA

THE ANNUAL MEETING OF THE METROPOLITAN NEW YORK SECTION

The third annual meeting of the Metropolitan New York Section of the Mathematical Association of America was held at New York University, Washington Square, New York City, on Saturday, April 22, 1944. Professor R. M. Foster, Chairman of the Section, presided at the morning session and at the business meeting. Mr. Max Peters, Vice-Chairman of the Section, presided at the afternoon session.

The attendance was one hundred and two, including the following forty-seven members of the Association: Claire F. Adler, R. G. Archibald, I. L. Battin, Brother Bernard Alfred (Welch), Frank Boehm, C. B. Boyer, A. B. Brown, Jewell Hughes Bushey, Ruth T. Coleman, H. R. Cooley, T. F. Cope, W. H. H. Cowles, D. R. Davis, J. E. Eaton, W. H. Fagerstrom, J. M. Feld, Edward Fleisher, R. M. Foster, P. H. Graham, Marion C. Gray, Mary W. Gray, Harriet M. Griffin, C. C. Grove, C. E. Heilman, J. H. Hlavaty, Joseph Jablonower, Nathan Lazar, C. H. Lehmann, H. F. Mac Neish, May Hickey Maria, Joseph Milkman, F. H. Miller, M. A. Nordgaard, Max Peters, Mina S. Rees, Moses Richardson, S. G. Roth, Charles Salkind, A. A. Schwartz, James Singer, E. R. Stabler, H. E. Wahlert, Israel Wallach, Alan Wayne, John Williamson, Jack Wolfe, R. C. Yates.

At the beginning of the morning session Dean Charles M. McConn of the Washington Square College of New York University welcomed the Section to New York University. At the business meeting the following officers were elected for the coming year: Chairman, Jewell Hughes Bushey, Hunter College; Vice-Chairman, Nathan Lazar, Midwood High School; Secretary, H. E. Wahlert, New York University; Treasurer, F. H. Miller, Cooper Union.

The following program was presented:

1. *Elementary mathematical theory of exterior ballistics*, by Professor H. F. Mac Neish, Brooklyn College.

Three aspects of the subject were considered. First, the parabolic trajectory in the ideal case in which there is no resistance. Many interesting geometric properties of the parabolic trajectory were considered. Second, the computation of the path of a projectile by the method of differences (due to F. R. Moulton), which takes into account the most important types of resistance. Third, the computation of the path of a projectile by a new procedure called the functional method. This method requires no knowledge of the method of differences, and is comparatively simple both in theory and in practice.

2. *Applications of mathematics in aerodynamics*, by Professor R. Paul Harrington, Polytechnic Institute of Brooklyn, introduced by Professor R. M. Foster.

Two mathematical procedures which may be used in the solution of certain aerodynamic problems were discussed. The first dealt with the use of conjugate functions to represent certain physical characteristics of fluid flow. These flows were later transformed into curves (in another plane) representing airfoil cross-sections. The second dealt with the mathematical representation of a fuselage in an airstream, and the calculation of the effects upon the angle of attack of the propeller blades. The fuselage was represented by an ellipsoid of revolution, and the problem resolved iself into the determination of the coefficients in a series representing the velocity potential. The series was found to be composed of Legendre and associated Legendre functions so chosen that the surface of the fuselage became a stream line of the flow. The theoretical effects were certain velocity changes of such a nature as to alter the angle of attack of the blade by two or three degrees. This might result in torsional and bending vibrations of the blades.

3. *Combinatorial statistics*, by Dr. Jacob Wolfowitz, Columbia University, introduced by Professor Jewell Hughes Bushey.

The speaker remarked that the control of quality in articles produced by mass production offers an important field for the application of combinatorial statistics. He pointed out that sampling methods are in order whenever the cost of inspection makes complete inspection uneconomical or prohibitive. The Dodge-Romig sampling method was discussed from the viewpoint of modern statistical theory. The Shewhart method of runs for control of quality during manufacture was also described. A brief description of the theory of runs above and below the median, and runs up and down, was given. The asymptotic normality of the distribution of runs up and down will be proved by the speaker in a forthcoming article.

4. *A guiding philosophy for teaching demonstrative geometry*, by Mr. Morris Hertzig, Forest Hills High School, introduced by Mr. Max Peters.

Mr. Hertzig stated that the high school geometry course cannot be successfully taught as an abstract mathematical science. If taught as an empirical science of space, the following objectives can be realized: (1) the student has an opportunity to engage in experimental inquiries and to study the logic of sound experimental procedure; (2) he learns that an induction obtained as a generalization from several observations is insufficient to give scientific knowledge of the principle observed; (3) the student is given his only opportunity to study the complete scientific method (because high school courses are not organized with this objective in mind); (4) through a study of the deductive aspect of the scientific method the student develops understanding of an important device for extending the limits of knowledge.

5. *Mathematics and empirical science*, by Professor C. G. Hempel, Queens College, introduced by Professor T. F. Cope.

The speaker examined the significance of geometric theories as they pertain

to our knowledge of physical space. Two meanings of the term "geometry" were recognized, namely, pure geometry and physical geometry. A pure geometry, euclidean or non-euclidean, is an uninterpreted deductive system. It does not concern physical space, and the mathematical certainty of its theorems is due to the fact that each theorem simply re-asserts part of the content of the postulates. A system of physical geometry is obtained by assigning to each primitive of pure geometry its customary physical meaning. This transforms the postulates and theorems into physical hypotheses which may be said to concern the structure of physical space. When combined with the nongeometric part of physics, a physical geometry can be tested empirically. Recent physical findings appear to support the hypothesis, developed in connection with the general theory of relativity, that the geometric structure of the universe at large is of a certain non-euclidean type.

H. E. WAHLERT, *Secretary*

THE ANNUAL MEETING OF THE MINNESOTA SECTION

The annual meeting of the Minnesota section of the Mathematical Association of America was held at Macalester College in St. Paul, Minnesota, on Saturday, May 6, 1944. Sessions were held in the forenoon, at luncheon, and in the afternoon. Professor C. H. Gingrich, Chairman of the Section, presided.

Forty-five persons attended the meeting, including the following twenty-one members of the Association: R. W. Brink, W. E. Brooke, L. E. Bush, W. H. Bussey, E. J. Camp, C. S. Carlson, R. W. Erickson, I. C. Fischer, Gladys Gibbens, C. H. Gingrich, W. L. Hart, Dunham Jackson, G. M. Jensen, W. R. McEwen, Helen K. Milleson, J. M. H. Olmsted, Abraham Spitzbart, F. J. Taylor, H. L. Turrittin, A. L. Underhill, G. L. Winkelmann, and Sister Thomas à Kempis (institutional representative).

At the business meeting the following officers were elected for the coming year: Chairman, L. E. Bush, College of St. Thomas; Secretary, A. L. Underhill, University of Minnesota; Executive Committee, E. J. Camp, Macalester College, C. S. Carlson, St. Olaf College, H. L. Turrittin, University of Minnesota, K. W. Wegner, Carleton College.

The following seven papers were presented:

1. *Sophus Lie*, by Professor C. S. Carlson, St. Olaf College.

This paper was devoted to a biographical sketch of Sophus Lie. Particular attention was directed to his early childhood, his education, and the first few years following the completion of his education. An attempt was made to show how he came to enter upon his mathematical studies, and his mathematical career was summarized briefly.

2. *A note on the evaluation of* $\int_0^\infty \cos mx\,(1+x^2)^{-1}dx$, by Professor E. J. Camp, Macalester College.

The function $u(m, \beta)$ was defined by the equation

$$u(m, \beta) = \int_0^\infty \frac{e^{-\beta x} \cos mx}{1 + x^2} dx.$$

It was shown that this function satisfies the differential equation

$$\frac{d^2u}{dm^2} - u = - \frac{\beta}{\beta^2 + m^2}.$$

After solving this differential equation, it was found that

$$\lim_{\beta \to 0} u(m, \beta) = \int_0^\infty \frac{\cos mx}{1 + x^2} dx = \frac{\pi}{2} e^{-|m|}.$$

3. *Peacetime education in the light of the war training programs*, by Professor R. W. Brink, University of Minnesota.

4. *Remarks on activities of the committee on war programs of the American Mathematical Society and the Mathematical Association of America*, by Professor W. L. Hart, University of Minnesota.

5. *Elliptical and ellipsoidal boundary value problems*, by Professor Dunham Jackson, University of Minnesota.

6. *A study: Caroline Herschel*, by Sister M. Thomas à Kempis, College of St. Teresa.

This address was devoted to a brief study of Caroline Herschel, the first woman to discover a comet. The speaker described how Caroline Herschel shared the labors, hardships, and fatigues of her brother, William Herschel. Her perseverance and devotion made possible the distinctions enjoyed by her brother and by her nephew Sir John Herschel.

7. *A simple application of the density of the values of the complex exponential function* 1^z, by Professor J. M. H. Olmsted, University of Minnesota.

Professor Olmsted gave a proof that corresponding to any $\epsilon > 0$ and any irrational number a, there exist integers p and q such that

$$\left| a - \frac{p}{q} \right| < \frac{\epsilon}{q}.$$

The proof was developed from a special case of a simple consequence of the fact that, if a is an irrational number, then the values of 1^a are a set of points dense on the circumference of the unit circle.

A. L. UNDERHILL, *Secretary*

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